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FINITE ORBIT ANALYSIS FOR LONG WAVELENGTH MODES IN A PLASMA WITH A HOT COMPONENT

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Abstract

The z-pinch model is used to calculate finite Larmor radius effects of a plasma with a hot component plasma annulus. The equations are analyzed for layer modes and the finite Larmor radius stabilization condition is calculated. Stability requires $k^2\rho_h^2 R\beta_h/\Delta \gtrsim 1$, where k is the wavenumber in the z-direction, ρ_h the hot species Larmor radius, β_h the hot particle beta and Δ the thickness of the pressure profile. In addition a new instability is found, caused by the interaction of the precessional modes associated with inner and outer edges of the hot particle pressure profile.

I. Introduction

We investigate finite orbit effects on long wavelength, curvature driven modes for a plasma with a hot component.[1-6] Our method is to construct a variational form for an axisymmetric, linear pinch using the Vlasov and Maxwell equations. Specializing to flute instabilities of a z-pinch, we introduce the orbit expansion, and recover results of zero-Larmor radius[5] theory and finite-Larmor radius[7,8] theory in the eikonal approximation. New analysis is presented to describe the finite-Larmor radius effects on long wavelength layer modes. One result of the analysis is that the finite Larmor radius terms become small for long axial wavelengths if the perturbed magnetic field, δB, is proportional to the radial gradient of the unperturbed field, $\partial B/\partial r$, just as is in the case for long wavelength modes in the conventional magnetohydrodynamic (MHD) ordering.[9] For these modes stability will arise if $k^2\rho_h^2 R/\Delta_B \gtrsim 1$, where k is the wave number in the z-direction, R the curvature radius, and Δ_b the magnetic field gradient in the hot pressure region.

In addition this paper improves on the analysis of layer modes,[6,10,11] which is here calculated for arbitrary beta rather than for $\beta_h < 1$, where β_h is the beta of the hot component. The improved dispersion contains a new mode of instability. For a thin annulus of hot particles the new mode is caused by the precessional oscillation of the plasma currents from the negative pressure gradient region interacting with the precessional oscillation from the positive pressure gradient region.

In Sec. II the derivation of the finite Larmor radius equations is presented. In Sec. III, these equations are analyzed for radially long wavelength modes assuming $k\Delta < 1$, where Δ is the thickness of the hot plasma annulus. In Sec. IV an interpretation of these results for realistic geometry is given.

II. Derivation of Equations

We begin by constructing a quadratic form that embodies the equation of motion.

$$\delta W = \int d\underset{\sim}{r}\, \frac{\underset{\sim}{E}^{+}}{i\omega} \cdot \left(\nabla\times\underset{\sim}{B}_1-\underset{\sim}{J}_1(\underset{\sim}{E})\right) , \tag{1}$$

where $\underset{\sim}{E}^{+}$ is the adjoint and $\underset{\sim}{J}_1(\underset{\sim}{E})$, the perturbed current, is found from the Vlasov equation. We use the following representation for $\underset{\sim}{E}$

$$\underset{\sim}{E} = -\nabla\Phi+i\omega A\underset{\sim}{b} + i\omega\underset{\sim}{b}\times\nabla a , \tag{2}$$

where $\underset{\sim}{b} = \underset{\sim}{B}/|B|$ and Φ, A, a are all proportional to $\exp[-i\omega t+ikz+i\ell\vartheta]$ for a linear, axisymmetric pinch. The perturbed current $\underset{\sim}{J}_1$ is simply

$$\underset{\sim}{J}_1 = \int d\underset{\sim}{v}\; e\underset{\sim}{v}f_1 , \tag{3}$$

where a sum over species is implied and f_1 is the solution to

$$\left.\frac{d}{dt}\right|_o f_1 = -\left.\frac{d}{dt}\right|_1 f_o , \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\nabla + \frac{e(E+\underset{\sim}{v}\times\underset{\sim}{B})\cdot\partial/\partial\underset{\sim}{v}}{m} . \tag{4}$$

f_o is a function of E, P_z, P_ϑ where $E = mv^2/2$, and P_z, P_ϑ are the canonical moments in the z and ϑ directions respectively. The zeroth order electrostatic potential is assumed to be zero. After considerable algebra we find

$$\frac{\delta W}{\pi L_z} = \int B_1^+ B_1 \; rdr$$

$$-e^2 \int rdrd\underset{\sim}{v} \{ [-\Phi^+\Phi - \underset{\sim}{v}\cdot\underset{\sim}{s}(a_r^+\Phi + \Phi^+ a_r)$$

$$+ \underset{\sim}{v}\cdot\underset{\sim}{b}(A^+\Phi+\Phi^+A) + v\cdot\underset{\sim}{b}v\cdot\underset{\sim}{s}(a_r^+A+A^+a_r)] f_\varepsilon$$

$$+ \underset{\sim}{\lambda}\cdot\underset{\sim}{b} \; v\cdot\underset{\sim}{b}A^+A \; + \underset{\sim}{\lambda}\cdot\underset{\sim}{s}\underset{\sim}{v}\cdot\underset{\sim}{s}a_r^+a_r \}$$

$$\frac{-e^2}{m^3} \int dP_z dP_\vartheta d\varepsilon T(\omega f_\varepsilon + \ell f_{P_\vartheta} + k_z f_{P_z}) \sum_n \frac{\zeta_n \zeta_n^+}{\omega - k_z \bar{v}_z - \ell\bar{\dot{\vartheta}} - n\Omega} \; , \qquad (5)$$

where

$$\underset{\sim}{\lambda} = rf_{P_\vartheta} \; \hat{\underset{\sim}{\vartheta}} + f_{P_z} \; \hat{\underset{\sim}{z}}$$

$$\underset{\sim}{s} = \hat{\underset{\sim}{r}}\times\underset{\sim}{b}$$

$$\zeta_n = \frac{1}{T} \oint d\tau \; \exp[-in\Omega\tau + i\ell\delta\vartheta + ik\delta z] (\Phi - \underset{\sim}{v}\cdot\underset{\sim}{b}A - \underset{\sim}{v}\times\underset{\sim}{b}\cdot\nabla a) \Big|_{\substack{r=r(\tau) \\ \underset{\sim}{v}=\underset{\sim}{v}(\tau)}}$$

$$T = \frac{2\pi}{\Omega} = \int d\tau = \int \frac{dr}{|v_r|}$$

$$z(\tau) = z_o + \bar{v}_z\tau + \delta z(\tau)$$

$$\vartheta(\tau) = \vartheta_o + \bar{\dot{\vartheta}}\tau + \delta\vartheta(\tau)$$

$L_z \equiv$ axial system length .

Use of the time-reflection symmetry of the orbits,

$$r(\tau) = r(-\tau),\ v_r(\tau) = -v_r(-\tau),\ \dot{\vartheta}(\tau) = \dot{\vartheta}(-\tau),\ v_z(\tau) = v_z(-\tau)$$

yields a simple relation between solutions of the adjoint operator

$$\Phi = \Phi(r)\ \exp\left[i(kz+\ell\vartheta-\omega t)\right]$$

$$\Phi^{+} = \Phi(r)\ \exp\left[-i(kz+\ell\vartheta-\omega t)\right]\ . \tag{6}$$

Identical relations hold for a and A. In terms of the r dependent amplitudes $\Phi(r)$, $a(r)$, $A(r)$, the quadratic form is then self-adjoint.

For flute modes in a z-pinch ($\underset{\sim}{b}=\hat{\underset{\sim}{\vartheta}}$, $\ell=0$), Eq. (5) simplifies to

$$\frac{\delta W}{\pi L_z} = \int rdr\{B_1^2+e^2\int d\underset{\sim}{v}[(\Phi^2+2v_z a_r\Phi)f_\varepsilon - v_z f_{P_z} a_r^2$$

$$- \sum_{n=-\infty}^{\infty} \frac{(\omega f_\varepsilon+kf_{P_z})}{\omega-k\bar{v}-n\Omega}\ (\Phi+v_z a_r-ikv_r a)_n^2\}]\ , \tag{7}$$

where $B_1 = a_{rr}-k^2a$ in this limit. A more useful form of Eq. (7), suitable for applying a finite Larmor radius expansion, can be obtained through the use of several identities:

$$v_z = -\frac{d}{dt}\left(\frac{v_r}{\Omega_c}\right) + v_D \tag{8a}$$

$$v_r = \frac{\dot{v}_z}{\Omega_c} \tag{8b}$$

$$\int mv_r^2 g_{P_z} d\underset{\sim}{v} = \int v_z g d\underset{\sim}{v} \tag{8c}$$

$$\int v_z g_\varepsilon d\underset{\sim}{v} = \int v_D g_\varepsilon d\underset{\sim}{v} = -\int\ g_{P_z} d\underset{\sim}{v}\ , \tag{8d}$$

where $\Omega_c \equiv eB(r)/mc$ and $v_D \equiv v_B + v_{\kappa\|}$, $v_B \equiv \frac{-v_r^2}{\Omega_c}\frac{B'}{B}$, $v_{\kappa\|} \equiv \frac{v_\vartheta^2}{\Omega_c r}$ and g is any function of ε, P_z and P_ϑ. Equations (8a,b) follow exactly from the unperturbed orbit equations. We also use the identity:

$$\sum_n \frac{\zeta_n^2}{\omega-k\bar{v}-n\Omega} = -\frac{1}{iT}\oint d\tau \int_0^\infty d\tau' \ \zeta(\tau)\zeta(\tau'-\tau)\exp[i(\omega-k\bar{v})\tau'] \ , \qquad (9)$$

where $\zeta = (\Phi+v_z a_r+ikv_r a)\exp(-ik\delta z)$. For manipulations of Eq. (9) a useful relation is

$$\oint d\tau\int_0^\infty d\tau'\alpha(\tau)\beta(\tau'-\tau)\exp[i(\omega-k\bar{v})\tau'] = \oint d\tau\int_0^\infty dt'\beta(\tau)\alpha(\tau'-\tau)\exp[i(\omega-k\bar{v})\tau'] \ ,(10)$$

for any periodic functions α,β which is easily shown through a change of variables in the integration. Noting that $\int rdrd\underset{\sim}{v}$ and m^{-3} $\int d\varepsilon dP_\vartheta dP_z \oint d\tau$ can be used interchangeably in Eq. (7), we obtain the following after use of the identities (8)-(10) and integrations by parts:

$$\frac{\delta W}{\pi L_z} = \int rdr[B_1^2 + e^2\int d^3v \ \{\frac{\omega^2}{k^2}\xi^2B^2f_\varepsilon$$

$$- \frac{(\omega f_\varepsilon+kf_{P_z})}{ik^2}\int_0^\infty d\tau' \ \Lambda(\tau) \ \Lambda(\tau'-\tau)\exp[i(\omega-k\bar{v})\tau']\}] \ , \qquad (11)$$

where $\Lambda \equiv (\omega\xi B+iv_rB_1)\exp(-ik\delta z)$ and the radial displacement $\xi \equiv \frac{-1}{B}(\frac{k\Phi}{\omega} + a_r)$ has been introduced.

To proceed further we introduce the orbit expansion $\delta=\rho_s/\Delta<<1$, where Δ=scale length. The frequency ω is assumed to be small

$[\frac{\omega}{\Omega} = \vartheta(\delta^2)]$. In the low frequency limit, the integral in Eq. (11) can be evaluated to $\vartheta(\delta^3)$ (the necessary order to pick up the FLR corrections) by expanding the exponential $\exp[i(\omega-k\bar{v})\tau'] \cong 1+i(\omega-k\bar{v})\tau'$ when it multiplies an oscillatory term. We find

$$\oint d\tau \int_0^\infty d\tau' \Lambda(\tau)\Lambda(\tau'-\tau)\exp[i(\omega-k\bar{v})\tau']$$

$$\cong \oint d\tau \; \{\frac{\bar{\Lambda}^2}{-i(\omega-k\bar{v})} - 2\tilde{\Lambda}_{even} \int^\tau d\tau' \; \tilde{\Lambda}_{odd} +$$

$$i(\omega-k\bar{v})[(\int^\tau d\tau' \; \tilde{\Lambda}_{odd})^2 - (\int^\tau d\tau' \; \tilde{\Lambda}_{even})^2]\} \;, \qquad (12)$$

where $\bar{\Lambda} = \frac{1}{T} \oint d\tau\Lambda(\tau)$, $\tilde{\Lambda} = \Lambda-\bar{\Lambda}$, $\tilde{\Lambda}_{even}(-\tau) = \tilde{\Lambda}_{even}(\tau)$ and $\tilde{\Lambda}_{odd}(-\tau) = -\tilde{\Lambda}_{odd}(\tau)$. Expanding Λ to $\vartheta(\delta^3)$ and inserting it into Eq. (12) yields the desired quadratic form containing the FLR corrections. To simplify the analysis we evaluate the FLR contributions for the hot species in the limit ω, $kv_{\kappa\perp} << k\bar{v}_B$ $\left(v_{\kappa\perp} = \frac{v_\perp^2}{2r\Omega},\; \bar{v}_B \cong \frac{-v_\perp^2 B'}{2\Omega B}\right)$ since this is the limit where the FLR terms are most significant. We also assume that the background ion species dominates the inertia $(m_i n_i >> m_h n_h)$ and satisfies the condition $\omega >> kv_{B_i}$. The result then is

$$\frac{\delta W}{\pi L_z} = \int r dr \; \{(1+G_1)(B_1+\xi B')^2 + \frac{2(1-G_2)}{r} \xi B(B_1+\xi B')$$

$$+ (P_\parallel' - BB'+G_3 \; B^2/r)\xi^2/r$$

$$- n_i m_i \omega(\omega-\omega^*)(\xi^2 + \frac{1}{k^2B^2} ((\xi B)' + B_1)^2)$$

$$+ \rho_h^2(B'^2(\frac{B_1}{B'})'^2 + k^2B_1^2)\} \;, \qquad (13)$$

where

$$\omega^* = k\int \frac{mv_\perp^2}{2} f_{p_{zi}} \, d\underset{\sim}{v}$$

$$G_1 = -\sum_s e^2 \int d\underset{\sim}{v} \frac{(\omega f_\varepsilon + k f_{p_z})}{\omega - k\bar{v}} \bar{v}_B v_B \frac{B^2}{B'^2}$$

$$G_2 = -\sum \frac{m^2}{B^2} \int d\underset{\sim}{v} \frac{(\omega f_\varepsilon + k f_{p_z})}{\omega - k\bar{v}} \frac{v_\perp^2 v_\parallel^2}{2}$$

$$G_3 = -\sum \frac{m^2}{B^2} \int d\underset{\sim}{v} \left[\frac{(\omega f_\varepsilon + k f_{p_z})}{\omega - k\bar{v}} v_\parallel^4 + r f_{p_\vartheta} v_\parallel^3\right]$$

$$\rho_h^2 = \frac{-m_h^2}{B^2} \int d\underset{\sim}{v} \frac{f_{p_z}}{\bar{v}_B} \frac{v_\perp^6}{16\Omega^2} .$$

The quantities $G_{1,2,3}$ summed over species, are the same as defined in Ref. 6. Note that G_1 is defined so that in the low frequency, weak curvature limit, the $\bar{v}_B$ terms cancel in the numerator and denominator. The leading terms in G_1 are then the same as found in the zero-orbit theory:

$$G_1 \cong e^2 \int d^3v \, f_{p_z} v_B \frac{B^2}{B'^2} = -e\int d^3v \frac{mv_r^2 f_{p_z}}{B'}$$

$$= \frac{-J_{zhot}}{B'} = -1 - \frac{B}{rB'} + \frac{J_{zcold}}{B'} . \tag{14}$$

Since the FLR corrections are neglected on terms proportional to ω or the curvature $\kappa \equiv -1/r$, the only FLR modification of the zero-orbit result is the term proportional to ρ_h^2. Details of the derivation of the FLR correction are included in the appendix. The eikonal results for the FLR correction can be recovered by letting

$$\rho_h^2 \left(B'^2 \left(\frac{B_1}{B'}\right)'^2 + k^2 B_1^2 \right) \rightarrow k_\perp^2 \rho_h^2 B_1^2$$

in Eq. (13). An interesting property of the FLR term is that it vanishes in the limit $k^2 \rightarrow 0$ if $B_1 \propto B'$. Here $B_1 \propto B'$ corresponds to a rigid motion of the entire plasma annulus, so it is physically plausible that the FLR effect vanishes for this case.

The Euler equations resulting from Eq. (13), found by varying δW with respect to ξ and B_1, are:

$$B'(1+G_1)(B_1+\xi B') + \frac{(1-G_2)}{r} B(B_1+\xi B')$$
$$+ (P_\parallel - G_2 BB' + G_3 \, B^2/r)\xi/r - n_i m_i \omega(\omega-\omega^*)\xi$$
$$+ \frac{1}{r}\left(r \, \frac{n_i m_i \omega(\omega-\omega^*)}{k^2 B^2} \left((\xi B)' + B_1\right)\right)' = 0 \qquad (15)$$

$$(1+G_1)(B_1+\xi B') + \frac{(1-G_2)}{r} \xi B - \frac{n_i m_i \omega(\omega-\omega^*)}{k^2 B^2} \left((\xi B)' + B_1\right)$$
$$- \frac{1}{r}\left(r \rho_h^2 B'^2 (B_1/B')'\right)' + k^2 \rho_H^2 B_1 = 0 \, . \qquad (16)$$

If the FLR terms can be neglected, Eq. (16) becomes an algebraic equation for B_1. Inserting the zero-FLR solution for B_1 into Eq. (15)

yields Eq. (39) of Ref. 6 in the limit $\omega/\Omega_{ci} \ll 1$. (Reference 6 neglects ω_i^*). In the finite FLR case, Eqs. (15) and (16) must be solved as two coupled second order differential equations.

Equation (16) shows the increased significance of the FLR correction in the weak curvature-low frequency limit. Since the term $(1+G_1) = \vartheta(\kappa)$, the FLR term is the same order as the other terms in Eq. (16) when $\rho_h^2/\Delta^2 \sim \kappa\Delta_B$ (Δ = pressure gradient scale length, Δ_B = magnetic field scale length $\cong \Delta/\beta$). A limit of interest is $1 \gg \rho_h^2/\Delta^2 \gg \kappa\Delta_B$, the "strong FLR" limit, in which case the FLR terms dominate Eq. (16). Both the strong and weak FLR limits will be analyzed in the next section.

III. Long-wavelength modes

A tractable limit in which Eq. (15) and (16) can be analyzed is the case $k\Delta \ll 1$, i.e., long axial wavelengths. This case is of special interest in the strong FLR limit when FLR can stabilize all of the short wavelength curvature driven modes. We will consider both the strong and weak FLR limits since the latter has not been carried out for arbitrary β.

We will examine stability for equilibria of the type shown in Fig. 1, i.e., a hot plasma annulus centered about radius R with a background of cooler plasma having a finite pressure gradient only in the outer half of the annulus. This is the model used in Ref. 6 to simulate an Elmo Bumpy Torus (EBT) type configuration. Similar results can be obtained for other profiles, such as disc-like hot plasma distributions to model tandem mirrors or other devices. For simplicity, we assume that $f(\varepsilon,P_\vartheta,p_z) = g(\varepsilon,P_z)\delta(P_\vartheta)$ so $G_2=G_3 \propto P_\parallel=0$

and set $\omega^*=0$ for the background plasma. Equations 15 and 16 then become

$$B'(1+G_1)(B_1+\xi B') + \frac{B}{r}(B_1+\xi B') - n_i m_i \omega^2 \xi + B\Phi' = 0 \qquad (17a)$$

$$(1+G_1)(B_1+\xi B') + \frac{\xi B}{r} - \Phi - \frac{1}{B'}\left(\rho_h^2 B'^2 \left(\frac{B_1}{B'}\right)\right)' + k^2 \rho_h^2 B_1 = 0 \ , \qquad (17b)$$

where the quantity $\Phi \equiv \frac{n_i m_i \omega^2}{k^2 B^2}\left((\xi B)' + B_1\right)$. In the long wavelength limit, two types of modes are generally found: $\omega \approx \gamma_{MHD}$ or $\omega \approx \omega_{\kappa\perp}$ where $\gamma_{MHD} \equiv \left(\frac{kP_h}{n_i m_i r}\right)^{1/2}$ and $\omega_{\kappa\perp} \equiv -\left\langle\frac{kv_{\perp h}^2}{2\Omega_h r}\right\rangle$. We are also usually interested in the limit $\omega_{\kappa\perp}/\gamma_{MHD} >> 1$, one of the decoupling conditions necessary for stability in systems with unfavorable curvature. If we use these estimates for ω in Eq. (17a) then we find Φ is $\vartheta(1/k\Delta)$. To lowest order in $k\Delta$, we have

$$\Phi \propto (\xi B)' + B_1 = 0 \ . \qquad (18)$$

To proceed further analytically, some ordering of the finite Larmor radius terms is necessary.

(a) <u>Weak FLR Limit</u>

Consider first the weak FLR limit where the terms $\propto \rho_h^2$ are neglected. Iterating on the solution for Φ using Eq. (18) in Eq. (17b) gives

$$\Phi = -(1+G_1)\xi' B + \frac{\xi B}{r} \ . \qquad (19)$$

Using Eqs. (18) and (19) in Eq. (17a) gives an equation for ξ:

$$[(1+G_1)\xi' B^2]' - \frac{\xi BB'}{r} = 0 \ . \tag{20}$$

We are considering the limit $\Delta_B/r \ll 1$, so r is treated as a constant in Eq. (20). A boundary condition on ξ is found from the continuity of Φ at the two edges of the hot particle annulus:

$$\Phi(r^+) = \frac{n_i m_i \omega^2}{k^2 B} \xi'(r^+) \equiv \frac{n_i m_i \omega^2 K \xi(r^+)}{k^2 B}$$

$$= \Phi(r^-) = -(1+G_1)\xi'(r^-) + \frac{\xi(r^- B)}{r} \ , \tag{21}$$

where with $\varepsilon>0$,

$$r^+ = \lim_{\varepsilon\to 0} R \pm (\Delta+\varepsilon) \ ,$$
$$r^- = \lim_{\varepsilon\to 0} R \pm (\Delta-\varepsilon) \ .$$

The quantity K is precisely determined from the solution outside of the hot particle annulus. If $\omega < kv_A$, $K \cong |k|$ for the inner edge of the annulus and we assume that n_i vanishes on the outer edge of the annulus.

The quantity $1 + G_1$ is given by

$$1+G_1 = -\frac{B}{rB'}\left(1-\frac{\omega}{\omega_{\kappa\perp}} - \tilde{\beta}\right) \ , \tag{22}$$

where $\omega_{\kappa\perp} = \frac{kP'_{\perp h}}{m\Omega r n'_h}$ and $\tilde{\beta} = \frac{-rP'_{\perp c}}{B^2}$

is the Nelson-Lee-Van Dam parameter.[3] For simplicity we take profiles such that $\omega_{\kappa\perp}$ is a constant and $\tilde{\beta}=0$ (i.e., P_c/B^2 constant) on the inner half of the annulus and $\tilde{\beta}$ is a constant in the outer half of the annulus (a ramp-like background pressure gradient). Inserting Eq. (22) in Eq. (20) gives

$$\left(\frac{B^3}{B'}\left(1 - \frac{\omega}{\omega_{k\perp}} - \tilde{\beta}\right)\xi'\right)' + BB'\xi = 0 \; . \tag{23}$$

Equation (23) has solutions of the form $\xi \propto B^{\lambda}$ with

$$\lambda = -1 \pm \left(\frac{\frac{-\omega}{\omega_{\kappa\perp}} - \tilde{\beta}}{1 - \frac{\omega}{\omega_{\kappa\perp}} - \tilde{\beta}}\right)^{1/2}$$

for the outer half of the annulus and

$$\lambda = -1 \pm \left(\frac{-\omega}{\omega_{\kappa\perp}-\omega}\right)^{1/2}$$

for the inner half of the annulus. A dispersion relation is found by requiring continuity of ξ and $\left(1 - \frac{\omega}{\omega_{\kappa\perp}} - \tilde{\beta}\right)\xi'/B'$ at the midpoint of the annulus and applying continuity of ξ at the annulus boundaries as well as the boundary conditions from Eq. (21). The result is a transcendental equation for ω:

$$\omega^2 = \frac{k^2 B_-^2}{KRn_i m_i} \frac{(\tilde{\mu}-\mu)}{1+\tilde{\mu}-\mu+\left(1 - \frac{\omega_{\kappa\perp}}{\omega}\right)\mu\tilde{\mu}} , \quad (24)$$

where

$$\mu = \left(\frac{-\omega/\omega_{\kappa\perp}}{1-\ \omega/\omega_{\kappa\perp}}\right)^{1/2} \left[\frac{1-(1+\beta_h)^{[-(\omega/\omega_{\kappa\perp})/(1-\omega/\omega_{\kappa\perp})]^{1/2}}}{1+(1+\beta_h)^{[-(\omega/\omega_{\kappa\perp})/(1-\omega/\omega_{\kappa\perp})]^{1/2}}}\right]$$

$$\tilde{\mu} = \left(\frac{-\omega/\omega_{\kappa\perp} - \bar{\beta}}{1-\ \omega/\omega_{\kappa\perp} - \bar{\beta}}\right)^{1/2} \left[\frac{1-(1+\beta_h)^{[(-\omega/\omega_{\kappa\perp}-\bar{\beta})/(1-\omega/\omega_{\kappa\perp}-\bar{\beta})]^{1/2}}}{1+(1+\beta_h)^{[(-\omega/\omega_{\kappa\perp}-\bar{\beta})/(1-\omega/\omega_{\kappa\perp}-\bar{\beta})]^{1/2}}}\right]$$

and $\beta_h \equiv 2P_{\perp h}(R)/B^2(R)$, $B_- = B(R-\Delta)$.

The dispersion relation Eq. (24) simplifies considerably for $\Delta/R \ll \beta_h \ll 1$. Expanding in β_h yields

$$\omega^2 = \frac{k^2 P_{\perp h}(R)}{Kn_i m_i R} \frac{\bar{\beta}(1+\beta_h)}{(1-\omega/\omega_{\kappa\perp})(1-\omega/\omega_{\kappa\perp}-\bar{\beta})+\beta_h\bar{\beta}/2} , \quad (25)$$

where the finite β_h correction has been kept. Setting $\beta_h = 0$ recovers the dispersion relation found in Refs. 6 and 8. Further simplifications occur in the strongly decoupled limit $\omega_{\kappa\perp}^2/\gamma_{MHD}^2 \gg 1$. In this limit two types of modes are found. The first is the curvature driven interchange mode with

$$\omega^2 = \frac{k^2 P_{\perp h}(R)}{K n_i m_i R} \quad \frac{\tilde{\beta}(1+\beta_h)}{1-\tilde{\beta}(1-\beta_h/2)} . \tag{26}$$

Finite β_h causes a slight shift of the Nelson–Lee–Van Dam stability condition with $\tilde{\beta} < 1/(1-\beta_h/2)$ for stability.

The second type of mode in the strongly decoupled limit are the precessional modes associated with the inner and outer halves of the annulus. The result, found to first order in β_{hot} is

$$\frac{\omega}{\omega_{\kappa\perp}} = 1 - \frac{\tilde{\beta}}{2} \pm \frac{\tilde{\beta}}{2} \left(1 - \frac{2\beta_h}{\tilde{\beta}}\right)^{1/2} , \tag{27}$$

showing instability when $\tilde{\beta} < 2\beta_h$. This new instability is due to finite-β_h induced coupling of the negative and positive energy precessional modes which are nearly degenerate at small $\tilde{\beta}$. A stabilization mechanism is "detuning" or removal of the near degeneracy through the radial variation of the curvature, so that the precessional modes on the inner and outer parts of the pressure gradient are at different frequencies even when $\tilde{\beta}$=0. This detuning mechanism is discussed in Sec. IV.

(b) Strong FLR Limit

The other tractable limit for analyzing Eqs. (17a,b) is when the radial FLR term dominates Eq. (17b). When $\rho_h^2/\Delta^2 \gg \kappa\Delta_B$ we have

$$\rho_h^2 B'^2 \left(\frac{B_1'}{B'}\right)' = 0 \ . \tag{28}$$

The non-singular solutions of Eq. (28) as $B'\to 0$, are of the form:

$$B_1 = C_\pm B' \ . \tag{29}$$

The ± subscripts on C in Eq. (29) denote the outer (+) and inner (−) halves of the annulus. To see that B_1/B' is allowed to be discontinuous at the point $B'=0$ we consider the variational form Eq. (13). If we approximate the discontinuity by a continuous function between $r = R+\delta$ and $R-\delta$ and then let $\delta\to 0$, we find a vanishing contribution to the energy. Combining Eq. (18) with Eq. (29) we find

$$\xi = (\xi_\pm + C_\pm)\,\frac{B_\pm}{B} - C_\pm \ , \tag{30}$$

where $\xi_\pm$, $B_\pm$ are ξ, B evaluated at the outer and inner edges of the annulus. These constants are partially determined by the continuity of ξ at the midpoint of the annulus and the inner and outer boundaries. Two more constraints come from annihilating the large radial FLR term in Eq. (17b) by multiplying by B' and integrating from the edge to the center:

$$\int_R^{R\pm\Delta} dr\left\{-\frac{BB_1}{r} - B'\Phi + B'\left(1+G_1 + \frac{B}{B'r}\right)(B_1+\xi B') + k^2\rho_n^2 B_1 B'\right\} = 0 \ . \tag{31}$$

Integrating the Φ term by parts and using Eq. (17a) gives the conditions

$$C_\pm \int_R^{R\pm\Delta} dr\left(-\frac{BB'}{r} + k^2\rho_h^2 B'^2\right) - B\Phi\Big|_R^{R\pm\Delta} = 0 \ . \tag{32}$$

From Eq. (17a) we can relate $\Phi(R\pm\Delta)$ to $\Phi(R)$

$$\Phi(R\pm\Delta) = \Phi(R) - \left[\int_R^{R\pm\Delta} dr\, \frac{B'^2}{B^2}\left(1 + \frac{B}{rB'} + G_1\right)\right](\xi_\pm + C_\pm)B_\pm \ . \tag{33}$$

Continuity of Φ at the edges of the annulus gives

$$\begin{aligned} &\Phi(R+\Delta) = 0 \quad (\text{assuming } n_i(R+\Delta) = 0) \\ &\Phi(R-\Delta) = \frac{n_i m_i \omega^2 K \xi_-}{k^2 B_-} \ . \end{aligned} \tag{34}$$

Combining Eqs. (32) through (34) along with continuity of the midplane from Eq. (30),

$$(\xi_+ + C_+)\frac{B_+}{B_o} - C_+ = (\xi_- + C_-)\frac{B_-}{B_o} - C_- \ , \tag{35}$$

($B_+ = B_-$ to lowest order in the curvature) gives a dispersion relation:

$$\omega^2 = \frac{k^2 P_{\perp h}(R)}{KRn_i m_i} \frac{B_+}{B_o} \frac{\hat{\beta}(1-\lambda)^2}{(1-\lambda-Y)(1-\lambda-Y-\hat{\beta})+(1-\lambda)\hat{\beta}(\frac{B_+}{B_o}-1)} \tag{36}$$

where

$$Y = \frac{B_+ B_o}{P_{\perp h}(R)} \int_R^{R\pm\Delta} dr \frac{B'}{B} \frac{\omega}{\omega_{\kappa\perp}} \approx \frac{\omega}{\omega_{\kappa\perp}} \quad \text{if} \quad \beta_h << 1$$

$$\hat{\beta} = \frac{B_+ B_o}{P_{\perp h}(R)} \int_R^{R+\Delta} dr \frac{B'}{B} \tilde{\beta} \approx \tilde{\beta}\ , \ \text{if} \quad \beta_h << 1$$

$$\lambda = \frac{k^2 r}{P_{\perp h}(R)} \int_R^{R\pm\Delta} \rho_h^2 B'^2 \approx \vartheta(k^2\rho_h^2 \frac{r}{\Delta_B}) \ .$$

Note that no special profiles have been assumed. Equation (36) simplifies in the strongly decoupled limit $\omega_{k\perp}^2/\gamma_{MHD}^2 >> 1$. As found for the weak FLR case, two types of modes result. The interchange mode dispersion relation is

$$\omega^2 = \frac{k^2 P_{\perp h}}{KRn_i m_i} \frac{B_+}{B_o} \frac{\hat{\beta}(1-\lambda)}{1-\lambda-\hat{\beta}(2-\frac{B_+}{B_o})} \ . \tag{37}$$

In the low β_h limit and for $\lambda \rightarrow 0$ Eq. (37) reduces to the low β_h, zero FLR case found in Eq. (26). Note that although the radial FLR operator was the dominant term in Eq. (17b), the stability criterion is unchanged by this effect in the limit $k^2 \rightarrow 0$ $(\delta \rightarrow 0)$. This is analogous to the stability problem for cylinders with $\underset{\sim}{B} = B\hat{\underset{\sim}{z}}$ where the

FLR effect is small if $m\Delta/r \ll 1$ (m = azimuthal mode number) and nearly rigid displacements of the plasma. One should also note that in a cylinder the FLR effects exactly cancel if m=1. The case $\lambda \neq 0$ is similar to the eikonal result. Stability is assured for all $\bar{\beta}$ when $\lambda > 1$ $(k^2 \rho_h^2 \gtrsim \Delta_B/R)$.

The dispersion relation for the precessional mode in the strongly decoupled limit is

$$Y = 1-\lambda-\frac{\hat{\beta}}{2} \pm \frac{\hat{\beta}}{2}\left[1-4\left(\frac{B_+}{B_o}-1\right)\frac{(1-\lambda)}{\hat{\beta}}\right]^{1/2} . \tag{38}$$

At low β_h, $\lambda=0$ and for the profile $\bar{\beta}$ = piecewise constant, Eq. (38) reduces to Eq. (25), again predicting instability for $\bar{\beta} < 2\beta_h$. The axial FLR term again stabilizes this mode when $\lambda>1$.

IV. Discussion

The description of the response of a hot component plasma including finite Larmor radius effects has been developed for the z-pinch model and long wavelength layer modes ($k\Delta<1$) have been analyzed. This work improves on past analysis in that we can solve the equations for arbitrary large values of β_h (the beta of the hot component), whereas previous analysis assumed $\beta_h < 1$. The resulting dispersion relation gives two types of modes. One is a mode near the precessional frequency and the other is at lower frequency and more closely related to MHD modes.

We find that the finite Larmor radius (FLR) effects change the character of the equations somewhat if $\beta_h \rho_h^2 R/\Delta^3 > 1$, but does not necessarily alter the zero FLR stability picture. The perturbed magnetic field is then constrained to have a piecewise rigid response on each side of the pressure gradient. As a result, previously derived decoupling conditions for the hot particle energy, or the Lee-Van Dam-Nelson critical beta criterion, are only modestly shifted from the results of zero FLR theory if $k^2\beta_h\rho_h^2R/\Delta < 1$. However, if $k^2\beta_h\rho_h^2R/\Delta > 1$, our theory predicts stability for all modes. In our model we can have $k\to 0$, and we cannot achieve a robust stability. This is similar to the result for a cylindrically shaped plasma with an imposed curvature, $-1/R$. In such a case robust stability requires $(m^2-1)\beta_h\rho_h^2R/r_p^2\Delta > 1$, where m is the mode number and r_p the plasma radius. (This result will be derived in a subsequent paper.) The cylindrical result shows that a robust stability cannot be achieved from the FLR terms for m=1. However, it was recently shown that wall effects[11,12] can stabilize the m=1 of a cylindrical disc-shaped plasma with finite axial extent. However, a thin annulus shaped plasma will not be stabilized in a robust manner.

The annular shaped plasma is subject to the Lee-Van Dam-Nelson critical core beta limit,[3,4] and even when the core beta is below the critical limit the precessional mode can destabilize. Equation (25) describes a new mechanism for the precessional mode to destabilize. If, $\tilde{\beta} \approx P_cR/B^2\Delta << 1$, Eq. (25) predicts that the precessional modes associated with the inner and outer parts of the hot particle pressure gradient interact and destabilize. Equation (25) is unstable for sufficiently small $\tilde{\beta}$. In order to observe a threshold, we should take into account that $\omega_{\kappa\perp}$ is slightly different on the inner and outer parts of the layer as $\underset{\sim}{\kappa}$ varies with radial position. The precessional mode in the strongly decoupled limit is found by setting the denominator of Eq. (25) to zero. Taking into account the shifts in $\omega_{\kappa\perp}$, the dispersion relation then becomes

$$[\omega_\kappa(r_0)-\omega]\,[\omega_\kappa(r_1)(1-\tilde{\beta})-\omega] = -\omega_\kappa(r_1)\omega_\kappa(r_0)\beta_h\tilde{\beta}/2 \ , \tag{39}$$

where r_0 is the center of the inner half of the layer and r_1 the center of the outer half of the layer. The stability condition, assuming β_h, $\frac{\Delta}{r} < 1$, is then,

$$\begin{aligned} &\frac{\Delta\omega_\kappa}{\omega_\kappa} > \tilde{\beta} + (2\beta_h\tilde{\beta})^{1/2}, \text{ if } \frac{\Delta\omega_\kappa}{\omega_\kappa} > 0 \\ &\tilde{\beta}_h + \left|\frac{\Delta\omega_\kappa}{\omega_\kappa}\right| > (2\beta_h\tilde{\beta})^{1/2} \ , \text{ if } \frac{\Delta\omega_\kappa}{\omega_\kappa} < 0 \ , \end{aligned} \tag{40}$$

where $\Delta\omega_\kappa = \omega_\kappa(r_1)-\omega_\kappa(r_0)$. In a realistic plasma configuration $\kappa \propto r$, where r is the plasma radius, and therefore $\Delta\omega_\kappa/\omega_\kappa > 0$. Thus, with a finite width annulus, the new precessional instability can be averted at small values of $\tilde{\beta}$, but should eventually appear when $\tilde{\beta} \gtrsim \Delta\omega_\kappa/\omega_\kappa$. In addition, a recent calculation[13] that accounts for line-bending in an axially dependent model shows that stability of the coupled precessional mode arises if, roughly, $\beta_h|m| < 1$, where m is the mode number.

It should be noted that our detailed calculation assumed that the hot pressure gradients were of equal magnitude on either side of the annulus. One can readily show that if one has a skewed hot pressure profile, such that $k\Delta^{-1}_{max} < 1$, where Δ^{-1}_{max} is the smaller pressure gradient, that our results are essentially unchanged. The main assumption is that there is a hot plasma annulus such that the hot pressure goes to zero on either side of the annulus. The FLR term, in Eq. (36) needs to be modified to differentiate between the appropriate pressure gradients on either side on the layer, but otherwise there is no essential difference from the results of the text as a result of skewness in the pressure profile. However, somewhat different conclusions would arise if $\kappa\Delta_b \gtrsim 1$, a case that violates the fundamental ordering of this paper. For present day experiments $\kappa\Delta_b \approx 1$ may be appropriate. This possibility will be treated in forthcoming investigations.

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Appendix: Derivation of the FLR Corrections

In the section we include some of the detailed calculations leading to the FLR corrections, starting with Eq. (12) and employing the orbit expansion. As mentioned in the text, the corrections are evaluated in the low frequency, weak curvature limit.

The FLR corrections appear only for terms involving the perturbed magnetic field B_1, so here we will consider only the B_1 terms in Eq. (12). The other terms, after relatively straightforward algebra, lead to forms derived previously, and will not be discussed in detail.

Defining $\lambda = iv_r B_1 e^{-ik\delta z}$, we need to calculate

$$\oint d\tau \int_0^\infty d\tau' \ \lambda(\tau)\lambda(\tau'-\tau)\exp[i(\omega-k\bar{v})\tau']$$

$$\cong \oint d\tau \ \{\frac{\bar{\lambda}^2}{-i(\omega-k\bar{v})} - 2\tilde{\lambda}_{even} \int^\tau d\tau'\tilde{\lambda}_{odd} \tag{A.1}$$

$$+ i(\omega-k\bar{v})[(\int^\tau d\tau\tilde{\lambda}_{odd})^2 - (\int^\tau d\tau'\tilde{\lambda}_{even})^2]\} + \vartheta[\frac{(\omega-k\bar{v})^2}{\Omega^2}]$$

where $\bar{\lambda} = \oint \frac{d\tau\lambda}{T}$ $\tilde{\lambda} = \lambda-\bar{\lambda}$. Expanding the exponent in the expression for λ through third order, we have

$$\lambda = iv_r B_1 + kv_r\delta z B_1 - \frac{ik^2}{2}\delta z^2 v_r B_1 - \frac{v_r\delta z^3}{6} k^3 B_1 \ . \tag{A.2}$$

From Eq. (A.2) we have

$$\overline{\lambda}^2 = \overline{kv_r \delta z B_1}^2 - \frac{1}{3} k^4 B_1^2 \ \overline{v_r \delta z} \ \overline{v_r \delta z^3} . \tag{A.3}$$

In the second term we can use $v_r \cong v_\perp \sin\Omega_c \tau$, $\delta z \cong -\frac{v_r}{\Omega_c}$ to sufficient accuracy giving:

$$\overline{\lambda}^2 = \overline{kv_r \delta z B_1}^2 - \frac{k^4 v_\perp^6}{16\Omega_c^4} B_1^2 . \tag{A.4}$$

δz is needed to higher accuracy in the first term. From Eq. (8a) we have

$$\delta z = \int d\tau (v_z - \overline{v}) = -\frac{v_r}{\Omega_c} + \int d\tau (v_D - \overline{v}) . \tag{A.5}$$

Using Eq. (A.5) we find

$$\overline{kv_r \delta z B_1} = -\frac{\overline{kv_r^2}}{\Omega_c} B_1 + \overline{kv_r \int d\tau (v_D - \overline{v}) B_1} . \tag{A.6}$$

To evaluate the second term we use $v_r = \frac{\dot{v}_z}{\Omega_c}$

$$\overline{kv_r \int d\tau (v_D - \overline{v}) B_1} = \overline{k\dot{v}_z \frac{1}{\Omega_c} \int d\tau (v_D - \overline{v}) B_1} = + \overline{kv_z v_r \int d\tau (v_D - \overline{v}) \left(\frac{B_1 B'}{\Omega_c B} - \frac{B_1'}{\Omega_c}\right)}$$

$$- \frac{\overline{kv_z}}{\Omega_c} \overline{(v_D - \overline{v}) B_1} . \tag{A-7}$$

Using $v_z = \frac{-\dot{v}_r}{\Omega_c}$ to lowest order in the curvature

$$\overline{kv_r\int d\tau(v_D-\bar{v})} = \overline{\frac{k\bar{v}_z v_r}{\Omega_c}\int d\tau(v_D-\bar{v})}\ (B_1B'/B - B_1') + \frac{k\mathring{v}_r}{\Omega_c^2}\ (\frac{-v_r^2}{\Omega_c}\frac{B'}{B} - \bar{v})B_1$$

$$= \frac{-kv_\perp^4}{16\Omega_c^3}\ [B_1\ \frac{B'^2}{B^2} - B_1'\ \frac{B'}{B} + \frac{4B'}{B}\ (\frac{B_1}{B^2})'\ B^2] - \frac{kB'B_1}{\Omega_c^3 B}\ \overline{(\frac{\mathring{v}_r^3}{3})}$$

$$= -\frac{kv_\perp^4}{16\Omega_c^3}\ (B_1(\frac{B'^2}{B^2} - \frac{2B''}{B}) + B_1'\ \frac{B'}{B})\ . \qquad (A.8)$$

Combining results gives

$$\bar{\lambda}^2 = (\overline{\frac{kv_r^2B_1}{\Omega_c}})^2 + \frac{k^2v_\perp^6B_1}{16\Omega_c^4}\ [B_1(\frac{B'^2}{B^2} - \frac{2B''}{B}) + \frac{B_1'B'}{B}] - \frac{k^4v_\perp^6B_1^2}{16\Omega_c^4}\ . \qquad (A.9)$$

The first term in Eq. (A.9) gives the term $\propto G_1$ in δW to lowest order in the orbit expansion. To extract the G_1 term we define $v_B = -v_r^2B'/\Omega_cB$ and we have,

$$(\overline{\frac{kv_r^2B_1}{\Omega_c}})^2 \equiv (\overline{kv_B\ \frac{BB_1}{B'}})^2 \qquad (A.10)$$

$$= \bar{v}_Bk^2\overline{v_B\ \frac{B^2B_1^2}{B'^2}} + \overline{k^2v_B\ \frac{BB_1}{B'}\ (\overline{v_B\ \frac{BB_1}{B'}} - \bar{v}_B\ \frac{BB_1}{B'})}\ ,$$

The first term gives the G_1 term defined in the text. Inserting the orbit expansion into the second term gives

$$\overline{\left(\frac{kv_r^2B_1}{\Omega_c}\right)^2} = \bar{v}_B \overline{k^2 v_B \frac{B^2B_1^2}{B'^2}} - \frac{k^2v_\perp^6}{16\Omega_c^4}\frac{B'^2}{B^2}\left(\frac{BB_1}{B'}\right)'^2 \tag{A.11}$$

Thus,

$$\bar{\lambda}^2 = \bar{v}_B \overline{k^2 v_B \frac{B^2B_1^2}{B'^2}} - \frac{k^2v_\perp^6}{16\Omega_c^4}\left[B'^2\left(\frac{B_1}{B'}\right)'^2 + B_1B_1' \frac{B'}{B} + k^2B_1^2\right] . \tag{A.12}$$

Next we need to evaluate the term:

$$\begin{aligned} \oint d\tau - 2\tilde{\lambda}_{even}\int^{\tau} d\tau'\, \tilde{\lambda}_{odd} &= \oint d\tau + 2\tilde{\lambda}_{odd}\int^{\tau} d\tau'\, \tilde{\lambda}_{even} \\ &\cong \oint d\tau\; 2iv_rB_1 \int^{\tau} d\tau' kv_r\delta zB_1 \end{aligned} \tag{A.13}$$

to sufficient accuracy. Using the orbit expansion in the right-hand side of (A.13) we find

$$\begin{aligned} \oint d\tau\; 2iv_rB_1 \int^{\tau} d\tau' kv_r\delta zB_1 = {} & 2ikB_1B_1'(\bar{r})\oint d\tau\left(v_r\delta r\int^{\tau} v_r\delta z d\tau'\right. \\ & \left. + v_r\int^{\tau} v_r\delta z\delta r d\tau'\right) \\ & + 2ikB_1^2(\bar{r})\oint d\tau v_r\left(\int^{\tau} v_r\delta z d\tau'\right) \end{aligned} \tag{A.14}$$

where $r = \bar{r}+\delta r$, $\bar{r}$ = mean radius. The term $\propto B_1B_1'$ in (A.14) can be evaluated using zero-order orbits:

$$2ikB_1B_1' \oint d\tau\left(v_r\delta r\int^{\tau} d\tau' v_r\delta z + v_r\int^{\tau} v_r\delta z\delta r d\tau'\right) \cong 2ikB_1B_1' \frac{v_\perp^4}{16\Omega_c^3} . \tag{A.15}$$

The term $\propto B_1^2$ requires greater care:

$$2ikB_1^2 \oint d\tau (v_r \int^{\tau} d\tau' v_r \delta z) = 2ikB_1^2 \oint d\tau (-\delta r v_r \delta z)$$

$$= 2ikB_1^2 \oint d\tau \frac{\delta r^2}{2} \delta \dot{z} = 2ikB_1^2 \oint d\tau [-\frac{d}{d\tau} (\frac{v_r}{\Omega_c}) + v_D - \bar{v}] \frac{\delta r^2}{2}$$

$$= 2ikB_1^2 \oint d\tau [\frac{v_r^2 \delta r}{\Omega_c} + \frac{1}{2} (v_D - \bar{v}) \delta r^2] \qquad (A.16)$$

where several integrations by parts have been used. We need δr to sufficient accuracy to evaluate (A.16)

$$v_r = \frac{\dot{v}_z}{\Omega} = \frac{d}{d\tau} (\frac{v_z}{\Omega}) - v_z (\dot{\frac{1}{\Omega_c}}) = \frac{d}{d\tau} (\frac{v_z}{\Omega}) + \frac{v_z v_r B'}{\Omega_c B} .$$

Then,

$$\delta r = \frac{v_z}{\Omega_c} + \int^{\tau} d\tau' \; v_z \frac{v_r B'}{\Omega_c B} = \frac{-\dot{v}_r}{\Omega_c^2} + \int^{\tau} d\tau' \; v_z v_r \frac{B'}{\Omega_c B} . \qquad (A.17)$$

Hence, we have

$$2ikB_1^2 \oint d\tau (v_r \int^{\tau} d\tau' v_r \delta z) = 2ikB_1^2 \oint d\tau [-\frac{d}{d\tau} (\frac{v_r^3}{3}) \frac{1}{\Omega^3} + \frac{v_r^2}{\Omega} \int^{\tau} d\tau' v_z v_r \frac{rB'}{\Omega B} + \frac{1}{2} (v_D - \bar{v}) \delta r^2] . \qquad (A.18)$$

Integrating once more by parts:

$$2ikB_1^2\oint d\tau(v_r\int^\tau d\tau' v_r\delta z) = 2ikB_1^2\oint d\tau[-\frac{v_r^4}{\Omega_c^3}\frac{B'}{B} +$$

$$\frac{v_r^2}{\Omega_c}\int^\tau d\tau' \frac{v_z v_r B'}{\Omega_c B} + \frac{1}{2}(v_D-\bar{v})\delta r^2] \quad . \tag{A.19}$$

The right-hand side of (A.19) can be evaluated using zero-order orbits:

$$2ikB_1^2\oint d\tau(v_r\int^\tau d\tau' v_r\delta z) = 2ikB_1^2\left(\frac{v_\perp^4}{8\Omega_c^3}\frac{B'}{B}\right) \quad . \tag{A.20}$$

Combining with (A.15) we have

$$-2\tilde{\lambda}_{even}\int^\tau d\tau' \tilde{\lambda}_{odd} = \frac{ikv_\perp^4}{8\Omega_c^3}\left(B_1B_1' + 2B_1^2\frac{B'}{B}\right)$$

$$= \frac{\left(\frac{k^2v_\perp^6}{16\Omega_c^4}\right)}{ik\bar{v}_B}\left(2B_1^2\frac{B'^2}{B^2} + B_1B_1'\frac{B'}{B}\right) \tag{A.21}$$

where we have written the result divided by $ik\bar{v}_B$ in order to combine with the FLR contributions from the $\bar{\lambda}^2$ term ($\omega-ik\bar{v}$ is replaced by $-ik\bar{v}_B$ in the FLR terms in the low-frequency, weak curvature limit).

Finally, we need the terms

$$i(\omega-k\bar{v})\left[\left(\int^\tau d\tau'\tilde{\lambda}_{odd}\right)^2 - \left(\int^\tau d\tau'\tilde{\lambda}_{even}\right)^2\right] \cong -ik\bar{v}_B\left[\int^\tau d\tau' iv_rB_1\right]^2 \quad . \tag{A.22}$$

The right-hand side of A.22 can be evaluated using zeroth order orbits

$$- \oint d\tau ik\bar{v}_B \left[\int^{\tau} d\tau' i v_r B_1\right]^2 \cong \oint ik\bar{v}_B \delta r^2 B_1^2 = \frac{-\frac{k^2 v_\perp^6}{16\Omega_c^4}}{ik\bar{v}_B} \left(2B_1^2 \frac{B'^2}{B^2}\right) . \qquad (A.23)$$

Combining results from (A.12), (A.21) and (A.23) we find

$$\oint d\tau \int_0^\infty d\tau' \lambda(\tau)\lambda(\tau'-\tau)\exp(i(\omega-k\bar{v})\tau')$$

$$\cong \oint d\tau \left\{ \frac{\bar{v}_B k^2 v_B \frac{B^2 B_1^2}{B'^2}}{-i(\omega-k\bar{v})} - \frac{1}{ik\bar{v}_B} \frac{k^2 v_\perp^6}{16\Omega_c^4} \left[B'^2 \left(\frac{B_1}{B'}\right)'^2 + k^2 B_1^2\right]\right\} . \qquad (A.24)$$

The first term on the right-hand side gives the $G_1 B_1^2$ term appearing in δW defined in Eq. (13) of the text. The remaining terms are the FLR correction.

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Figure Caption

Figure 1 — Radial pressure and magnetic field profiles for a hot plasma annulus containing core pressure.

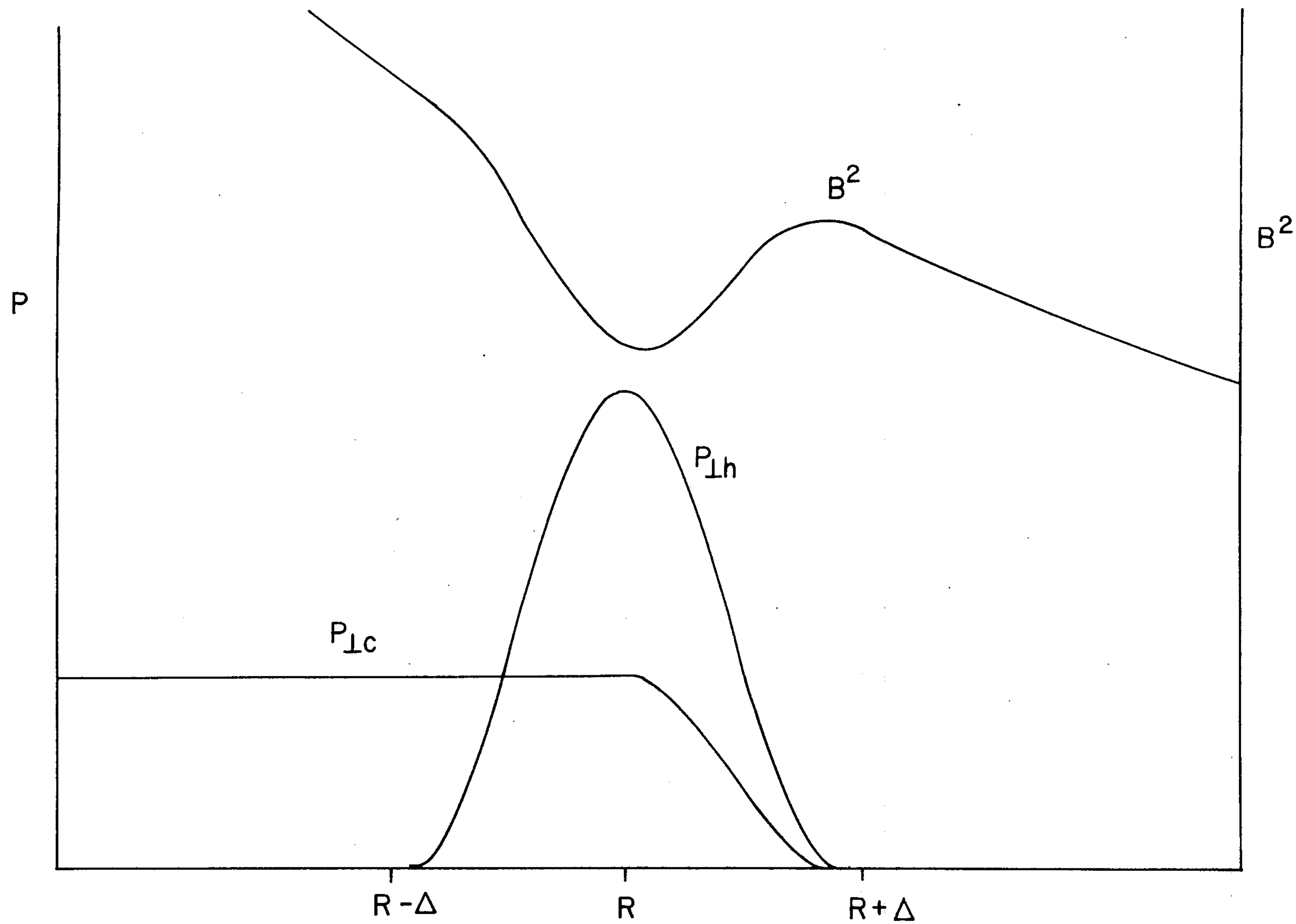

Fig. 1