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EQUILIBRIUM CURRENT-DRIVEN TEARING MODE IN THE HYDRODYNAMIC REGIME

Franco Cozzani
Swadesh Mahajan
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

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Abstract

The effect of the parallel equilibrium current on the linear stability of the drift-tearing mode in the collisional regime is investigated analytically.

In the appropriate parameter regime, a new unstable mode, driven by equilibrium current, is found and its relevance to tokamak discharges is discussed.

Introduction

In the linear analysis of the tearing instability,[1] the dispersion relation is obtained by matching the external solution for the perturbed magnetic field in the ideal (MHD) region with the internal solution, valid in a narrow boundary layer centered around a particular mode rational surface.[1,2] The plasma response inside the layer is either derived from kinetic theory[3,4,5] or from fluid equations.[6,7] A common feature of such analyses is the neglect of the parallel equilibrium field $E_{\parallel}^{(0)}$ inside the tearing layer, to make the eigenmode equations analytically tractable.

The scope of the present paper is to discuss the effect of the equilibrium current on the drift tearing mode in the collisional (hydrodynamic) regime, which pertains to the cooler, outer regions of some present day tokamak discharges.

The extension of this analysis to the semi-hydrodynamic regime of the tearing instability[5,7] will be the subject of a future paper.

The present study shows that the general analysis of the effect of the equilibrium current can be carried out analytically; in a most interesting case the calculation turns out to be quite simple. The appropriate "classical" drift tearing mode result is recovered neglecting the effect of the current term in the final dispersion relation, and a new strong instability, driven by the equilibrium current, is found whenever the current term exceeds the usual Δ' term.

The present article is organized as follows: in Sec. I the notation and some relevant approximations are discussed; in Sec. II, fluid equations are used to derive the relevant eigenmode equations in the inner tearing layer; Sec. III is devoted to the setting up of the mathematical formulation in the general case of arbitrary equilibrium current; in Sec. IV, the dispersion relation for the "weak" current case is derived; concluding remarks are presented in Sec. V.

I. General System

Let $\hat{b}_o \equiv \underset{\sim}{B}_o/B_o$ be a unit vector in the direction of the unperturbed magnetic field; $\hat{r}$ be a unit vector in the radial direction, normal to a flux surface; $\hat{e} \equiv \hat{b}_o \times \hat{r}$. In the slab model of the tokamak, these unit vectors, as well as the equilibrium quantities, depend only on the radius.

A usual Fourier representation is introduced for any linearly perturbed quantity $\tilde{g}$, so to have

$$\frac{\partial}{\partial t}\tilde{g} = -i\omega\tilde{g} \quad ; \quad \hat{b}_o \cdot \nabla\tilde{g} = i\kappa_{\parallel}(x)\tilde{g} \; ;$$
$$\hat{e} \cdot \nabla\tilde{g} = i\kappa_{\perp}(x)\tilde{g} \quad ; \quad \hat{r} \cdot \nabla\tilde{g} = \tilde{g}' \; , \tag{1}$$

where $x = r - r_s$, r_s being the radius of the mode rational surface of interest. In the vicinity of such surface, where $\kappa_{\parallel}(0) = 0$, we have

$$\kappa_{\parallel}(x) \simeq \kappa_{\parallel}' \cdot x \; . \tag{2}$$

It is convenient to formulate the eigenmode problem in terms of the perturbed electrostatic potential $\tilde{\Phi}$ and the parallel component of the magnetic vector potential $\tilde{A}_{\parallel}$, since for sufficiently small β (ratio between plasma pressure and magnetic pressure) it is consistent to assume

$$\underset{\sim}{\tilde{A}} = \hat{b}_o \tilde{A}_{\parallel} \; . \tag{3}$$

The radial magnetic perturbation is therefore

$$\tilde{B}_r = i\kappa_\perp \tilde{A}_\parallel \; , \tag{4}$$

whereas the components of the perturbed electric field are[(*)]

$$\begin{aligned} \hat{b}_o \cdot \tilde{\underset{\sim}{E}} &= -i\kappa_\parallel \tilde{\Phi} + i\omega \tilde{A}_\parallel \; ; \\ \hat{e} \cdot \tilde{\underset{\sim}{E}} &= -i\kappa_\perp \tilde{\Phi} \quad ; \quad \hat{r} \cdot \tilde{\underset{\sim}{E}} = -\tilde{\Phi}' \; . \end{aligned} \tag{5}$$

Also, the parallel component of Ampere's law is

$$\tilde{J}_\parallel = \frac{1}{4\pi} \left(-\tilde{A}''_\parallel + \kappa_\perp^2 \tilde{A}_\parallel \right) \approx -\frac{\tilde{A}''_\parallel}{4\pi} \; . \tag{6}$$

In Eq. (6), as well as in the forthcoming equations, we assumed the radial wavelength to be much shorter than the azimuthal wavelength.

(*) the speed of light c is taken to be unity for convenience

II. Eigenmode Equations

The electron parallel response is conveniently summarized by a generalized Ohm's law of the form[6,7]

$$\alpha'' \frac{\partial}{\partial t}(nm_e\hat{b}\cdot\underset{\sim}{u}) = -ne\hat{b}\cdot\underset{\sim}{E} - \hat{b}\cdot\nabla p_e$$

$$-(ne)^2\eta\hat{b}\cdot\underset{\sim}{u}\left[1+\frac{3}{2}\alpha''\left(\frac{m_e}{\eta ne^2}\right)\frac{\partial}{\partial t}\ln T_e\right] \tag{7}$$

$$-\alpha n\hat{b}\cdot\nabla T_e\left[1-\frac{3\alpha'}{\nu}\frac{\partial}{\partial t}\ln T_e - \frac{\alpha'}{\nu}\frac{\partial}{\partial t}\ln(\hat{b}\cdot\nabla T_e)\right] .$$

α,α',α'' are numerical transport coefficients defined in Ref. 6 and 7; η is the Spitzer–Braginskii resistivity; ν is the electron collision frequency; $\hat{b}$ a unit vector in the direction of the total magnetic field $\underset{\sim}{B}$. The physical origin and relevance of the time-dependent thermal force terms in Eq. (7) have been discussed in detail in Ref. 6; here, we shall only remark that they are responsible for the ∇T_e-driven tearing mode growth rate in the final dispersion relation, and they are not necessary to obtain the new instability discussed in the present paper.

Linearization of Eq. (7) is carried out, through Eq. (1) and (6), neglecting terms of order $\kappa_\parallel^2 D/\omega$, $D \equiv T_e/m_e\nu$, to give

$$-i\eta\frac{\tilde{A}_\parallel''}{4\pi} + \left[\omega-\omega_n^* - (1+\alpha)\omega_T^* - i\alpha\alpha'\frac{\omega}{\nu}\omega_T^*\right]\tilde{A}_\parallel$$

$$+\kappa_\parallel' x\left(\frac{\omega_n^*}{\omega}-1\right)\tilde{\Phi} - \frac{3}{2}i\eta j_\parallel^{(0)}\frac{\tilde{T}_e}{T_e} = 0 . \tag{8}$$

In Eq. (8) the "drift" frequencies ω_n^*, ω_T^* are defined to be

$$\omega_n^* \equiv -\frac{\kappa_\perp}{eB}\frac{n'}{n}T_e$$

$$\omega_T^* \equiv -\frac{\kappa_\perp}{eB}T_e' \; . \qquad (9)$$

The equilibrium parallel current appears in the last term of Eq. (8) and is related to the electron drifting velocity u_e by

$$j_\parallel^{(0)} = -neu_e \; . \qquad (10)$$

In Eq. (8), perturbations in density have been eliminated by continuity equation

$$\frac{\tilde{n}}{n} = \frac{\omega_n^*}{\omega}\frac{e\tilde{\Phi}}{T_e} \; . \qquad (11)$$

Temperature perturbations in Eq. (8) are now eliminated through energy balance equation

$$\frac{\tilde{T}_e}{T_e} = \frac{\omega_T^*}{\omega}\frac{e\tilde{\Phi}}{T_e} \; , \qquad (12)$$

to get a first coupled equation in $\tilde{A}_\parallel$, $\tilde{\Phi}$:

$$\psi'' = \sigma_*[\psi - (x - \frac{iR}{\Omega_2})\phi] \; . \qquad (13)$$

In "Ohm's law" Eq. (13), we defined

$$\psi \equiv \frac{\omega \tilde{A}_{\parallel}}{\kappa_{\parallel}'}$$

$$\sigma_{*} = -\frac{i\Omega_2 \tau_s}{a^2}$$

$$\Omega_2 \equiv \omega - \omega_n^* - (1+\alpha)\omega_T^* - i\alpha\alpha'\omega_T^* \frac{\omega}{\nu}$$

$$\tau_s \equiv \frac{\omega_p^2 a^2 s}{\nu} \tag{14}$$

a ≡ plasma radius

$$s = \eta(Z=1)/0.51\eta(Zeff)$$

$$R \equiv \frac{3}{s} \frac{u_e}{v_e} \frac{\nu}{\kappa_{\parallel}' v_e} \omega_T^* .$$

The effect of the equilibrium current appears in R through the drifting velocity u_e. v_e is the electron thermal velocity; ω_p is the plasma frequency. Ignoring this term, Eq. (13) would reduce to the familiar expression[3,4] $\tilde{j}_{\parallel} = \sigma \tilde{E}_{\parallel}$. The crucial point is to realize that, although the effect of the equilibrium current is usually neglected due to the smallness of the factor u_e/v_e, the parameter R appears in Eq. (13) divided by the frequency Ω_2, which is essentially the "residual" tearing mode growth (damping) rate, after the ∇T_e-driven growth rate contribution has been subtracted out. From the classical analysis of the tearing mode,[7] Ω_2 turns out to be also quite small; therefore it is not correct to neglect the term R/Ω_2 a priori, although of course Ω_2 has to be determined self-consistently. A second coupled equation in Φ, ψ is obtained by momentum balance equation[4]

$$m_i n_i \frac{\partial}{\partial t} \underset{\sim}{V} + \nabla p_i = e n_i (\underset{\sim}{E} + \underset{\sim}{V} \times \underset{\sim}{B}) , \tag{15}$$

which, upon linearization, give the standard result[4]

$$x_A^2 \phi'' = x\psi'' , \tag{16}$$

where

$$x_A^2 \equiv \frac{\omega(\omega+\omega_i^*)}{(\kappa_\parallel' v_A)^2}$$
$$\omega_i^* \equiv -\frac{\kappa_\perp}{eB}\frac{p_i'}{n_i} . \tag{17}$$

v_A is the Alfvén velocity. In deriving Eq. (16), radial gradients of the equilibrium current have been neglected.

We note that the equilibrium current term alters significantly the structure of the eigenmode Eq. (13), (16).

By defining a new radial variable

$$z \equiv x - \lambda , \tag{18}$$

where the "shift" λ is defined to be

$$\lambda \equiv \frac{1}{2} \frac{R}{-i\Omega_2} , \tag{19}$$

the dimensionless conductivity becomes then a symmetric function of z. Also, it is convenient to define

$$Q \equiv \frac{\psi}{z-\lambda} - \phi \ . \tag{20}$$

The eigenmode equations become then

$$\phi'' = \frac{\sigma(z)}{x_A^2} Q \tag{21}$$

$$\psi'' = \frac{\sigma(z)}{z+\lambda} Q \ , \tag{22}$$

where
$$\sigma(z) \equiv (z^2-\lambda^2)\sigma_* \ . \tag{23}$$

We note that, in the absence of equilibrium current (i.e., $\lambda=0$), Eq. (21) and (22) are identical with Eq. (1) and (2) of Ref. 3, and the variable $Q(z,\lambda)$ reduces to $Q(x) \equiv E_{\parallel}(x)/x$ of Ref. 3. Also, one realizes that the equilibrium current introduces a scale length in the conductivity given by Eq. (23) even in the hydrodynamic regime. Finally, $E_{\parallel}(x)$ does not go to zero outside the tearing layer, defined as the region in which $\psi''\neq 0$; it is rather the variable $Q(z,\lambda)$ which goes to zero when $\psi''\rightarrow 0$.

Equation (21) and (22) are now combined to obtain a single second-order differential equation for Q:

$$\left(\frac{x_A^2(z-\lambda)^2}{(z-\lambda)^2-x_A^2} Q'\right)' + \sigma(z)\ Q = \frac{2E_o x_A^4(z-\lambda)}{[(z-\lambda)^2-x_A^2]^2} \ , \tag{24}$$

where E_o, an integration constant, is related to Δ', the well-known stability parameter of the tearing mode theory, by

$$E_o\Delta' = \frac{1}{x_A^2}\int_{-\infty}^{\infty} \frac{\sigma(z)Q}{z+\lambda}\,dz \quad . \tag{25}$$

Here, we recall, $\Delta' \equiv (\psi'_+ - \psi'_-)/\psi_o$ and the asymptotic behavior of ψ has been taken to be $\psi = \psi_o + \psi'_\pm x$ for $|x| \to \infty$. The value of Δ' is presumed known from the solution of the external problem, and appears here as a free parameter.

In order to derive Eq. (24), a "constant ψ-like" approximation has been employed in the form

$$x^2\left(\frac{\psi}{x}\right)' \approx (x-2\lambda)^2\left(\frac{\psi}{x-2\lambda}\right)' \quad . \tag{26}$$

From Eq. (25) we recognize that E_o is really an integral operator; multiplying Eq. (24) by $1/x_A^2\,(1/z+\lambda)$ and integrating, we obtain an integro-differential equation for Q:

$$\left[\frac{x_A^2(z-\lambda)^2}{(z-\lambda)^2-x_A^2}\,Q'\right]' + \sigma(z)Q = -\left[\Delta' + \frac{i\pi x_A}{(x_A+2\hat{\lambda})^2}\right]^{-1} \frac{4x_A^4(z-\lambda)}{[(z-\lambda)^2-x_A^2]^2} \tag{27}$$

$$\cdot \int_{-\infty}^{\infty} \frac{(z'-\lambda)\left[2\lambda x_A^2+(z'-\lambda)^3\right]}{(z'+\lambda)^3\left[(z'-\lambda)^2-x_A^2\right]^2}\,Q(z')dz' \quad .$$

In the integration, $\mathrm{Im}x_A > 0$ has been assumed, and $\hat{\lambda}$ has been defined to be

$$\hat{\lambda} \equiv (\text{sign Im}\lambda)\lambda \ . \tag{28}$$

Equation (27) is the main result of this section. Let us note the following:

(1) whenever the equilibrium current is neglected, i.e., $\lambda=0$, Eq. (24), (25), and (27) reduce to Eq. (3), (4), and (6) of Ref. 3, and the constant ψ approximation (26) is no longer necessary.

(2) Since the kernel at RHS of Eq. (27) is not symmetric with respect to z,z', it is not straightforward to derive a variational principle formulation to obtain the mode dispersion relation; this is presented in the following section.

(3) The presence of the current term in the quantity $[\Delta'+i\pi x_A/(x_A+2\hat{\lambda})^2]$, however, makes the special case of a mode much broader than the scale of the conductivity quite simple to treat.

Indeed, expanding for small λ, one finds corrections of order λ^2 from all terms in the equation, with the exception of the term $[\Delta'+i\pi x_A/(x_A+2\hat{\lambda})^2]$ which gives a contribution of order λ.

Therefore, in the small λ limit, it is sufficient to keep the effect of the current in this single term only, therefore performing a very simple variational calculation with a trial function of definite parity. Let us stress that it is precisely because the equilibrium current alters the "boundary condition term" $[\Delta'+i\pi x_A/(x_A+2\hat{\lambda})^2]$ that such simple calculation is possible. This calculation is presented in Sec. IV.

III. Variational Formulation for Eq. (27)

Let

$$Q_+ \equiv Q(z,\lambda) \quad ; \quad Q_- \equiv Q(z,-\lambda) \ . \tag{29}$$

$Q_\pm$ satisfy

$$(p_+(z)Q_+')' + \sigma(z)Q_+ = -\hat{\Delta}^{-1} \int_{-\infty}^{\infty} K_+(z,z')Q_+(z')dz \tag{30}$$

$$(p_-(z)Q_-')' + \sigma(z)Q_- = -\hat{\Delta}^{-1} \int_{-\infty}^{\infty} K_-(z,z')Q_-(z')dz' \ , \tag{31}$$

where

$$p_\pm(z) = \frac{x_A^2(z\mp\lambda)^2}{(z\mp\lambda)^2-x_A^2} \tag{32}$$

$$\hat{\Delta} \equiv \Delta' + \frac{i\pi x_A}{(x_A+2\hat{\lambda})^2} \tag{33}$$

$$K_\pm(z,z') \equiv \frac{4x_A^4(z\mp\lambda)}{\left[(z\mp\lambda)^2-x_A^2\right]^2} \ \frac{(z'\mp\lambda)\left[\pm 2\lambda x_A^2+(z'\mp\lambda)^3\right]}{(z'\pm\lambda)^3\left[(z'\mp\lambda)^2-x_A^2\right]^2} \ . \tag{34}$$

Noting that

$$\begin{aligned} Q_+(z) &= Q_-(-z) \\ Q_-(z) &= Q_+(-z) \ , \end{aligned} \tag{35}$$

we can construct the functions:

$$\begin{aligned} \psi(z) &\equiv 1/2\left(Q_+(z)+Q_-(z)\right) \\ \varphi(z) &\equiv 1/2\left(Q_+(z)-Q_-(z)\right) \ . \end{aligned} \tag{36}$$

The function $\psi(\varphi)$ is of even (odd) parity with respect to the "shifted" radial variable z.

From Eq. (30) and (31), a coupled system of equations for the definite parity set Ψ, φ is obtained:

$$M\Psi = N\varphi \tag{37}$$

$$M\varphi = N\Psi \ , \tag{38}$$

where the operators M,N are defined to be

$$M\zeta \equiv \left[(p_- + p_+)\zeta'\right]' + 2\sigma\zeta + \hat{\Delta}^{-1}\int_{-\infty}^{\infty}(K_- + K_+)\zeta(z')dz' \tag{39}$$

$$N\zeta \equiv \left[(p_- - p_+)\zeta'\right]' + \hat{\Delta}^{-1}\int_{-\infty}^{\infty}(K_- - K_+)\zeta(z')dz' \ , \tag{40}$$

for an arbitrary function ζ.

Since the kernels $K_\pm$ are not (z,z')-symmetric, the operators M,N are not self-adjoint, in the sense

$$\langle f, Mg\rangle \neq \langle Mf, g\rangle$$

$$\langle f, Ng\rangle \neq \langle Nf, g\rangle \ ,$$

where $$\langle f, g\rangle \equiv \int_{-\infty}^{\infty} fg\ dz$$

for suitably behaving trial functions f,g.

The operators M,N can however be made self-adjoint by invoking a widely used approximation in the $m \geqslant 2$ tearing mode theory; namely, let us assume the radial width of the mode to be much larger than the Alfvén layer x_A. The kernels $K_\pm$ are then approximated as

$$K_-\pm K_+ \rightarrow \overline{K}_-\pm\overline{K}_+ \equiv 4x_A^4\left[\frac{1}{(z'-\lambda)^3(z+\lambda)^3} \pm \frac{1}{(z'+\lambda)^3(z-\lambda)^3}\right] . \qquad (41)$$

Denoting by $\overline{M},\overline{N}$ the corresponding self-adjoint operators, Eq. (37) and (38) reduce to

$$\overline{M}\Psi = \overline{N}\varphi \qquad (42)$$

$$\overline{M}\varphi = \overline{N}\Psi . \qquad (43)$$

It is now possible to formulate a variational principle for Eq. (42) and (43).

It is straightforward to show that the bilinear functional

$$S(\Psi,\varphi) \equiv \langle\Psi,\overline{M}\Psi\rangle - \langle\varphi,\overline{M}\varphi\rangle - 2\langle\Psi,\overline{N}\varphi\rangle \qquad (44)$$

is variational, in that

$$\left.\delta S\right|_{\varphi\mathrm{const}} = 0$$

$$\left.\delta S\right|_{\Psi\mathrm{const}} = 0 \qquad (45)$$

reproduce Eq. (42), (43) and that S=0, for Ψ,φ solution of (42) and (43). Here, for the purpose of lightening the formulae, the same notation Ψ,φ is used to denote both the solution of Eqs. (42), (43) as well as the trial functions, containing variational parameters, which appear in Eq. (44).

Let, for example,

$$\Psi = \beta\Psi_o(\alpha) \quad ; \quad \varphi = \varphi_o(\alpha) \ , \tag{46}$$

α,β being variational parameters. The mode dispersion relation is then obtained by solving the system:

$$\frac{\partial S}{\partial \alpha} = 0 \quad ; \quad \frac{\partial S}{\partial \beta} = 0 \quad ; \quad S = 0 \ . \tag{47}$$

Denoting by $S_1(\alpha) \equiv \langle\Psi_o,\overline{M}\Psi_o\rangle$; $S_2(\alpha) \equiv \langle\varphi_o,\overline{M}\varphi_o\rangle$; $S_3 \equiv \langle\Psi_o,\overline{N}\varphi_o\rangle$ the variational Eq. (44) reduces in this case to:

$$S(\alpha,\beta) \equiv \beta^2 S_1(\alpha) - S_2 - 2\beta S_3 \ . \tag{48}$$

The system Eq. (47) gives therefore:

$$\begin{aligned} &\beta^2 S_1 - S_2 - 2\beta S_3 = 0 \\ &\beta S_1 = S_3 \\ &\beta^2 S_1' - S_2' - 2\beta S_3' = 0 \ , \end{aligned} \tag{49}$$

where $S_j' \equiv \frac{\partial}{\partial \alpha} S_j$ $(j=1,2,3)$. We note that, when $\lambda \to 0$, the operator $\overline{N}$ becomes identically zero, i.e., the integral S_3 vanishes. From Eq. (49) we conclude that $\beta=0$ and the system Eq. (49) reduces to

$$S_2 = 0 \quad ; \quad S_2' = 0 \ . \tag{50}$$

Of course, the system Eq. (50) is precisely the usual variational principle formulation for the one parameter - definite parity trial function corresponding to the classical analysis in which the equilibrium current is neglected. Our variational formulation, Eq. (42) through (49), is therefore capable of treating analytically the general λ case, making continuous contact with the conventional analysis.

The choice of the trial functions Eq. (46) depends, of course, on the class of modes under consideration. In particular,[3] for solutions which retain tearing mode symmetry in the $\lambda=0$ case, we shall choose, for the odd parity function

$$\varphi(z) = \frac{e^{-\frac{\alpha z^2}{2}}}{z} \quad ; \quad \mathrm{Re}\ \alpha>0 \ . \tag{51}$$

We remark incidentally that this choice for the odd part of the trial formulation requires to retain terms in x_A^2 in the variational integrals, whenever such integrals are calculated to zeroth order in $|\alpha^{1/2}\lambda|$, in the case of a mode broader than the shift λ, to insure the convergence of the integrals.

IV. Dispersion Relation

The dispersion relation for the small λ limit is easily derived, as pointed out at the end of Sec. II, by expanding the quantity $\hat{\Delta}$ to first order in λ and evaluating the variational integrals with the trial function (51). Explicitly the variational functional reduces to

$$S(\varphi) \simeq \hat{\Delta}(I_1+I_2) + I_3^2 , \tag{52}$$

where
$$\hat{\Delta} \simeq \Delta' + \frac{i\pi}{x_A} - 4i\pi \frac{\hat{\lambda}}{x_A^2} \tag{53}$$

and

$$\begin{aligned} I_1 &\equiv \int_{-\infty}^{\infty} \varphi\left(\frac{x_A^2 z^2 \varphi'}{z^2-x_A^2}\right)' dz \\ I_2 &\equiv \int_{-\infty}^{\infty} \sigma_* z^2 \varphi^2 dz \\ I_3 &\equiv 2x_A^2 \int_{-\infty}^{\infty} \frac{z\varphi \, dz}{(z^2-x_A^2)^2} . \end{aligned} \tag{54}$$

The calculation of S, Eq. (52), to lowest non-vanishing order in $|\alpha^{1/2}x_A|$, is carried out in detail in Ref. 3; we shall just quote the final result. The main difference is, of course, the λ term in Eq. (53).

The variational $S = S(\alpha)$ is, to lowest order in $|\alpha^{1/2}x_A|$,

$$\begin{aligned} S(\alpha) &\simeq \sigma_*\alpha^{-1/2} - b\alpha^{3/2}x_A^2 - \frac{\Delta'}{\pi^{1/2}} + 4i\pi^{1/2}\frac{\hat{\lambda}}{x_A^2} \\ b &\equiv \frac{4\sqrt{2}-5}{3} . \end{aligned} \tag{55}$$

The variational parameter α is obtained by $\frac{\partial S}{\partial \alpha} = 0$ to be

$$\alpha^{1/2} = \frac{4/3\ \sigma_*}{\frac{\Delta'}{\pi^{1/2}} - 4i\pi^{1/2}\frac{\hat{\lambda}}{x_A^2}} . \tag{56}$$

The mode dispersion relation is

$$\sigma_*^3 x_A^2 = -\left(\frac{3}{4}\right)^4 \left(\frac{1}{4\sqrt{2}-5}\right)\left[\frac{\Delta}{\pi^{1/2}} - 4i\pi^{1/2}\frac{\hat{\lambda}}{x_A^2}\right]^4 . \tag{57}$$

The roots in Eq. (57) must consistently satisfy Re $\alpha^{1/2} > 0$; Re $\alpha > 0$, as given by Eq. (56). Let us now proceed to discuss the dispersion relation Eq. (57).

Contact with the classical case is established by setting $\lambda=0$ in Eq. (57); we therefore obtain the collisional drift tearing mode dispersion relation[3,4]

$$\sigma_*^3 x_A^2 = -\left(\frac{3}{4}\right)^4 \left(\frac{1}{4\sqrt{2}-5}\right)\left(\frac{\Delta'}{\pi^{1/2}}\right)^4 . \tag{58}$$

We find, in complete agreement with Ref. 7, that in this hydrodynamic regime, the real frequency and the ∇T_e driven part of the tearing mode growth rate are given by

$$\omega_o = \omega_n^* + (1+\alpha)\omega_T^* \tag{59}$$

$$\gamma_{\nabla T_e} \equiv \alpha\alpha' \frac{\omega_o}{\nu}\, \omega_T^* . \tag{60}$$

In Eq. (59) and (60), α, α' are the numerical transport coefficients of Ref. 6. The RHS of Eq. (58) containing Δ' provides, in this drift ordering $\gamma/\omega_0 < 1$, a small damping to the growth rate Eq. (60); we recall from Eq. (56) that the mode is consistent when $\Delta'<0$. Continuous departure from the usual result Eq. (58) follows by assuming $\lambda \neq 0$ but

$$|\Delta' x_A| > 4\pi |\lambda / x_A| \quad , \tag{61}$$

in Eq. (57). We can then treat the λ-term as a perturbation of the classical result; in particular the $(i\Omega_2)$-factor in the definition of λ, is given by the usual magnetic part of γ. Expanding to the lowest significant order we find

$$\sigma_*^3 x_A^2 = - \left(\frac{3}{4}\right)^4 \left(\frac{1}{4\sqrt{2}-5}\right) \left(\frac{\Delta'}{\pi^{1/2}}\right)^4 \left[1 \pm \frac{i16\pi\lambda}{\Delta' x_A^2} - \frac{96\pi^2\lambda^2}{(\Delta' x_A^2)^2}\right] \quad . \tag{62}$$

From the definition of σ_*, given by Eq. (14), we see that the first order change in the eigenvalue affects the real frequency of the mode (the $\pm$ sign depending on the sign of $\mathrm{Im}\lambda$, which in the scaling Eq. (61) depends in turn whether the classical mode is stable or unstable), whereas the growth rate is affected only to second order in the small quantity $\lambda/(\Delta' x_A^2)$. It appears that the effect of the equilibrium current is to (slightly) destabilize the mode.

However, the most interesting physics appears to be described by the scaling opposite to Eq. (61), namely

$$|\Delta' x_A| < 4\pi |\lambda/x_A| \ . \tag{63}$$

Indeed, even though all this section is based on the assumption $|\lambda/x_A| < 1$, the $m \geqslant 2$ tearing mode is characterized by $|\Delta' x_A| < 1$ and therefore the approximate scaling to use whenever current is included is likely to be Eq. (63) rather than Eq. (61).

Let us then neglect Δ' in Eq. (56) and (57); now, of course, the $-i\Omega_2$ factor appearing in λ has to be computed self-consistently. The dispersion relation reduces to

$$\sigma_*^7 x_A^{10} = -\pi^2 \left(\frac{3}{2}\right)^4 \left(\frac{1}{4\sqrt{2}-5}\right) \frac{R^4 \tau_s^4}{a^8} \ . \tag{64}$$

Of the seven roots of unity, one finds from Eq. (56) that $e^{\pm i\pi/7}$ (growing roots) are consistent.

In addition to the growth rate Eq. (60), one therefore finds a new, rather strong, growth rate driven by equilibrium current,

$$\gamma = \frac{1.67}{s} \left(\frac{u_e}{v_e} \frac{v_A}{v_e} \frac{3\omega_T^*}{\omega_o}\right)^{4/7} \left(\frac{c}{\omega_p} \frac{\kappa_\parallel' v_A}{\omega_o}\right)^{6/7} \nu \ . \tag{65}$$

For clarity, the speed of light has been set in Eq. (65) equal to c. As a function of the relevant plasma parameters, the new growth rate scales as

$$\gamma \sim \left(\frac{u_e}{v_e}\right)^{4/7} \left(\frac{m_e}{m_i}\right)^{5/7} \left(\frac{L_n}{L_s}\right)^{6/7} \frac{\nu}{\beta^{8/7}} \ . \tag{66}$$

In Eq. (66), L_n, the density scale length, is taken to be comparable to the electron temperature scale length; β is the usual plasma $\beta \equiv 8\pi p/B^2$, L_s is the shear length.

For simplicity let us define, in addition to x_A, the scales[3] x_e, x_r to be the distances from the mode rational surface where the mode frequency equals $\kappa_\| v_e$; $\kappa_\|^2 v_e^2/\nu$, respectively. Typically $|x_e| < |x_r| \ll a$, a being the plasma radius. We recognize $x_e(x_r)$ to be the scale over which the collisionless (semi-collisional) "conductivity"[3] varies.

Let us now list the scalings used in deriving the new result Eq. (64) and (65), which define the region of parameter space in which the mode exists:

$$|\frac{x_A}{x_r}| < |\alpha^{1/2} x_A| < 1 \tag{67}$$

$$|\Delta' x_A| < |\lambda/x_A| < 1 \tag{68}$$

$$\frac{\gamma}{\omega_o} < 1 \; . \tag{69}$$

With the variational $\alpha^{1/2}$ standing for the (mode width)$^{-1}$, Eq. (67) requires the mode to be broader than x_A as discussed in the calculation, but narrower than x_r to justify the neglect of $\kappa_\|^2 D/\omega$ terms in the fluid equations. Equation (68) has already been emphasized; Eq. (69) has been invoked in taking x_A^2 to be basically real and positive in going from Eq. (64) to (65). Substituting for $\alpha^{1/2}$, λ, γ in Eq. (67) through (69), we find the independent constraints:

$$\left(\frac{\omega}{\nu}\right)^{1/2} \frac{v_e}{v_A} < p^{1/7} < 1 \tag{70}$$

$$|\Delta' x_A| < p^{3/7} \tag{71}$$

$$p^{4/7} < \omega\tau_s\left(\frac{x_A}{a}\right)^2 , \tag{72}$$

where the adimensional parameter p is

$$p \equiv \frac{u_e}{v_e}\frac{x_e}{a}\frac{x_A}{a}\tau_s\nu . \tag{73}$$

The plasma radius a is not, of course, a parameter in the problem, since in Eq. (72) and (73), a^2 in the denominator cancels out with the a^2 factor appearing in the skin time τ_s.

The parameter p scales with respect to the relevant plasma parameters as

$$p \sim \frac{u_e}{v_e}\left(\frac{L_s}{L_n}\right)^2 \beta^{3/2}\left(\frac{m_i}{m_e}\right)^{1/2} . \tag{74}$$

V. Conclusions

The most relevant result of the present paper is given by Eq. (64), which follows from Eq. (57) whenever Eq. (63) is satisfied. Whenever the set of conditions Eq. (70) through (72) are satisfied, the presence of equilibrium parallel current, usually neglected in the inner tearing analysis, is the primary driving source of instability for the tearing mode, with growth rate given by Eq. (65). For some choices of the parameters, such growth rate can even exceed the one driven by T_e gradients.[5-8] A detailed comparison with a particular machine is avoided since the new growth rate is strongly dependent on the plasma parameters which can usually vary among different discharges for the very same machine. Furthermore, many tokamaks operate nowadays in a "mixed" collisionality regime, in that although most of plasma column is in the semi-hydrodynamic regime, its outer part is more in the collisional regime.

Perhaps the most dramatic departure from conventional theory is that the equilibrium current "takes on" the role of sustaining the mode (see Eq. (56)) previously played by the Δ'-term. It is precisely because Δ' is typically very small in the $m \geqslant 2$ tearing mode that one should not neglect the effect of the equilibrium current, although small in the sense $|\lambda / x_A| < 1$.

As we already remarked in the discussion following Eq. (13), the equilibrium current term appears in a rather subtle way, since, although small in some ordering, it turns out to compare with terms which are also very small in the conventional analysis.

We identify the driving physical mechanism in the coupling between equilibrium current and temperature (resistivity) perturbations in Ohm's law, Eq. (8).

It is perhaps worthwhile mentioning that, although the same term in Ohm's law is responsible for the "rippling mode" of Ref. 1, the present instability is a completely new one, and in particular, it does have tearing solution symmetry.

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