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THE FAST ALPHA PARTICLE DISTRIBUTION FUNCTION IN AN OPEN FIELD-LINE PLASMA WITH ELECTROSTATIC CONFINING POTENTIAL

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Abstract

An approximate solution to the Fokker-Planck equation for the alpha-particle distribution function in an open field-line fusion plasma with electrostatic confining potential is presented. Particle and energy pitch-angle scattering losses are computed. The electrostatic confining potential is found to substantially reduce particle and energy losses.

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I. INTRODUCTION

In a reacting deuterium-tritium plasma, the charged fusion reaction products (alpha particles) are born with an energy near 3.5 MeV. In an open field-line plasma, those alphas born in the loss region of velocity space will be lost very rapidly. Those born outside the loss region will slow down via collisions with the background ions and electrons, and some will be pitch-angle scattered into the loss cone. The fast alpha particle velocity space distribution is needed to calculate the particle and energy loss due to pitch-angle scattering, and the fraction of alpha energy deposited in ions and electrons during slowing down. The particle and energy loss and the energy deposition are needed for reactor studies [1,2]. The distribution function itself is needed for microinstability studies where the anisotropy of the alphas is a source of free energy for destabilizing waves [3].

Previous calculations of the fast alpha particle distribution function [3-6] have assumed that the loss region is a cone in velocity space. However, tandem mirror reactors will have a sizeable electrostatic confining potential. This electrostatic confining potential changes the loss region from a cone to a hyperboloid. In this paper we present an approximate method of calculating the fast alpha distribution function including the effects of the electrostatic potential. The method of solution uses fictitious particle sources in the loss region [7]. Including the electrostatic potential makes a significant difference in the alpha particle and energy loss fractions and in alpha particle microinstability integrals.

In Sec. II we present the Fokker-Planck equation that determines the alpha distribution function. Section III is devoted to the approximate solution. Section IV compares the distribution function of Sec. III to an approximate distribution function with a loss cone rather than a loss hyperbola.

II. THE FOKKER-PLANCK EQUATION

The steady-state Fokker-Planck equation for the alpha distribution f is [5]

$$-\frac{1}{\tau_s v^2}\frac{\partial}{\partial v}\left[(v^3+v_c^3)f\right] - \frac{\kappa}{\tau_s}\frac{v_c^3}{v^3}\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial f}{\partial x}\right] = S \quad , \qquad (1)$$

where

$$\tau_s^{-1} = \frac{16\sqrt{2\pi}}{3}\frac{\sqrt{m_e}}{m_\alpha}\frac{e^4 n_e}{(kT_e)^{3/2}} \ln\Lambda \quad ,$$

$$v_c^3 = 3\sqrt{\pi}\,\frac{m_e}{m_\alpha}\,[Z]\,u_e^3 \quad ,$$

$$[Z] = \sum_{\text{ions}} \frac{(Z_i^2\, n_i/A_i)}{n_e} \quad ,$$

$$\kappa = \frac{Z_{eff}}{(2A_\alpha[Z])} \quad ,$$

v is the speed, x the cosine of the pitch angle, $x = v_{\parallel}/v$. T is temperature, m is particle mass, n is density, A the mass number, u the thermal speed $(2kT/m)^{1/2}$, and Z is the charge in units of the proton charge. The subscript e refers to electrons, α to alpha particles, and i to the ion species, mainly deuterium and tritium. The Coulomb logarithm is $\ln\Lambda$, e is the electron charge, k is Boltzmann's constant, and $Z_{eff} = \sum_i Z_i^2(n_i/n_e)$. The source term S represents the birth of alpha particles at 3.5 MeV, and is assumed to be $S = S_0\delta(v-v_\alpha)H(x)$ where S_0 is determined by the fusion reaction rate coefficient, $v_\alpha = (2E_\alpha/m_\alpha)^{1/2}$, E_α = 3.5 MeV., and H(x) is assumed to be 1 (i.e., isotropic birth). The first term in Eq. (1) represents the frictional slowing down of the alphas, and the second term represents pitch-angle scattering by the background ions. The ions and electrons are assumed to have Maxwellian distributions, and the inequality $u_i < v << u_e$ has been used in simplifying the equation. Fast alpha-fast alpha collisions are ignored, as is the diffusion in the v coordinate [8].

One boundary condition is that the distribution function be zero on the boundary of the loss region. This boundary is given by

$$x_L = x_0\left[1 + \left(\frac{v_\varphi}{v}\right)^2\left(\frac{1-x_0^2}{x_0^2}\right)\right]^{1/2} , \qquad (2)$$

where x_0 is the cosine of the loss cone angle when the electrostatic potential is ignored, $x_0 = (1-1/R)^{1/2}$, R is the mirror ratio, and $v_\varphi = (2q_\alpha\varphi_c/m_\alpha)^{1/2}$, where φ_c is the electrostatic confining potential.

Using the transformation of variables,

$$\psi = \frac{(v^3+v_c^3)f}{S_0 v_\alpha^2 \tau_s} \tag{3}$$

$$t = \frac{\kappa}{3} \ln \left[\frac{(1+v_c^3/v^3)}{(1+v_c^3/v_\alpha^3)}\right] \tag{4}$$

Equation (1) becomes

$$\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial\psi}{\partial x}\right] \quad . \tag{5}$$

The boundary conditions are then

$$\psi(x,0) = 1 , \qquad |x| \leqslant x_L(0) \tag{6}$$

$$\psi(\pm x_L(t) , t) = 0 , \qquad t \geqslant 0 \tag{7}$$

$$x_L(t) = x_0\left\{1 + \left(\frac{1-x_0^2}{x_0^2}\right)\left(\frac{v_\varphi^2}{v_c^2}\right)\left[\left(1 + \frac{v_c^3}{v_\alpha^3}\right)\exp\left(\frac{3t}{\kappa}\right) - 1\right]^{2/3}\right\}^{1/2} \quad . \tag{8}$$

Equation (4) has been inverted to find $v(t)$, and this result used in Eq. (2) to obtain Eq. (8).

III. APPROXIMATE SOLUTION

The major difficulty in solving Eqs. (5-8) comes from the boundary condition in the pitch angle, Eq. (7). The solution region is oddly shaped, and a separation of variables is not feasible. Using a technique similar to that of Pastukhov [7], we solve exactly a related problem, in which we remove the boundary condition in the pitch angle, and add a source term to Eq. (5).

$$\frac{\partial\tilde{\psi}}{\partial t} = \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial\tilde{\psi}}{\partial x}\right] + \tilde{S} . \tag{9}$$

The solution region is now nicely shaped ($-1 \leq x \leq 1$) so that we can use separation of variables to solve Eq. (9). The fictitious source S lies within the loss region, and is chosen so that $\tilde{\psi}(s,t)$ is zero near the loss boundary described in Eq. (8). A good choice for $\tilde{S}$ is then one where $\tilde{\psi}(x,t)$ has zeros very close to the loss boundary.

To proceed, we expand both $\tilde{\psi}$ and $\tilde{S}$,

$$\begin{aligned} \tilde{\psi}(x,t) &= \sum_n P_n(x) q_n(t) \\ \tilde{S}(x,t) &= \sum_n P_n(x) s_n(t) \end{aligned} \tag{10}$$

where the P_n are Legendre polynomials, satisfying

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right] = -n(n+1)P_n$$

and we make the ansätz

$$\tilde{S} = \begin{cases} 2\alpha(t_\varphi - t)[\delta(x-1) + \delta(x+1)] , & 0 \leqslant t \leqslant t_\varphi \\ 0 & t > t_\varphi \end{cases} \tag{11}$$

where α will be determined later. The initial condition for $\tilde{\psi}$ in Eq. (6) must be extended for $|x| > x_{L0}$ where $x_{L0} = x_L(0)$. We choose

$$\tilde{\psi}(x,0) = \begin{cases} 1 & |x| \leqslant x_{L0} \\ -1 & |x| > x_{L0} \end{cases} \tag{12}$$

so that the boundary $\tilde{\psi}(x,t) = 0$ will move smoothly near $t=0$.

Since the initial condition Eq. (12) is even, and Eq. (9) is unchanged by the operation $x \to -x$, $\tilde{\psi}$ will be an even function of x. Only even terms will enter the sums, and so from here on we assume n even.

Standard techniques yield

$$q_0(0) = 2x_{L0} - 1 \tag{13}$$

$$q_n(0) = \frac{2(2n+1)}{(n+1)}[x_{L0}P_n(x_{L0}) - P_{n-1}(x_{L0})] \qquad (n>0) \tag{14}$$

$$\frac{dq_n}{dt} = -n(n+1)q_n + s_n \tag{15}$$

$$s_n(t) = \begin{cases} (2n+1)\alpha(t_\varphi - t) & 0 \leqslant t \leqslant t_\varphi \\ 0 & t > t_\varphi . \end{cases} \tag{16}$$

The solution to Eq. (15) for $t \leqslant t_\varphi$ is

$$q_0(t) = q_0(0) + \alpha t(t_\varphi - t/2)$$

$$q_n(t) = q_n(0)\exp[-n(n+1)t] + \frac{\alpha(2n+1)}{n(n+1)}$$

$$[(t_\varphi + \frac{1}{n(n+1)})(1-\exp[-n(n+1)t]) - t] \ , \ n>1 \ . \tag{17}$$

The solution for $t > t_\varphi$ is

$$q_n(t) = q_n(t_\varphi)\exp[-n(n+1)(t-t_\varphi)] \ . \tag{18}$$

At t=0, the approximate loss boundary is located where the actual loss boundary is, at $x = x_{L0}$. We choose α such that the approximate loss boundary closes at $t = t_\varphi$, i.e., $\tilde{\psi}(1,t_\varphi) = 0$. Expanding $\tilde{\psi}(x,t_\varphi)$ in Legendre polynomials, and using the fact that $P_n(1) = 1$ for all n we obtain

$$\tilde{\psi}(1,t_\varphi) = A + \alpha B \tag{19}$$

where

$$A \equiv q_0(0) + \sum_{n>0} q_n(0)\exp[-n(n+1)t_\varphi] \tag{20}$$

$$B \equiv t_\varphi^2/2 + \sum_{n>0} \frac{2n+1}{n(n+1)} \{\frac{1}{n(n+1)} (1-\exp[-n(n+1)t_\varphi])$$

$$- t_\varphi\exp[-n(n+1)t_\varphi]\} \ . \tag{21}$$

Thus, to insure that $\psi(1,t_\varphi) = 0$, we choose

$$\alpha = -A/B \ . \tag{22}$$

IV. COMPARISON

For all of the results shown, we have used the parameters $x_0 = 0.9$, $v_\varphi/v_\alpha = 0.293$, $v_c/v_\alpha = 0.435$, and $\kappa = 0.075$. They correspond to a mirror ratio of 5.3, an electrostatic potential of $e\varphi = 150$ keV, $T_e = 20$ keV, and a 50-50 mixture of deuterium and tritium. Figure 1 shows a comparison of the desired hyperboloid loss boundary with the approximate loss boundary (i.e., the line of zeros of the approximate $\tilde{\psi}$). The two loss boundaries are quite close together, indicating that the $\tilde{\psi}$ calculated in Sec. III is a good approximation to the solution of Eqs. (5-8). Figure 2 shows a contour plot of $\tilde{\psi}$.

The fractional particle ($\overline{P}$) and energy (Q) losses due to pitch angle scattering into the loss region are given by [5]

$$\overline{P} = P(t_\varphi) \tag{23}$$

$$P(t) = \int_0^{x_L(t)} [\psi(x,0) - \psi(x,t)]dx \tag{24}$$

$$Q = \int_0^{P} \frac{v^2}{v_\alpha^2}\, dP = \int_0^{t_\varphi} \frac{v^2(t)}{v_\alpha^2} \frac{dP}{dt}\, dt \ . \tag{25}$$

Note that the upper limits of integration correspond to the closing up of the loss region at $t = t_\varphi$. $\overline{P}$ can be computed exactly for the $\tilde{\psi}$ of Section III:

$$\overline{P} = 1-x_{L0} - \alpha t_\varphi^2/2 \ . \tag{26}$$

A numerical quadrature has to be performed to find Q. Table I compares the per cent losses calculated from a distribution with 1) a loss cone (Ref. 5), and 2) a loss hyperbola (Sec. III). Including the electrostatic confining potential clearly has a significant effect on these loss fractions.

Because of their anisotropy, the alpha particles in a tandem mirror reactor may drive plasma waves unstable [3]. The growth rate is partly determined by a resonant integral over the alpha distribution function. Define

$$\mathcal{I}(v_\parallel) = -\int_0^\infty dv_\perp \frac{v_\perp^3}{v} \frac{\partial f}{\partial x} (x,v) \ . \tag{27}$$

The growth rate of the Alfvén Ion-Cyclotron instability [7], with $k_\perp = 0$, is proportional to $\mathcal{I}[(\omega-\omega_{ca})/k_\parallel]$ when the growth rate $\gamma << \omega$. Figure 3 shows $\mathcal{I}(v_\parallel)$ calculated from f with 1) a loss cone (Ref. 5), and 2) a loss hyperbola (Sec. III). The loss hyperbola distribution function is generally more isotropic than the loss cone distribution function.

For Figs. 1 and 2, 25 terms were kept in the Legendre polynomial expansion. For Figure 3 and Table I, 100 terms were kept. Even with 100 terms, a slight oscillation in the loss hyperbola results can be seen. The oscillations come from the Gibbs phenomena associated with the polynomials approximation to the discontinues initial condition [Eq. (12)]. If computational speed is important, a slightly smoothed initial condition could be used to decrease the size of the Gibbs phenomena, and thus reduce the number of terms needed in the Legendre polynomial expansion.

V. CONCLUSION

We have presented an efficient method of calculating the steady-state alpha distribution function in a tandem mirror reactor that includes the effect of the electrostatic confining potential. The approximate distribution function involves only sums over elementary functions and Legendre polynomials. The electrostatic confining potential has a significant effect on the particle and energy pitch-angle scattering losses and on typical microinstability integrals over the distribution function.

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REFERENCES

[1] G. A. Carlson, B. Arfin, W. L. Barr, B. M. Boghosian, J. L. Erickson, J. H. Fink, G. W. Hamilton, B. G. Logan, J. O. Myall, and W. S. Neff, Jr., UCRL-52836 (Lawrence Livermore National Laboratory, 1979).

[2] J. F. Santarius and R. W. Conn, UWFDM-340 (University of Wisconsin, Madison, 1980).

[3] J. D. Hanson and E. Ott, Phys. Fluids 27, 150 (1984).

[4] L. C. Steinhauer, Nucl. Fusion 20, 69 (1980).

[5] D. Anderson, H.-G. Gustavsson, H. Hamnén, and M. Lisak, Nucl. Fusion 22, 403 (1982).

[6] J. F. Santarius and J. D. Callen, Phys. Fluids 26, 1037 (1983).

[7] V. P. Pastukhov, Nucl. Fusion 14, 3 (1974).

[8] H. Saito, T. Sekiguchi, M. Katsurai, and S. Maekawa, Nucl. Fusion 17, 919 (1977).

[9] D. C. Watson, Phys. Fluids 23, 2485 (1980); R. C. Davidson and J. M. Ogden, Phys. Fluids 18, 1045 (1975); T. A. Casper and G. R. Smith, Phys. Rev. Lett. 48, 1015 (1982).

TABLE I

Pitch Angle Scattering Loss Percentages

	Particle	Energy
Loss cone	25.7	9.0
Loss hyperbola	14.5	5.8

Table Caption

Pitch angle scattering particle and energy loss percentages [Eqs. (24) and (25)] for alpha distribution functions computed in two different ways: loss cone, from Ref. 5, and loss hyperbola, from Sec. III.

Figure Captions

Fig. 1. Loss boundary location in velocity space. The solid line is a hyperbola, the desired loss boundary. The dashed line is the approximate loss boundary, the line of zeros of $\tilde{\psi}$, as calculated in Section III.

Fig. 2. Contour plot of the alpha distribution function with an electrostatic confining potential. The outermost contour is at f=0. The other contours are linearly spaced between $f(v_\alpha)$ and the maximum value of f.

Fig. 3. Microinstability integral [Eq.(26)] as function of $v_\parallel$. Solid line is for a loss cone distribution (Ref. 5); dashed line is for a loss hyperbola distribution, (Sec. III).

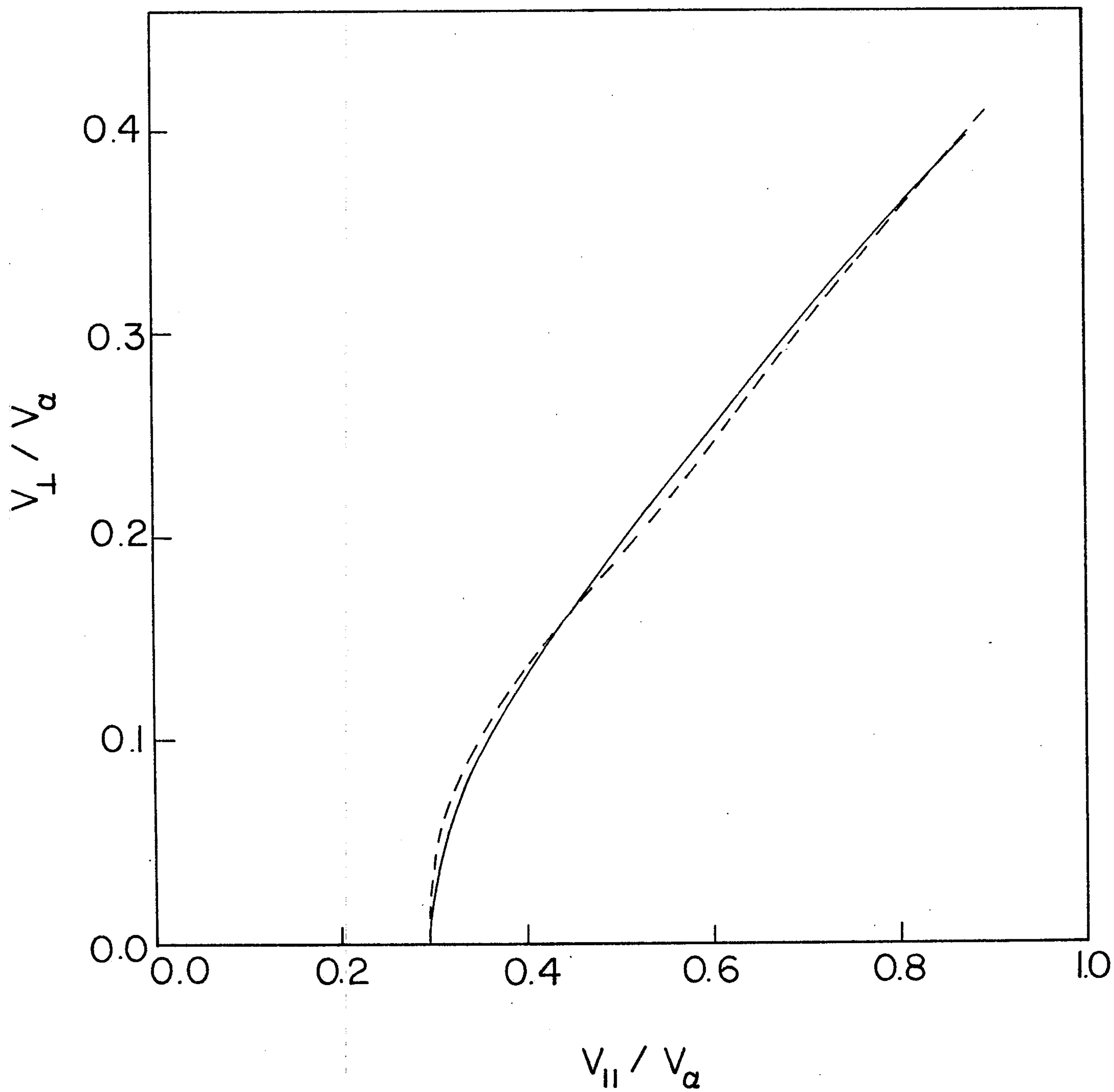

FIG. 1

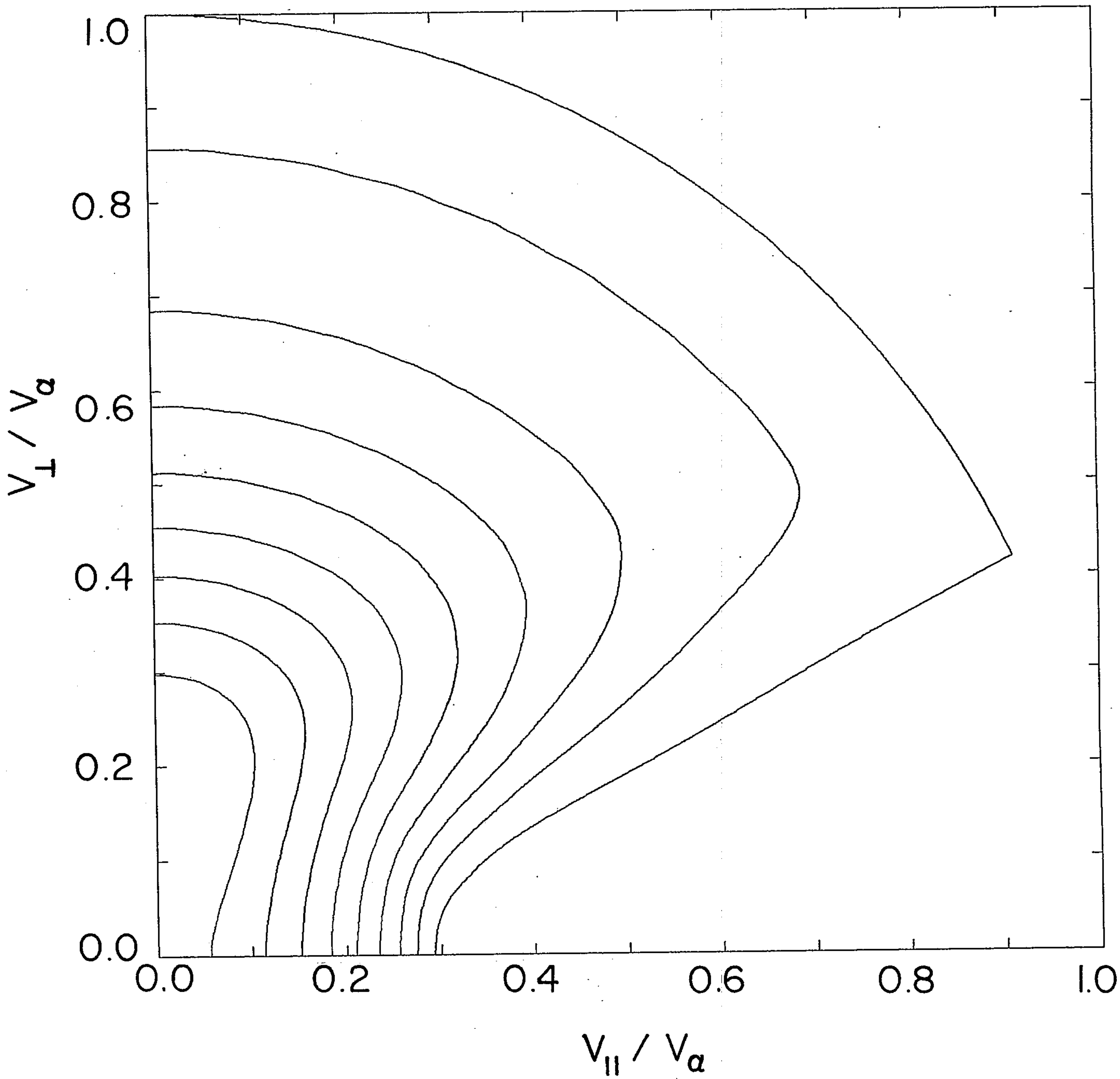

FIG. 2

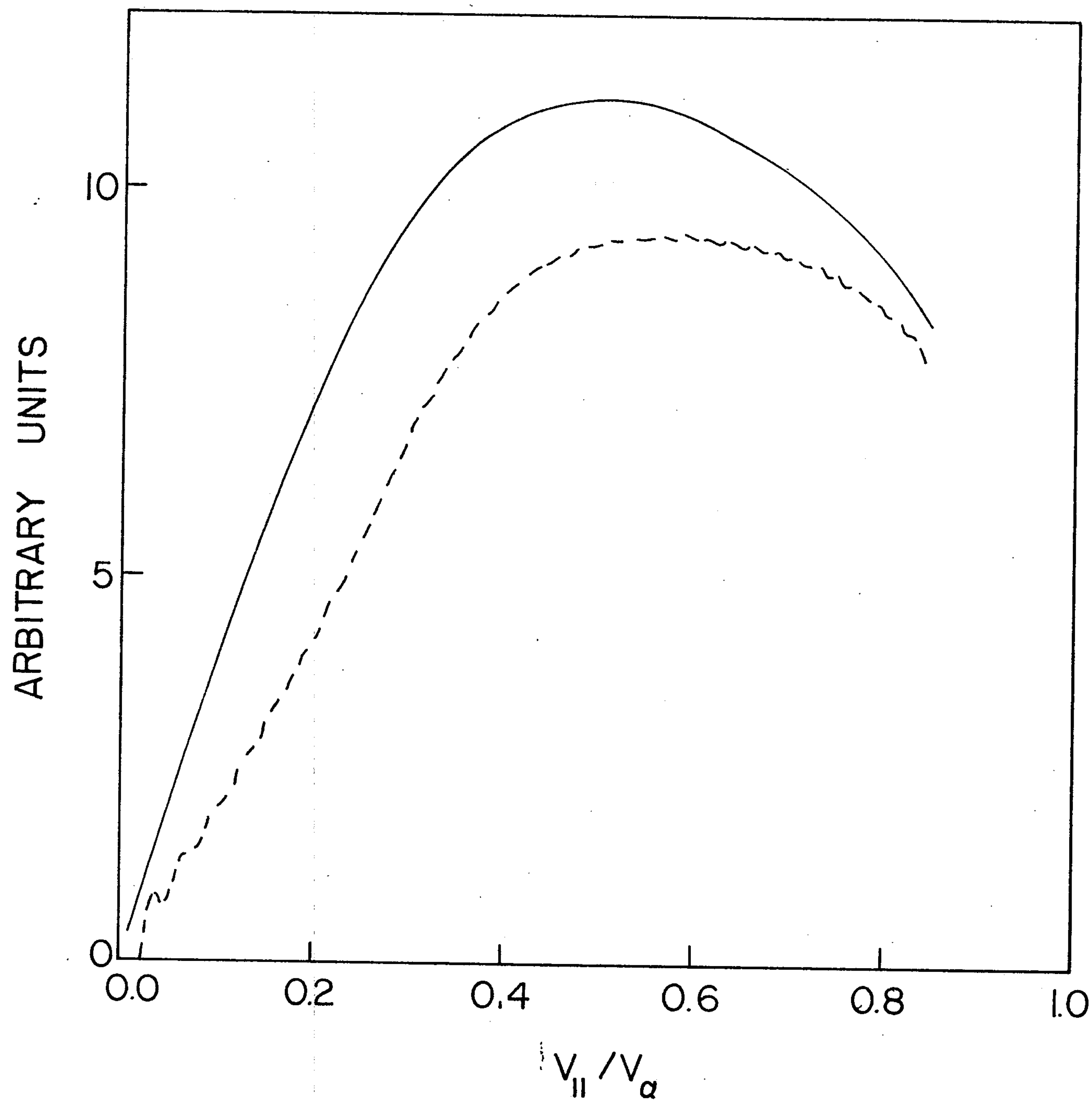

FIG. 3