Renormalized theory of dissipative dispersive turbulent systems

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A systematic perturbation theory to deal with stationary, homogenous turbulence in a dispersive dissipative system is developed, and is shown to be renormalizable. General properties of the renormalized equations are discussed, and for the specific case of Vlasov–Poisson turbulence, it is shown that the current theory reduces to the conventional weak turbulence theory. Kramers–Kronig kind of dispersion relations are derived for the nonlinear dielectric.

I. Introduction

There is a variety of models used to describe the nonlinear behavior of physical phenomena in different areas of physics and engineering. Some of these systems, like the Korteweg-DeVries equation (k-dV), have been extensively studied to elucidate the properties of coherent nonlinear states, for example, the solitons. On the other hand, Navier-Stokes equations and Vlasov-Poisson equations, etc., have been primarily investigated to study the turbulent phenomena associated with these systems. It may be noted that very specific conditions have to be specified for a nonlinear system to allow a coherent solution. Thus one expects that the time asymptotic or steady state solution of a large variety of nonlinear systems will be a turbulent state; a small amount of dissipation could, in general, disturb the delicate balance between nonlinearities and dispersion required to give rise to a coherent state. To deal with the general problems, a very elegant and formally powerful systematalog has been developed by Martin, Siggia and Rose[1] (MSR). However, the MSR formalism is not easily amenable to applications, and only lowest-order (in perturbation) solutions have been obtained so far. Specifically in Krommes[2], application to the Vlasov-Poisson system, the nonlinear source (defined in the text) has not been included in a solvable form.

In this paper, we take a less formal approach and develop a systematic perturbation theory to deal with stationary homogenous turbulence described by dispersive dissipative systems with a quadratic nonlinearity. The theory will contain Vlasov-Poisson, Navier-Stokes, k-dV, Boussinesq equation, etc., as its special cases.

To deal with a general statistical ensemble of turbulence, we use a formal method called correlation expansion[3] to decompose the product of fluctuating quantities into correlated and uncorrelated parts. This decomposition is carried out order by order using a diagrammatic technique.

The existence of a formal reliable solution for the fluctuating field requires that the theory be renormalizable, which in the present context means that the compensating term added to renormalize the linear part of the operator must be cancelled by appropriate contributions from the nonlinear terms to each order. Thus, to each order in the perturbation theory, the nonlinear term is uniquely split into the coherent part, which is used to cancel the compensating (or renormalizing) term, and the incoherent part, which is to be interpretted as a nonlinear source. We show, in this paper, that this is indeed possible. This proof of renormalizability puts the perturbation theory on a firm footing, and one can use it with great confidence.

The renormalized system, thus obtained, is quite complicated and not readily solvable. However, some qualitative properties of the system can be discussed. For example, the renormalized operator contains the entire information about the response of the medium, and can be seen as a response function.[1] This response function is shown to be explicitly so for the Vlasov-Poisson case. This further suggests that we could obtain relationships between the real and imaginary parts of the response function (dielectric function) by invoking causality, and thus obtain a Kramers-Kronig kind of dispersion relation. This can be quite important, because it allows us to determine the real (imaginary) part of the response if the other is given (say experimentally). The constraint of causality is also necessary for the self-consistency of the system.

In Sec. II, we develop the general theoretical framework for dealing with a dissipative, dispersive system with a quadratic nonlinearity. This kind of nonlinearity corresponds to a Yukawa type interaction. In Sec. III, we discuss the general properties of the renormalized equations, and derive Kramers-Kronig dispersion relation for the nonlinear dielectric function. Section IV is devoted to a discussion of Vlasov-Poisson system as an application of our formalism and Sec. V contains a brief summary and conclusions.

II. General Theory

The nonlinear system under investigation could be written as

$$L\ \varphi(x) = N\ \varphi(x) \tag{1}$$

where $\varphi(x)$ is a typical field variable (flow velocity, temperature, electromagnetic fields, etc.), $x = (\underset{\sim}{x},t)$ denotes the space-time coordinates, and L and N are respectively the linear and the quadratically nonlinear operators. As usual, we transform this equation to Fourier space, where all differential operators become algebraic. The transformed equation becomes

$$G_k^{(0)^{-1}} \varphi_k = \sum_{k=k_1+k_2} V_{k_1,k_2}\, \varphi_{k_1}\, \varphi_{k_2} \tag{2}$$

where $G_k^{(0)^{-1}}$ is the transform of the linear operator L, and the right hand side is the convolution term obtained by transforming the nonlinear terms. V_{k_1,k_2} is the strength of the coupling between different modes, and $k=(\underset{\sim}{k},\omega)$ is the wave four vector. The superscript (0) for G_k^{-1} is to explicitly display that it is unrenormalized or the bare linear propagator. The wave φ_k is diveded into two parts, a random part $\tilde{\varphi}_k$, and what we call an unrandom (for want of a better term) part Φ_k

$$\varphi_k = \Phi_k + \tilde{\varphi}_k$$

Notice that Φ_k is unrandom but has k-dependence. The ensemble averaging (< >) yields the following equation:

$$G_k^{(0)^{-1}} \Phi_k = \sum_{k_1+k_2=k} V_{k_1,k_2} \Phi_{k_1} \Phi_{k_2} + \sum_{k_1+k_2=k} V_{k_1,k_2} \langle \tilde{\varphi}_{k_1} \tilde{\varphi}_{k_2} \rangle \, . \tag{3}$$

The equation obeyed by the random part will then be

$$G_k^{(0)^{-1}} \tilde{\varphi}_k = 2 \sum_{k_1+k_2=k} V_{k_1,k_2} \Phi_{k_2} \tilde{\varphi}_{k_1} + \sum_{k_1+k_2=k} V_{k_1,k_2} \tilde{\varphi}_{k_1} \tilde{\varphi}_{k_2} \tag{4}$$

Defining the renormalized propagator

$$G_k = (G_k^{(0)^{-1}} + i\Gamma_k)^{-1} \, , \tag{5}$$

we express Eq. (4) as

$$\tilde{\varphi}_k = G_k \, 2 \sum_{k_1+k_2=k} V_{k_1,k_2} \Phi_{k_1} \tilde{\varphi}_{k_2} + G_k \sum_{k_1+k_2=k} V_{k_1,k_2} \tilde{\varphi}_{k_1} \tilde{\varphi}_{k_2} + G_k \, i\Gamma_k \, \tilde{\varphi}_k \tag{6}$$

Equation (6) is the basis of developing a systematic perturbation theory.

For this purpose, we very closely follow the methodology of Ref. 3. Essential steps in this process are

1) Successive iterations of Eq. (6) to an appropriate order,

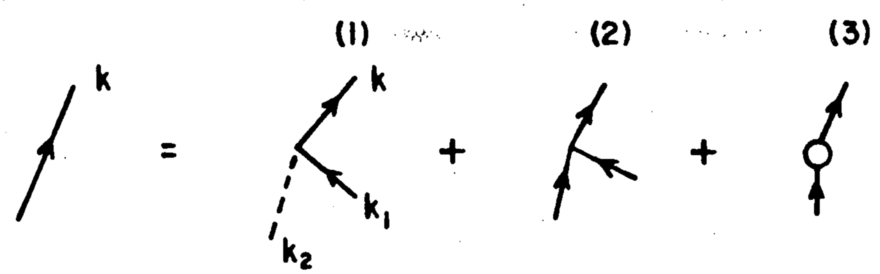
2) Make use of "correlation expansion" to separate the terms into correlated and uncorrelated parts

3) To each order, cancel the compensating term with the correlated parts, thus obtaining an expression for $i\Gamma_k$ to the appropriate order,

4) Show that this cancelling can be indeed accomplished to any desired order, thus proving the renormalizability of the theory

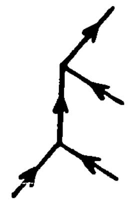
5) Obtain expression for physically interprettable quantities, for example, the nonlinear dielectric response, etc.

This program is best accomplished using diagrammatic techniques (Ref. 3). Let a solid line be used to denote either G_k (when it occurs as a trunk including the top line) or $\tilde{\varphi}_k$ (all others), the dotted line $2\varphi_k$, and a bubble $i\Gamma_k$. Then Eq. (6) is formally equivalent to

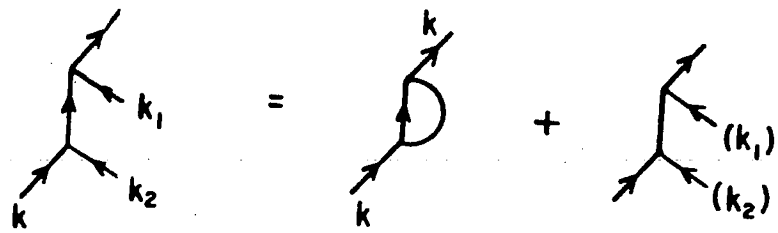


(7)

where the regular vertex represents the strength V_{k_1,k_2} of the interaction. Successive iterations of Eq. (7) will generate higher and higher order terms. We display a typical term which arises when the term (2) in Eq. (7) is iterated,

(2) $\longrightarrow$  + • • • (8)

The displayed term has two vertices, and consequently represents a second order process, that is, the strength of the process is quadratic in V. In addition, the process represented by Eq. (8) has two distinct parts; one in which the inner lines are contracted, and the other in which they have no correlation. That is



(9a)

where the first term on right hand side denotes that part of the process for which $k_1+k_2=0$, and contribute to the self-energy of the system. The second term is the rest of the process with (k_1) and (k_2) implying that $k_1+k_2\neq 0$ for this diagram. This is essentially the essence of the "Correlation Expansion" we described earlier, and in terms of field amplitudes this reads

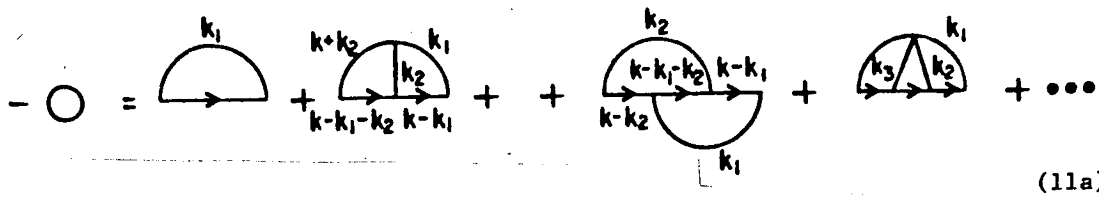
$$\tilde{\varphi}_{k_1}\tilde{\varphi}_{k_2} = \langle\langle\tilde{\varphi}_{k_1}\ \tilde{\varphi}_{k_2}\rangle\rangle + (\tilde{\varphi}_{k_1})(\tilde{\varphi}_{k_2}) \tag{9b}$$

where $\langle\langle\ \ \rangle\rangle$ is the correlation function to the second order (in this case), and $(\tilde{\varphi}_{k_1})$ and $(\tilde{\varphi}_{k_2})$ are uncorrelated, i.e.

$$\langle(\tilde{\varphi}_{k_1})(\tilde{\varphi}_{k_2})\rangle = \langle\tilde{\varphi}_{k_1}\rangle\ \langle\tilde{\varphi}_{k_2}\rangle = 0 \tag{10}$$

where < > is the ensemble average.

Following this procedure, we can prove the full cancellations of the diagrams containing $(i\Gamma_k)$ with those diagrams containing self-energy, which yields



(11a)

or

$$-i\Gamma_k = \sum_{k_1} V_{k_1,k-k_1} G_{k-k_1} V_{k-k_1,k} \langle\langle \tilde{\varphi}_{k_1} \tilde{\varphi}^*_{k_1} \rangle\rangle$$

$$+ \sum_{k_1,k_2} V_{k_1,k-k_1} G_{k-k_1} V_{k_2,k-k_1-k_2} G_{k-k_1-k_2} V_{-k_1-k_2,k} \langle\langle \tilde{\varphi}_{k_1} \tilde{\varphi}_{k_2} \tilde{\varphi}^*_{k_1+k_2} \rangle\rangle$$

$$+ \sum_{k_1,k_2} V_{k_1,k-k_1} G_{k-k_1} V_{k_2,k-k_1-k_2} G_{k-k_1-k_2} V_{k_1,k-k_2} G_{k-k_2} V_{-k_2,k}$$

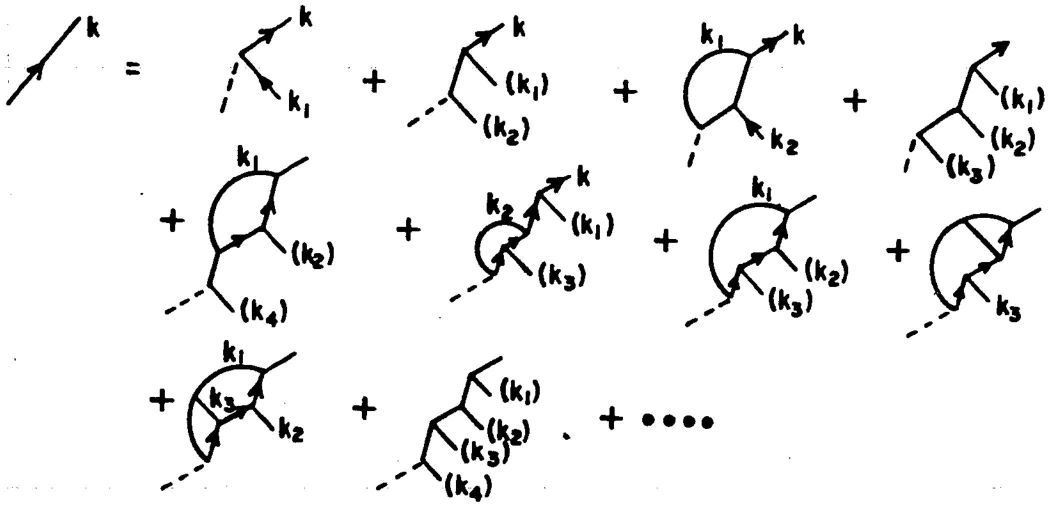
$$\langle\langle \tilde{\varphi}_{k_1} \tilde{\varphi}^*_{k_1} \rangle\rangle \langle\langle \tilde{\varphi}_{k_2} \tilde{\varphi}^*_{k_2} \rangle\rangle$$

$$+ \sum_{k_1,k_2,k_3} V_{k_1,k-k_1} G_{k-k_1} V_{k_2,k-k_1-k_2} G_{k-k_1-k_2} V_{k_3,k-k_1-k_2-k_3} G_{k-k_1-k_2-k_3}$$

$$V_{-k_1-k_2-k_3,k} \langle\langle \tilde{\varphi}_{k_1} \tilde{\varphi}_{k_2} \tilde{\varphi}_{k_3} \tilde{\varphi}^*_{k_1+k_2k_3} \rangle\rangle$$

$$+ \ldots \qquad (11b)$$

After the full cancellation, the remaining part will be



(12a)

or

$$
\begin{aligned}
\tilde{\varphi}_k = {} & G_k \sum_{k_1} V_{k_1,k-k_1} \tilde{\varphi}_{k_1} \, 2\Phi_{k-k_1} \\
& + G_k \sum_{k_1,k_2} V_{k_1,k-k_1} G_{k-k_1} V_{k_2,k-k_1-k_2} \, 2\Phi_{k-k_1-k_2} (\tilde{\varphi}_{k_1})(\tilde{\varphi}_{k_2}) \\
& + G_k \sum_{k_1,k_2} V_{k_1,k-k_1} G_{k-k_1} V_{k_2,k-k_1-k_2} \, 2\Phi_{k-k_2} \langle\langle \tilde{\Phi}_{k_1} \tilde{\Phi}^*_{k_1} \rangle\rangle \, \tilde{\varphi}_{k_2} \\
& + G_k \sum_{k_1,k_2,k_3} V_{k_1,k-k_1} G_{k-k_1} V_{k_2,k-k_1-k_2} G_{k-k_1-k_2} V_{k_3,k-k_1-k_2-k_3} \\
& \qquad 2\Phi_{k-k_1-k_2-k_3} (\tilde{\varphi}_{k_1})(\tilde{\varphi}_{k_2})(\tilde{\varphi}_{k_3}) \\
& + G_k \sum_{k_1,k_2,k_4} V_{k_1,k-k_1} G_{k-k_1} V_{k_2,k-k_1-k_2} G_{k-k_1-k_2} V_{-k_1,k-k_2} G_{k-k_2} \\
& \qquad V_{k_4,k-k_2-k_4} \, 2\Phi_{k-k_2-k_4} \langle\langle \tilde{\varphi}_{k_1} \tilde{\varphi}^*_{k_1} \rangle\rangle (\tilde{\varphi}_{k_2})(\tilde{\varphi}_{k_4}) + \ldots
\end{aligned}
\qquad (12b)
$$

We notice that in Eq. (12), which is an expression for $\tilde{\varphi}_k$, no sub-self-energy terms [see Eq. (13)] appear. In fact, the general rule for constructing terms in Eq. (12) is to retain all kinds of topologically allowed diagrams which do not contain self-energy structures. All external lines are uncorrelated. The correlated parts have been drawn explicitly in terms of internal lines and multi-wave structures. The general construction rules, as well as the proof of renormalizability to any arbitrary order very closely follows the procedure discussed in Ref. 3. Referring the reader to Ref. 3 for all detail, we simply sketch the major steps in the renormalization program. The proof is based on the following observations which are a manifestation of our iteration procedure:

Observation I. To a given order all possible self-energy structures must appear except those (and <u>never</u> those) that contain self-energy sub-structures. Henceforth, an allowable self-energy structure (in which no self-energy sub-structure can be isolated), will be called a completely overlapping diagram.

Observation II. A given diagram can appear only once. There is no repeated diagram.

Observation III. For non-self-energy diagrams containing self-energy sub-structures, all types of self-energy structures produced in the lower perturbative order are reproduced totally in the higher perturbative order.

Observation IV. In the higher order diagrams there exists no new type of self-energy sub-structure which has the same order as the self-energy structure that has already appeared in a lower order diagram. The detail of proof for these observations can be seen in Ref. 3.

Observation I suggests the choice of $-i\Gamma_k$ to be the sum of all types of possible self-energy structures that must be overlapping in topology. For example, the following diagrams, which we call sub-self-energy diagrams, are prohibited.

(13)

The combination of the observations II, III, and IV means that as soon as the cancellation takes place for the lowest order diagram containing $i\Gamma_k$, the same cancellation occurs for the higher order diagrams that have structure plus $i\Gamma_k$ with the diagram with that structure plus all self-energy structures.

The spectrum equation to the lowest order is readily found to be

$$I_k = 4|G_k|^2 \sum_{k_1} |V_{k,k-k_1}|^2 I_{k-k_1} I_{k_1} \tag{14}$$

where $I_k = \langle\langle \varphi_k \varphi_k^* \rangle\rangle$ in Eq. (14) is the fluctuation spectrum. Notice that G_k is a functional of the spectrum I_k. For higher order solution, it will turn out to be a functional of the spectrum as well as higher order correlations.

III. General Properties, Kromers-Kronig Relations

An examination of Eqs. (14) reveals that even in the lowest order, the system is quite complicated and not easy to solve analytically. To obtain the spectrum I_k, and other higher correlation functions one will eventually have to resort to numerical techniques. In this paper, however, we discuss only the general aspects of the nonlinear equations. We begin with categorizing the nonlinear systems which can be studied within the framework of our equations. There are two principal categories:

Self-Excited Turbulence: In this case the source of turbulence is some store of free energy in the system which drives one or more of the unrandom modes of oscillations unstable. For example, spatial inhomogeneties or velocity space anisotropy can provide free energy for a host of plasma instabilities. Notice that the unrandom modes of oscillation need not be linear; they could be nonlinear coherent modes like the solitons provided the original equation $L\varphi = N\varphi$ allows a soliton solution. In this case, the solitons solutions are to be interpretted as nonlinear normal modes, and the turbulence built around them will be called soliton-excited turbulence. We remind the reader that one of the principle aims of turbulence studies is to determine the effect of turbulence on the properties of the ambient system; it could be to determine turbulent transport, or the effect of turublence on the structure of the solitons. It is simple to find the conditions under which turbulence does affect initial unrandom state. Making use of Eq. (9b), we write Eq. (3) as

$$G_k^{(0)^{-1}} \phi_k = \sum_{k_1+k_2=k} V_{k_1,k_2} \phi_{k_1}\phi_{k_2} + \sum_{k_1+k_2=k} V_{k_1,k_2} I_{k_1} \delta(k_1+k_2)$$

$$= \sum_{k_1+k_2=k} V_{k_1,k_2} \phi_{k_1}\phi_{k_2} + \sum_p V_{p,-p} I_p \delta_{k,0} \qquad (15)$$

where the second term on the right hand-side determines the back reaction of turbulence on the unrandom state [plasma distribution function, or a kdV soliton for example]. Clearly, there is a back reaction only for k=0 or the d-c component of the original state. However, if $v_{p,-p}=0$, the initial state is unaffected by the presence of turbulence. Thus we conclude that $v_{p,-p}\neq 0$ is required for the turbulence to cause either anamoulous transport or distort the structure of the soliton. To determine these effects, we need to know the spectrum I_p, and it is to determine I_p that the renormalized equations are derived.

Externally Excited Turbulence: In several problems of interest, the turbulence is not due to an internal source of energy, but is caused by an external stirring source. This situation occurs, for example, in plasma heating experiments where high amplitude electromagnetic waves are excited in antennas or waveguides placed near the plasma edge. The propagation characteristics of these waves in a turbulent plasma (the turbulence is created by these waves only) can also be studied within the framework established in Sec. II.

Thus these is a large body of physically interesting problems which can be studied by making use of the renormalized equations derived in Sec. II. Continuing with our attempt to delineate general structural properties of the equations, we now derive for the nonlinear system an appropriate version of the Kramers-Kronig dispersion relation.[4] In its simplest form, the Kramers-Kronig dispersion relations are nothing but an expression of causality. The final result of Sec. II can be schematically written as

$$\tilde{\Phi}(k,\omega) = G(k,\omega)\ S(k,\omega) \tag{16}$$

where $\tilde{\varphi}(k,\omega)$ is the response, $S(k,\omega)$ is the nonlinear source, and $G(k,\omega)$ plays the part of the dielectric function. Since G is a function of ω, it implies that the relationship between $\tilde{\varphi}(t)$ and $S(t)$ will be

$$\tilde{\varphi}(t) = \int_{-\infty}^{+\infty} G(\tau)S(t-\tau)d\tau \tag{17}$$

The above response can be made causal, i.e., the value of $\tilde{\varphi}(t)$ at any time is determined solely by the value of the source at times previous to the time of observation, by demanding that $G(\tau)=0$ for $\tau<0$. Thus

$$\tilde{\varphi}(t) = \int_{0}^{\infty} G(\tau)S(t-\tau)d\tau\ , \tag{18}$$

and we can use the above equation to obtain the constraints on $G(\omega)$. for our purpose, it turns out to be more convenient to deal with the inverse causal equation

$$S(t) = \int_{-\infty}^{0} G^{-1}(\tau)\ \tilde{\varphi}(t-\tau)d\tau \tag{19}$$

because we are interested in determining the constraints on $G^{-1}(\omega)$. The fourier transform of Eq. (19) is obviously

$$S(\omega) = G^{-1}(\omega)\ \tilde{\varphi}(\omega) \tag{20}$$

with

$$G^{-1}(\omega) = \int_{-\infty}^{0} d\tau \ \ G^{-1}(t)\ \exp(i\omega\tau)\ . \tag{21}$$

We notice that under reasonably mild conditions, i.e., $G^{-1}(\tau)\to 0$ sufficiently fast as $\tau\to\infty$, $G^{-1}(\omega)$ is an analytic function of ω in the lower half plane including the real axis. Cauchy's theorem now leads us to

$$G^{-1}(\omega) = \frac{1}{2\pi i} \oint \frac{G^{-1}(\omega')d\omega'}{\omega'-\omega} \tag{22}$$

where the closed contour consists of the real axis and the great semicircle at infinity in the lower half plane. It is useful to subtract the asymptotic value $[G^{-1}(\omega)]_{asy} \equiv G^{-1}(\omega)_{\omega\to\infty}$ from both sides of Eq. (22). This step followed by the substitution of the expression for $G^{-1}(\omega)$ [Eq. (5)] leads to

$$i\bar{\Gamma}(\omega,k) \equiv G^{-1}(\omega) - G^{-1}(\omega)_{asy} = i\Gamma(\omega,k) - i\Gamma(\omega,k)_{asy} = \frac{1}{2\pi i} \oint \frac{i\bar{\Gamma}(\omega',k)}{\omega'-\omega}\, d\omega' \tag{23}$$

Realizing that $\bar{\Gamma}(\omega,k)$ goes to zero as $\omega\to\infty$, we can follow the usual procedure to obtain the required dispersion relations [P denotes the Cauchy principal value]

$$\mathrm{Re}[i\bar{\Gamma}(\omega,k)] = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\mathrm{Im}[i\bar{\Gamma}(\omega,k)]}{\omega'-\omega}\ d\omega' \tag{24}$$

$$\mathrm{Im}[i\bar{\Gamma}(\omega,k)] = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\mathrm{Re}[i\bar{\Gamma}(\omega,k)]}{\omega'-\omega} d\omega' \tag{25}$$

relating the real and imaginary parts of the renormalizing factor $i\Gamma(\omega,k)$. We remind the reader that Re $i\Gamma(\omega,k)$ implies a frequency shift (dispersion), and Im $i\Gamma(\omega,k)$ represents resonance broadening (absorption). The essence of the Kramer-Kronig dispersion relations (a direct consequence of causality) is that any resonance broadening must necessarily be accompanied by a frequency shift. In addition, if either of these is known, the other can be automatically calculated. Thus these relations can be used as an important practical tool when some empirical knowledge of the system is available. At the very least, they are an important constraint which must be obeyed by any physically reasonably theory. We end this section by obtaining an explicit expression for $i\Gamma_{asy}$. From Eq. (11a-b), we notice that as $\omega\to\infty$, only the first term will contribute to $i\Gamma(\omega,k)$ because all other terms have higher powers of ω in the denominator. Therefore

$$\underset{\omega\to\infty}{i\Gamma(\omega,k)} = \underset{\omega\to\infty}{} \int d\omega_1 dk_1 \frac{V_{k,k_1} |\varphi(\omega_1,k)|^2 V_{k_1-k_1}}{\omega-\omega_1-f(\underset{\sim}{k}-\underset{\sim}{k}_1)+i\Gamma(\omega-\omega_1,\underset{\sim}{k}-\underset{\sim}{k}_1)} \tag{26}$$

$$\simeq \frac{1}{\omega} \int d\omega_1 dk_1 \; V_{k,k_1} \; V_{k,-k_1} \; |\varphi(\omega_1,k_1)|^2 \equiv \frac{A(k)}{\omega} \tag{27}$$

where $A(k)$ is real, and which helps us rewrite Eqs. (24), (25) in the form

$$\mathrm{Re}[i\Gamma(\omega,k)] = \frac{A(k)}{\omega} - \frac{1}{\pi} p \int d\omega' \frac{\mathrm{Im}[i\Gamma(\omega',k)]}{\omega'-\omega} \tag{28}$$

$$\mathrm{Im}[i\Gamma(\omega,k)] = \frac{1}{\pi} p \int d\omega' \frac{\mathrm{Re}[\Gamma(\omega',k)]-A(k)/\omega'}{\omega'-\omega} \tag{29}$$

Sec. IV Vlasov Poisson System and Nonlinear Dielectric Response

In this section, we demonstrate the scope of our formalism by applying it to the Vlasov-Poisson system. The perturbed Vlasov equation ($k \neq 0$)

$$(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}) f_k = \hat{L}(k)\, f_o\, \phi_k + \sum_{k_1 \neq k} \hat{L}(k_1) f_{k-k_1} \phi_{k_1}, \tag{30}$$

and the Poisson equation

$$\varphi_k = O_k f_k, \tag{31}$$

where $\hat{L}(k) = (-q/m) \underset{\sim}{k} \cdot \underset{\sim}{\partial}$, and $O_k = \frac{4\pi q}{k^2} \int d\underset{\sim}{v}$ (with q and m as the charge and mass of the particles, $\underset{\sim}{\partial}$ as the gradient in velocity space) can be combined to yield

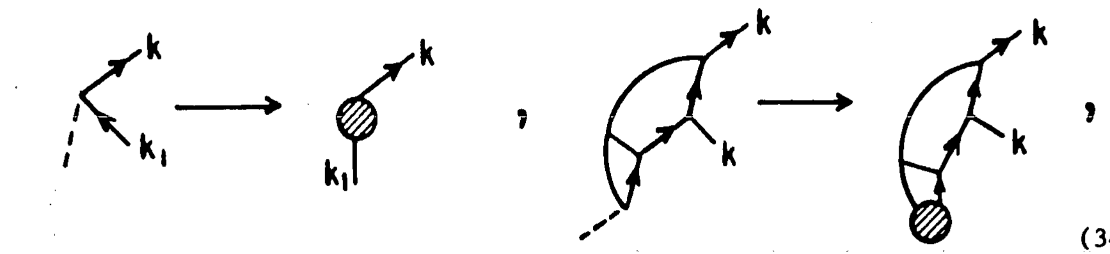
$$f_k = G_k \hat{L}(k) f_o O_k f_k + \sum_{k_1 \neq k} \hat{L}(k_1) f_{k-k_1} O_{k_1} f_{k_1} + G_k i \Gamma_k f_k \tag{32}$$

which is of the same form as Eq. (6) if we make the following transformation

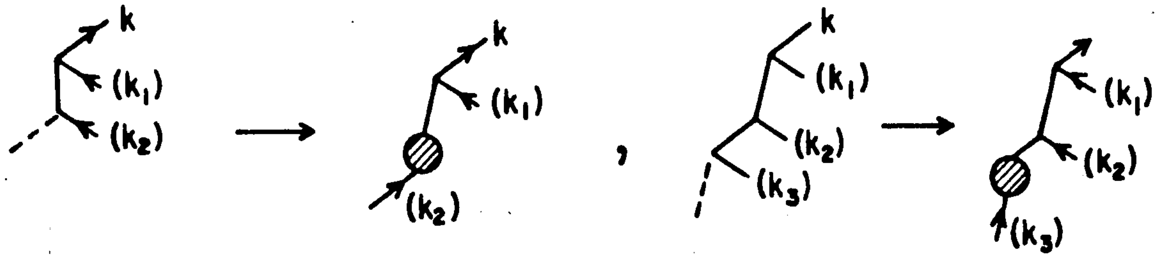
$$\varphi_k \rightarrow f_k , \qquad \phi_k \rightarrow f_o , \qquad V_{k_1, k_2} = \hat{L}(k_1) O_{k_1} . \tag{33}$$

We must keep in mind that O_k does not act on the propagators which are produced by iteration. As usual $G_k = (G_k^{(0)^{-1}} + i\Gamma_k)^{-1}$, and in this case $G_k^{(0)^{-1}} = \omega - \underset{\sim}{k} \cdot \underset{\sim}{v}$.

Diagramatically, the dotted line used in Sec. II is now changed to a shaded bubble to denote the zero momentum quantity f_o with the operator $\hat{L}(k)$ acting on it. For example, typical terms in Eq. (12a) change as



(34)



(35)

We remind the reader that the terms depicted in Eq. (34) have only one external line which must carry momentum k while the terms depicted in Eq. (35) belong to the class of terms which contain more than one external line, none of which could carry momentum k. It is indeed, in this sense, that the former (with only one external line) are called the coherent terms, and the letter, the intrinsically incoherent terms. Formally we break up f_k into a coherent (f_k^c) and an incoherent part ($\tilde{f}_k$), i.e.

$$f_k = f_k^c + \tilde{f}_k \equiv \hat{A}\, f_k + \tilde{f}_k \tag{36}$$

Taking all the coherent part to the left hand side, we obtain

$$(1-\hat{A}) f_k = \tilde{f}_k \tag{37}$$

Making use of Eqs. (12a), (34), (35) and (37), we can write down the diagrammatic structure of $\hat{A}$.

(38)

Operating both sides of Eq. (37) with O_k, and using Eq. (31), we obtain

$$O_k f_k - O_k \hat{A} f_k \equiv \varphi_k - O_k \hat{A}\, O_k^{-1} \varphi_k \equiv (1 - O_k A)\varphi_k = O_k \tilde{f}_k \qquad (39)$$

where $\hat{A} = AO_k$.

Symbolically, Eq. (39) can be case into a form

$$\mathscr{E}_k \varphi_k = \tilde{\varphi}_k \qquad (40)$$

where $\mathscr{E}_k \equiv 1-O_k A_k$ is the dielectric function used by Dupree[5]. $\tilde{\varphi}_k$ is thought to be the clump function[6]. However, this definition for the dielectric function given by Eq. (40) is not the same as that in statistical mechanics by[7],

$$\hat{\mathscr{E}}_k^{-1} \equiv \frac{\vec{k}^2}{4\pi} \left< \frac{\delta\varphi_k(\varphi^{(e)})}{\delta\varphi_k^{(e)}} \right>_{\varphi^{(e)} = 0}$$

$$= 1 + O_k \left< \frac{\delta f_k(\varphi^{(e)})}{\delta\varphi_k^{(e)}} \right>_{\varphi^{(e)} = 0} \qquad (41)$$

where $\varphi^{(e)}$ is unrandom external field associated with the bare source $\rho^{(e)}$ <u>within</u> the plasma. The plasma, thus, shields the bare source. Because the source is unrandom, the shielding, or the induced field, is coupled to the background plasma. This coupling gives a contribution to $\hat{\mathscr{E}}_k$ [defined in

Eq. (41)] which results from the correlation between the incoherent waves. In Appendix A we shall show a derivation from Eq. (41), in which $\hat{\mathscr{E}}_k$ goes over to $\mathscr{E}_k$ [Eq. (40)] in the limit of weak turbulence, when this coupling is turned off. In the general case, this coupling does exist, i.e., the dielectric function defined by Eq. (40) is not the dielectric function defined in statistical mechanics. The point of defining this dielectric [in Dupree's sense] is that it is a measure of the average dielectric response of a plasma to internally excited fluctuation which could be thermally excited or be due to an instability. For example, the simulation has shown a granulation structure for $<\tilde{f}_k \tilde{f}^*_k>$[6]. In the approximation of clump model the separation to the coherent and incoherent part still makes sense, of course, not in a sense of $\hat{\mathscr{E}}_k$.

In the limit of weak turbulence we write down $\mathscr{E}_k$ for the Vlasov-Poisson system.

$$\varepsilon_k = \varepsilon_k^{(\ell)} + 2\sum_{k_1} \varepsilon^{(3)}_{k_1,-k,k} \, I_{k_1} \tag{42}$$

where $\varepsilon_k^{(\ell)} \equiv 1 + \frac{4\pi}{\vec{k}^2} \int d\vec{v} \, G_k^{(0)} \, \vec{k}\cdot\vec{\partial} f_o$, is the usual linear dielectric function, and

$$\varepsilon^{(3)}_{k_1,-k_1,k} \equiv -\frac{2\pi}{\vec{k}^2} \int d\vec{v} \, G_k^{(0)} \, \vec{k}_1\cdot\vec{\partial} \, G^{(0)}_{k-k_1} \, [\vec{k}\cdot\vec{\partial} G_k^{(0)} \vec{k}_1 + \vec{k}_1\cdot\vec{\partial} G_k^{(0)} \vec{k}] \cdot \vec{\partial} f_o \tag{43}$$

with $I_k \equiv <\phi_k \phi_k^*>$ as the spectrum of the perturbation.

Generally, wave-particle systems cannot be treated by the simple formalism described here if the propagating characteristic of the wave is taken into account. For example, the system

$$i \frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} - g^2 \Phi \varphi = 0$$

$$\Box \Phi = - \frac{\partial^2}{\partial x^2} (\varphi^* \varphi)$$

which can be used as a model to describe Langmuir turbulence cannot be treated by our theory because for this system the renormalization of "particle" propagator is not sufficient. Any attempt to shrink the wave propagator would cause a Fermi type interaction which is unrenormalizable in the systematology given in this paper.

Sec. V Summary and Conclusions

We have developed a systematic renormalized theory to deal with stationary homogenous turbulence described by dispersive, dissipative systems with a quadratic nonlinearity. The nonlinearity is broken into two parts; a coherent part which modifies the "dielectric function", and an incoherent part which serves as a nonlinear source. Several general and structural properties of the renormalized system are disussed including the constraints imposed by causality which leads to Kramers-Kronig type of dispersion relation for the nonlinear dielectric function. Thus, nonlinear dissipation (or frequency broadening) is related to nonlinear dispersion (or frequency shift) and either of these can be calculated when the other is known. We have also demonstrated that the nonlinear dielectric function ε_k defined in our formalism is generally not equivalent to the standard definition in statistical mechanics, but is a meaningful description of the average dielectric response to internal plasma fluctuation. We have obtained an expression for ε_k in the weak turbulence limit for the Vlasov–Poisson system. We have also shown that our results in the appropriate limit are in complete agreement with the results of the conventional weak turbulence theory. We believe that the set of Eqs. (11), (12), and Eqs. (27)–(29) can provide a very firm starting point to deal with a broad class of turbulent problems.

Acknowledgements

This work was supported by Department of Energy Contract #DE-FG05-80ET-53088, and #DE-AC05-79ET-53036.

Appendix A. The nonlinear dielectric function in the limit of weak turbulence.

In the first part of this appendix we shall show that the dielectric function defined by Eq. (40) is not the same as defined by Eq. (41). The difference comes out of the coupling of induced wave to the background wave. If this coupling is turned off, the contributions to $\hat{\mathscr{E}}_k$ only comes from the coherent part. When the coupling is resumed, the calculation started from Eq. (41) gives the result in agreement with Ref. 2,8. Obviously, $\mathscr{E}^{(2)}$ term in the expression for $\hat{\mathscr{E}}_k$ in Ref. 2,8 stands for the correlation of induced wave to the background wave.

In the second part we derive the spectrum equations in weak turbulence starting from Eq. (40). The same scheme can be generalized to the second order in the renormalized theory. It is found that Eq. (A22) still holds for the renormalization with the same definitions for $\mathscr{E}_k$, $\mathscr{E}^{(2)}_{k_1,k_2}$, $\mathscr{E}^{(2)}_{k_1,k_2,k_3}$, $\mathscr{E}^{(e)}_k$, in which $G^{(0)}_k$ is replaced by the renormalized propagator G_k.

The Vlasov equation in the presence of an unrandom external source $\phi^{(e)}$ is

$$[\partial t + \vec{v}\cdot\nabla + \nabla\Phi(\phi^{(e)}) \cdot \vec{\partial}]f(\phi^{(e)}) = 0 \tag{A1}$$

(q=m=1 for simplicity). We define $f' \equiv f(\phi^{(e)})-f_o$ where $f_o \equiv \langle f(\phi^{(e)} = 0)\rangle$, as usual, satisfies $(\partial t + \vec{v}\cdot\nabla)f_o = 0$. The equation for f' in the Fourier representation (with $k\neq0$) is

$$f'_k = -G_k^{(0)}\vec{k}\cdot\vec{\partial}\ f_o\ \phi'_k - \sum_{k_1} G_k^{(0)}\vec{k}_1\cdot\vec{\partial}\ f'_{k-k_1}\phi'_{k_1} \tag{A2}$$

with $G_k^{(0)} = (\omega-\vec{k}\cdot\vec{v}+i\varepsilon)^{-1}$, $k\neq 0$, $\phi'_k \equiv \phi_k(\varphi^{(e)})$. The Poisson equation in the presence of an unrandom source is[7]

$$\phi'_k = \phi_k^{(e)} - \frac{4\pi}{\vec{k}^2}\int d\vec{v} f'_k \quad . \tag{A3}$$

For an infinitesimal $\phi_k^{(e)}$, f'_k is expanded to get

$$\phi'_k = \phi_k^{(e)} - \frac{4\pi}{\vec{k}^2}\int d\vec{v}\ \left[f'_k(0) + \sum_k \left(\frac{\delta f'_k}{\delta\phi_{k'}^{(e)}}\right)_{\phi^{(e)}=0} \varphi_{k'}^{(e)}\right]$$

$$\equiv \phi_k + \tilde{\phi}_k \quad . \tag{A4}$$

where

$$f'_k(0) \equiv f'_k(\phi^{(e)}=0), \qquad \phi_k \equiv \phi'_k(\phi^{(e)}=0)$$

and

$$\tilde{\phi}_k \equiv \sum_{k'} \left[\delta_{kk'} - \frac{4\pi}{\vec{k}^2}\int d\vec{v}\ \left(\frac{\delta f'_k}{\delta\phi_{k'}^{(e)}}\right)_{\phi^{(e)}=0}\right] \phi_{k'}^{(e)} \tag{A5}$$

Substituting Eq. (A4) into Eq. (A2) yields

$$f'_k = -G_k^{(0)}\vec{k}\cdot\vec{\partial}\ f_o(\phi_k+\tilde{\phi}_k) - \sum_{k_1\neq 0,k} G_k^{(0)}\vec{k}_1\cdot\vec{\partial}\ f'_{k-k_1} \cdot (\phi_{k_1}+\tilde{\phi}_{k_1}) \tag{A6}$$

Iterating Eq. (A6) to the second order gives

$$f_k' = -G_k^{(0)}\vec{k}\cdot\vec{\partial} f_o \phi_k - G_k^{(0)}\vec{k}\cdot\vec{\partial} f_o \tilde{\Phi}_k$$

$$+ \sum_{k_1\neq 0,k} G_k^{(0)}\vec{k}_1\cdot\vec{\partial}\, G_{k-k_1}^{(0)}(\vec{k}-\vec{k}_1)\cdot\vec{\partial} f_o(\phi_{k-k_1}+\tilde{\Phi}_{k-k_1})(\phi_{k_1}+\tilde{\Phi}_{k_1})$$

$$- \sum_{\substack{k_1\neq 0,k \\ k_2\neq 0}} G_k^{(0)}\vec{k}_1\cdot\vec{\partial} G_{k-k_1}^{(0)}\vec{k}_2\cdot\vec{\partial} G_{k-k_1-k_2}^{(0)}(\vec{k}-\vec{k}_1-\vec{k}_2)\cdot\vec{\partial}\, f_o(\phi_{k_1}+\tilde{\Phi}_{k_1})$$

$$\cdot(\phi_{k_2}+\tilde{\Phi}_{k_2})(\phi_{k-k_1-k_2} + \tilde{\Phi}_{k-k_1-k_2})$$

+ higher orders which are neglected in the standard weak turbulence theory).

(A7)

Ensemble averaging both sides of Eq. (A7), we obtain

$$\langle f_k'\rangle = -G_k^{(0)}\vec{k}\cdot\vec{\partial}\, f_o \langle\tilde{\Phi}_k\rangle + \sum_{k_1\neq 0,k} G_k^{(0)}\vec{k}_1\cdot\vec{\partial} G_{k-k_1}^{(0)}(\vec{k}-\vec{k}_1)$$

$$\cdot\, \vec{\partial} f_o\langle\phi_{k-k_1} + \phi_{k_1}\rangle - \sum_{\substack{k_1\neq 0,k \\ k_2\neq 0}} G_k^{(0)}\, \vec{k}_1\cdot\vec{\partial}\, G_{k-k_1}^{(0)}\, \vec{k}_2\cdot\vec{\partial} G_{k-k_1-k_2}^{(0)}(\vec{k}-\vec{k}_1-\vec{k}_2)$$

$$\cdot\, \vec{\partial} f_o\, \Big(\langle\phi_{k_1}\phi_{k_2}\phi_{k-k_1-k_2}\rangle + \langle\tilde{\Phi}_{k_1}\rangle\, \langle\phi_{k_2}\phi_{k-k_1-k_2}\rangle + \langle\tilde{\Phi}_{k_2}\rangle\, \langle\phi_{k_1}\phi_{k-k_1-k_2}\rangle$$

$$+ \langle\tilde{\Phi}_{k-k_1-k_2}\rangle\, \langle\phi_{k_1}\phi_{k_2}\rangle\Big) + \text{nonlinear terms in } \tilde{\Phi}. \qquad \text{(A8)}$$

In obtaining Eq. (A8) we used the decoupling condition

(i) $\langle\phi_k\tilde{\Phi}_{k'}\rangle = \langle\phi_k\rangle\, \langle\tilde{\Phi}_k\rangle$

$\langle\tilde{\Phi}_k\phi_{k_2}\phi_{k_3}\rangle = \langle\tilde{\Phi}_k\rangle\, \langle\phi_{k_2}\phi_{k_3}\rangle$

(ii) $\langle\phi_k\rangle = 0.$

Because of the homogeneity in space and time $\langle\phi_{k_1}\phi_k\rangle = I_{k_1}\delta(k_1+k_2)$, thus, Eq. (A8) is reduced to

$$\langle f_k'\rangle = -G_k^{(0)}\vec{k}\cdot\vec{\partial}f_o\,\langle\tilde{\phi}_k\rangle - \sum_{\substack{k_1\neq 0,k\\ k_2\neq 0}} G_k^{(0)}\,\vec{k}_1\cdot\vec{\partial}\,G_{k-k_1}^{(0)}\vec{k}_2\cdot\vec{\partial}$$

$$G_{k-k_1-k_2}^{(0)}(\vec{k}-\vec{k}_1-\vec{k}_2)\cdot\vec{\partial}f_o\,\cdot\,\langle\phi_{k_1}\phi_{k_2}\phi_{k-k_1-k_2}\rangle$$

$$-\sum_{k_1\neq 0,k} G_k^{(0)}\,\vec{k}_1\cdot\vec{\partial}\,G_{k-k_1}\,\vec{k}\cdot\vec{\partial}\,G_{-k_1}(-\vec{k}_1)\cdot\vec{\partial}f_o\,I_{k_1}\langle\tilde{\phi}_k\rangle$$

$$-\sum_{k_1\neq 0,k} G_k^{(0)}\,\vec{k}_1\cdot\vec{\partial}\,G_{k-k_1}(-\vec{k}_1)\cdot\vec{\partial}\,G_k\,\vec{k}\cdot\vec{\partial}f_o\,I_{k_1}\langle\tilde{\phi}_k\rangle$$

$$+\text{ nonlinear terms in }\tilde{\phi} \qquad \text{(A9)}$$

In obtaining Eq. (A9), we noticed that the choice $k_1=k$ has been excluded.

Taking the functional derivative of f_k' with respect to $\varphi_k^{(e)}$, we have

$$\left(\frac{\delta\langle f_k'\rangle}{\delta\phi_k^{(e)}}\right)_{\phi^{(e)}=0} = -G_k^{(0)}\,\vec{k}\cdot\vec{\partial}f_o - \sum_{k_1} G_k^{(0)}\vec{k}_1\cdot\vec{\partial}\,G_{k-k_1}^{(0)}$$

$$\cdot\,[\vec{k}\cdot\vec{\partial}G_{-k}(-\vec{k}_1) + (-\vec{k}_1)\cdot\vec{\partial}G_k\vec{k}]\,\cdot\,\vec{\partial}\,f_o\,I_{k_1}\frac{\delta\langle\tilde{\varphi}_k\rangle}{\delta\phi_k^{(e)}}\Big|_{\phi^{(e)}=0} \qquad \text{(A10)}$$

The nonlinear terms in $\tilde{\phi}$ vanish when $\phi^{(e)}=0$. We rewrite Eq. (A10) in the form

$$\frac{\delta\langle f_k'\rangle}{\delta\phi_k^{(e)}}\Big|_{\phi(e)=0} = -Q_k\,\frac{\delta\langle\tilde{\phi}_k\rangle}{\delta\phi_k^{(e)}}\Big|_{\phi^{(e)}=0} \qquad \text{(A11)}$$

where

$$Q_k \equiv G_k^{(0)} \{\vec{k}\cdot\vec{\partial} f_o + \sum_{k_1} \vec{k}_1\cdot\vec{\partial}\, G_{k-k_1}^{(0)} [\vec{k}\cdot\vec{\partial} G_{-k}^{(0)}(-\vec{k}_1) + (-\vec{k}_1)\cdot\vec{\partial} G_k^{(0)}\vec{k}] \cdot \vec{\partial} f_o\, I_{k_1}\} \tag{A12}$$

Substituting Eq. (A5) into Eq. (A11) yields

$$-\frac{\delta\langle f_k'\rangle}{\delta\phi_k^{(e)}} = Q_k \left[1 - \frac{4\pi}{\vec{k}^2} \int d\vec{v} \left\langle \frac{\delta f_k'}{\delta\phi_k^{(e)}} \right\rangle_{\varphi^{(e)}=0}\right] \tag{A13}$$

Integrating Eq. (A13) over velocity space, and making use of Eq. (38), we obtain

$$\frac{\vec{k}^2}{4\pi} (\varepsilon_k^{-1} - 1) = \int d\vec{v}\; Q_k \varepsilon_k^{-1} \;. \tag{A14}$$

Here we noticed that

$$\frac{\delta}{\delta\phi_k^{(e)}} \langle f_k'\rangle = \frac{\delta}{\delta\phi_k^{(e)}} \langle f_k(\varphi^{(e)}) \rangle \;.$$

Equation (14) leads to

$$\varepsilon_k = 1 + \frac{4\pi}{\vec{k}^2} \int d\vec{v}\; Q_k \;, \tag{A15}$$

where Q_k is defined by Eq. (A12). This derivation justifies Eq. (42). However, when the coupling is resumed, i.e., $\langle\varphi_k\tilde{\varphi}_{k'}\rangle$ can not be decorrelated, a further iteration gives the same result as obtained in Ref. 2,8, for $\hat{\mathcal{E}}_k$.

In the following, we show that the expression of ε_k given by Eq. (39) is in agreement with the wave energy equation of weak turblence.

From Sec. IV, the wave equation is

$$\varepsilon_k \phi_k = (1-\phi_k A)\phi_k = O_k \tilde{f}_k \tag{A16}$$

Writing the r.h.s. of Eq. (A16) explicitly in the limit of weak turbulence, we have

$$\varepsilon_k \phi_k = \frac{4\pi}{\vec{k}^2} \int d\vec{v}\, G_k^{(0)} \sum_{k_1} \vec{k}_1 \cdot \vec{\partial}\, G^{(0)}_{\vec{k}-\vec{k}_1}) \cdot \vec{\partial} f_o \phi_{k_1} \phi_{k-k_1} \tag{A17}$$

Multiplying ϕ_k^* on both sides, and ensemble averaging yields

$$\varepsilon_k\; I_k = \frac{4\pi}{\vec{k}^2} \int d\vec{v}\, G_k^{(0)} \sum_{k_1} \vec{k}_1 \cdot \vec{\partial}\, G^{(0)}_{k-k_1} (\vec{k}-\vec{k}_1) \cdot \vec{\partial} f_o \; \langle \phi_{k_1} \phi_{k-k_1} \phi_k^* \rangle \tag{A18}$$

The term $\langle \phi_{k_1} \phi_{k-k_1} \phi_k^* \rangle$ is calculated in the standard weak turbulence by using the quasi-Gaussian approximation:

$$\langle \phi_{k_1} \phi_{k-k_1} \phi_k^* \rangle = \langle \phi_{k_1}^{(2)} \phi_{k-k_1} \phi_k^* \rangle + \langle \phi_{k_1} \phi_{k-k_1}^{(2)} \phi_k^* \rangle + \langle \phi_{k_1} \phi_{k-k_1} \phi_k^{*(2)} \rangle$$

with

$$\phi_k^{(2)} = - \sum_{k_1+k_2=k} \frac{\varepsilon^{(2)}_{k_1,k_2}}{\varepsilon_k^{(\ell)}} \phi_{k_1}\phi_{k_2} \quad . \tag{A19}$$

Defining

$$\varepsilon^{(2)}_{k_1,k_2} \equiv - \frac{4\pi}{\vec{k}^2} \int d\vec{v}\; G_k^{(0)}\; \vec{k}_1\cdot\vec{\partial}\; G_{k_2}^{(0)}\; \vec{k}_2\cdot\vec{\partial} f_o \tag{A20}$$

and substituting Eqs. (A19), (A20) into Eq. (A18), we obtain

$$\varepsilon_k I_k = \sum_{\substack{k_1+k_2=k \\ k_1'+k_2'=k}} \varepsilon^{(2)}_{k_1,k_2} \frac{\varepsilon^{(2)*}_{k_1'k_2'}}{\varepsilon_k^{*(\ell)}} \langle \phi_{k_1}\phi_{k_2}\phi^*_{k_1'}\phi^*_{k_2'} \rangle$$

$$+ \sum_{\substack{k_1+k_2=k \\ k_1'+k_2'=k_1}} \varepsilon^{(2)}_{k_1,k_2} \frac{\varepsilon^{(2)}_{k_1'k_2'}}{\varepsilon^{(\ell)}_{k_1}} \langle \phi_{k_1'}\phi_{k_2'}\phi_{k_2}\phi^*_k \rangle$$

$$+ \sum_{\substack{k_1+k_2=k \\ k_1'+k_2'=k}} \varepsilon^{(2)}_{k_1,k_2} \frac{\varepsilon^{(2)}_{k_1',k_2'}}{\varepsilon^{(\ell)}_{k_2}} \langle \phi_{k_1}\phi_{k_1'}\phi_{k_2'}\phi^*_k \rangle \tag{A21}$$

The correlation function $\langle \phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4} \rangle$ is determined by the Gaussian process, and then Eq. (A21) becomes

$$\left(\varepsilon_k - \sum_{k_1} \frac{4\varepsilon^{(2)}_{k_1,k-k_1}\varepsilon^{(2)}_{k,-k_1}}{\varepsilon^{(\ell)}_{k-k_1}} I_{k_1} \right) I_k = \sum_{k_1+k_2=k} \frac{2|\varepsilon^{(2)}_{k_1,k_2}|^2}{\varepsilon_k^{*(\ell)}} I_{k_1} I_{k_2} \tag{A22}$$

Substituting the expression for ε_k of Eq. (39) into Eq. (A22), we just obtain the wave energy equation in the standard weak turbulence theory. The above exposition clearly shows that the $\varepsilon^{(2)}$-term in the coefficient of I_k of Eq. (A22), i.e.

$$4 \sum_{k_1} \frac{\varepsilon^{(2)}_{k_1,k-k_1} \varepsilon^{(2)}_{k,-k_1}}{\varepsilon^{(\ell)}_{k-k_1}} I_{k_1}$$

is irrelevant to the dielectric function ε_k. It is contributed from the interaction of source term with the wave.

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