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DIFFUSION OF PARTICLES IN A SLOWLY MODULATED WAVE

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ABSTRACT

Particle motion in a slowly modulated wave as studied. Recent results for the change in the adiabatic invariant due to separatrix crossing are used to calculate the diffusion coefficient for the adiabatic invariant of a particle moving in an amplitude modulated wave. It is argued that the calculated scaling also holds for the case of a narrow spectrum wave field. The present scaling law due to adiabatic invariant jumps, $D_J \sim k_0 \Delta v_\phi^3$, differs substantially from the resonance broadening theory result $D_{RB} \sim W_0^{3/4}/k_0^{1/2}$, where Δv_ϕ is the spectral width of the phase velocity, and W_0 is the energy density of the wave field.

I. Introduction

A particle under the influence of a Hamiltonian with slowly varying parameters has an adiabatic invariant J to all orders[1] in the slowness parameter ε. In practice this means that the phase function given by the first few terms in the adiabatic invariant series is well conserved except when the particle is near a separatrix. When the particle passes through a separatrix, the adiabatic invariant changes by a discrete amount which has been calculated through order ε for the special Hamiltonian of Sec. II in Ref. 2 and for the general case in Ref. 3. In the adiabatic limit successive crossings are uncorrelated because of the many oscillations that occur between crossings and the sensitivity of the adiabatic invariant change to the crossing parameter. Therefore, the result of many crossings is a diffusion of the adiabatic invariant.

The present work is concerned with the crossing diffusion for a particle moving in a wave, i.e., $H = \frac{1}{2} p^2 + A(t)\cos[k_0 q - \varphi(t)]$, where A and φ are slowly varying in the sense that $\dot{A}/A,\ \dot{\varphi} << k_0 A^{1/2}$. That is, the Hamiltonian changes little in one bounce period. The diffusion is calculated for the specific case of $\dot{\varphi}$ being constant. (Best[4] previously calculated the order-ε corrections to the adiabatic invariant for this case.) The diffusion constant scales like $\Omega^3 A_1^2/A_0^2$, where Ω is the modulation frequency and A_1/A_0 is the relative modulation. The scaling with Ω agrees with that deduced by Menyuk.[5]

This theory is also useful for understanding "strong-turbulence", by which is meant the motion of a particle in a stochastic wave field with a narrow phase-velocity spectrum. For the case of a field with mean energy density $W_0 \equiv \frac{1}{2} \langle A^2(t)/k_0^2 \rangle$, central wave-vector k_0, and spectral phase-velocity width Δv_ϕ, it is well known that quasilinear diffusion[6,7] is valid provided the spectral width as sufficiently broad, $\Delta v_\phi \gg k_0^{1/2} W_0^{1/4}$. The quasilinear diffusion coefficient scales like $D_{QL} \sim W_0/k_0 \Delta v_\phi$. To handle the opposite limit, Dupree[8] developed a resonance broadening theory that predicts a coefficient that scales like $D_{RB} \sim W_0^{3/4}/k_0^{1/2}$. However, subsequent analytical[9] and numerical[10-13] investigation have disagreed with this prediction.

The present theory has relevance to at least the particular case of fixed wave vector and finite frequency spectral width. In this case the present theory indicates a scaling of $D_J \sim k_0 \Delta v_\phi^3$. This result is quite different from the previous D_{RB} which scales as $W_0^{3/4}$ but is independent of Δv_ϕ. Another important difference between the two theories is that the present theory predicts diffusion in the adiabatic invariant while the previous theory deals with diffusion in the velocity variable.

The structure of this paper follows. In Sec. II the adiabatic invariant diffusion constant is calculated for a particle in an amplitude modulated wave. In Sec. III the relation of this problem to the problem of "strong turbulence" as discussed and the scaling of the general diffusion constant is given.

II. Diffusion in an Amplitude Modulated Wave

A. Definition of the problem

The problem of interest is the motion of a particle with Hamiltonian,

$$H(q,p,t) = \frac{1}{2} p^2 + A(t)\cos(q) \ , \tag{1}$$

where A(t) is slowly varying in the sense that

$$\frac{1}{\omega_0} \frac{d\ell nA}{dt} << 1 \ . \tag{2}$$

In this expression $\omega_0 = A^{1/2}$ is the oscillation frequency of a deeply trapped particle. Note that we have chosen units such that $k_0 = 1$.

In this situation the adiabatic invariant J is well conserved for particles not near the separatrix. The adiabatic invariant is given by a series,[1]

$$J = J_0 + J_1 + \cdots \ , \tag{3}$$

with terms ordered in the slowness parameter of (2). Following previous convention,[2,4] the lowest order term J_0 is taken to be the action for untrapped particles and half the action for trapped particles. This way J_0 is a continuous function. The action,

$$I(H,t) = \oint p(q,H,t)dq \ , \tag{4}$$

is the phase-space area enclosed by an orbit of the frozen Hamiltonian. For later reference we define the separatrix action,

$$I_{sx} \equiv \int_0^{2\pi} dq\; p(q,H{=}A) = 8A^{1/2} \;, \tag{5}$$

and a particular slowness parameter,

$$\varepsilon \equiv \frac{1}{\omega_0}\frac{d\ell nI_{sx}}{dt} = \frac{\dot{A}}{2A^{3/2}} \;. \tag{6}$$

B. Adiabatic invariant jumps

While the adiabatic invariant is well conserved when a particle is not near the separatrix, it is poorly conserved when the particle is at or near the separatrix. Recent calculations for the present specific case[1] and a much more general case[3] show that J changes by an amount of order ε when the particle crosses the separatrix. The results of these authors follows.

Particles lying on the line q=0 (see Fig. 1) get trapped before they complete their next oscillation, i.e., come close to $(p=0,\ x=2\pi n)$, if their energy lies in the range,

$$A < H < A+\Delta h \;,$$

where $\Delta h \equiv -\dot{I}_{sx}$. (7)

The change in the adiabatic invariant due to separatrix crossing is given by

$$\Delta J = -\varepsilon J_{sx}\ell n(2\sin|\pi h/\Delta h|) \;, \tag{8}$$

where $h \equiv H-A$. (9)

C. Statistics

To determine the mean and mean-square changes in J one must know how particles are distributed in the variable h. Before crossing, the particle distribution is assumed to be slowly varying in the adiabatic invariant and uniform in the conjugate angle variable. Then, in the small region shown in Fig. 1, the density can be taken to be uniform. This allows us to find the number of particles crossing the line q=0 between p and p+dp by simply calculating the flux,

$$\Delta n = n_0 p dp \; \Delta t = \frac{1}{2} n_0 dh \; \Delta t \quad .$$

Since the probability density $p(h)dh$ is proportional to $\Delta n/\Delta t$, we find

$$\rho(h) = |\Delta h|^{-1} \quad , \tag{10}$$

where ρ is normalized according to

$$\int_0^{|\Delta h|} \rho(h) dh = 1 \quad .$$

That is, the probability density is uniform in h.

With this result we can calculate the mean action jump,

$$\overline{\Delta J} \equiv \int_0^{|\Delta h|} dh \; \rho(h) \Delta J(h) \quad ,$$

and the mean-square action jump,

$$\overline{\Delta J^2} \equiv \int_0^{|\Delta h|} dh \; \rho(h) \; \Delta J^2(h) \quad .$$

We find

$$\overline{\Delta J} = 0 \tag{11a}$$

and
$$\overline{\Delta J^2} = (\pi\ \varepsilon\ I_{sx})^2/12\ . \tag{11b}$$

D. Diffusion Coefficient

These results can be the diffusion coefficient for a particle in an amplitude modulated wave,

$$A(t) = A_0 + A_1 \sin\Omega t\ .$$

In this situation the separatrix action oscillates in the range,

$$8|A_0 - A_1|^{1/2} < I_{sx} < 8|A_0 + A_1|^{1/2}\ . \tag{12}$$

It is assumed that $A_1 < A_0$. Thus, particles with adiabatic invariant in the range (12) continually cross and recross the separatrix.

Each time the particle crosses the separatrix, the adiabatic invariant changes by a small amount. The successive crossings are uncorrelated in the limit $\Omega \ll \omega_0$ for two reasons: (1) The jump depends sensitively on the crossing parameter h. (2) The particle motion undergoes many oscillations between crossings. Thus, the diffusion constant is given by adding the effect of uncorrelated changes,

$$D = \frac{1}{2}\,\nu\ \overline{\Delta J^2}\ , \tag{13}$$

where $\nu = \Omega/\pi$ is the crossing rate. (There are two crossings per modulation period $2\pi/\Omega$.)

Combining Eqs. (6), (11), and (13) and solving for $\dot{A}$ in terms of $A = (I_{sx}/8)^2$, we find the coefficient for adiabatic invariant diffusion,

$$D(J) = 2\pi\Omega^3\{A_1^2 - [(J/8)^2 - A_0]^2\}/(J/8)^4 \ . \tag{14}$$

We see that the maximum value of the diffusion coefficient is

$$D_{max} = 2\pi\Omega^3(A_1/A_0)^2 \ . \tag{15}$$

Furthermore, we note that D diverges at J=0 when there is full modulation, $A_1 \geqslant A_0$. For these particles the theory breaks down because the theory of a single jump loses validity since the smallness parameter ε of Eq. (6) is no longer small when the amplitude vanishes. Even so the diffusion constant is still valid for a large class of particles, those having

$$J >> 8(\Omega A/2)^{1/3} \ ,$$

where A is a typical value for the amplitude.

III. Application to Narrow Spectrum Stochastic Fields

The present theory is useful for understanding the motion of particles in narrow spectrum stochastic fields. In this case the relevant Hamiltonian is

$$H = \frac{1}{2} v^2 + V(x,t) \ ,$$

for which the force is defined by

$$F(x,t) = - \frac{\partial V}{\partial x} \ .$$

Previous work dealt with an electric force acting on a particle of mass m. To relate this discussion to those, one simply replaces F by eE/m.

A stochastic force is described by its correlation function, $\langle F(x,t)F(x',t')\rangle$, and various other statistical moments. For the present case of homogeneous, stationary turbulence the correlation function depends on only $x-x'$ and $t-t'$, and so the spectral density,

$$\mathcal{F}(k,\omega) \equiv \int \frac{d\tau}{2\pi} \frac{d\xi}{2\pi} e^{-ik\xi+i\omega\tau} \langle F(x,t)F(x+\xi,t=\tau)\rangle \ ,$$

is independent of x and t. For scaling purposes we assume that k_0 and ω_0 are typical values of the wave vector and frequency in the support of $\mathcal{F}$, while Δk and $\Delta\omega$ are the spectral widths. We further define

$$W_0 \equiv \frac{1}{2} \int dk d\omega \, \mathcal{F}(k,\omega) = \frac{1}{2} \langle F^2(x,t)\rangle \ .$$

In the case of an electric force, W_0 would be the mean energy density of the electric field.

The result of quasilinear theory[6,7] is that particles diffuse in momentum space with the quasilinear diffusion constant,

$$D_{QL}(v) = \pi \int dk \, \mathscr{F}(k,kv) \ , \tag{16}$$

provided

$$k_0^2 \, \Delta v_\phi^4 \gg W_0 \ , \tag{17}$$

where Δv_ϕ is the width of $D_{QL}(v)$. That is, the phase velocity spectrum must be sufficiently broad. In terms of k_0, Δv_ϕ, and W_0, the quasilinear diffusion coefficient scales like

$$D_{QL} \sim W_0/k_0 \Delta v_\phi \ . \tag{18}$$

Resonance broadening theory[8] was developed to work outside the validity range (17) of quasilinear theory. Resonance broadening theory states that particles diffuse in velocity space, but the diffusion coefficient is modified outside the validity range (17). In the narrow spectrum limit,

$$k_0^2 \, \Delta v_\phi^4 \ll W_0 \ , \tag{19}$$

it scales like

$$D_{RB} \sim W_0^{3/4} \, k_0^{-1/2} \ . \tag{20}$$

As a case study of diffusion we consider the potential

$$V(x,t) = A(t) \cos[k_0 x - \varphi(t)] \ ,$$

where A and φ vary stochastically. In terms of the spectral function

$$\mathscr{E}(\omega) \equiv k_0^2 \int \frac{d\tau}{2\pi} e^{i\omega\tau} <A(t)e^{i\varphi(t)}A(t+\tau)e^{-i\varphi(t+\tau)}> ,$$

the spectral density of the corresponding force is given by

$$\mathscr{F}(k,\omega) = \frac{1}{4} \mathscr{E}(\omega)\delta(k-k_0) + \frac{1}{4} \mathscr{E}(-\omega)\delta(k+k_0) .$$

(One can show that for stationary turbulence $\mathscr{E}(\omega)$ is real.) This model is special in that only single wave vector is present. Nevertheless, as this model contains a phase velocity spread, quasilinear theory and resonance broadening theory should apply.

Now in the narrow spectrum limit a theory analogous to that of Sec. II should apply. In the narrow spectrum limit, one can use a Galilean transformation to go to the frame in which $\mathscr{E}(\omega)$ is centered around ω=0. In this frame the characteristic rates of change are roughly given by the spectral width,

$$\frac{d\ln A}{dt} , \frac{d\varphi}{dt} , \left|\frac{d^2\varphi}{dt^2}\right|^{1/2} \sim \Delta\omega .$$

With the narrow spectrum limit (19) this implies

$$\frac{d\ln A}{dt} , \frac{d\varphi}{dt} , \left|\frac{d^2\varphi}{dt^2}\right|^{1/2} << k_0 <A^2(t)>^{1/4} \sim \bar{\omega}_0 ,$$

where $\bar{\omega}_0$ is the typical bounce frequency. Thus, the narrow spectrum limit is one in which the adiabatic invariant is well conserved except during separatrix crossings.

Unfortunately, the calculation in Sec. II of the diffusion coefficient cannot be applied directly to the present case since that calculation assumed $\ddot{\varphi}=0$. When $\ddot{\varphi}$ is nonzero new processes are present, and the statistical descriptions become much more complicated. An example of a new process is shown in Fig. 3, in which a particle is trapped when the phase velocity is at a positive fluctuation, $\dot{\varphi} = +\Delta\omega$, and detrapped when the phase velocity is at a negative fluctuation. Nevertheless, these additional processes yield contributions to the adiabatic invariant diffusion of order

$$D_J \sim k_0^2 \Delta\omega^3 \quad , \tag{21}$$

as was found in Sec. II. Thus, we infer the scaling law (21) for the adiabatic invariant diffusion in a narrow spectrum stochastic field.

This scaling law is completely different from that of resonance broadening theory. Resonance broadening theory predicts a diffusion constant depending on the spectral energy but not the spectral width (20), whereas the present has the opposite behavior.

Footnotes and References

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Figure Captions

Fig. 1 Phase flow near the x-point of the Hamiltonian. Particles with $H-H_{sx} < \Delta h$ are trapped before they complete their next oscillation.

Fig. 2 Separatrix action versus time. Particle with adiabatic invariant J between $\min(I_{sx}) = 8|A_0-A_1|^{1/2}$ and $\max(I_{sx}) = 8|A_0+A_1|^{1/2}$ experiences a change in the value of J at crossing times t_1 and t_2.

Fig. 3 New process present when $\ddot{\varphi}\neq 0$. In Fig. 3a the separatrix is centered at $v=\dot{\varphi} = k_0\Delta\omega$ and growing so that a particle becomes trapped as shown in Fig. 3b. Then the separatrix remains constant in size but moves down in phase space carrying the particle with it as shown in Fig. 3c. Finally, in Fig. 3d the separatrix shrinks in size, releasing the particle.

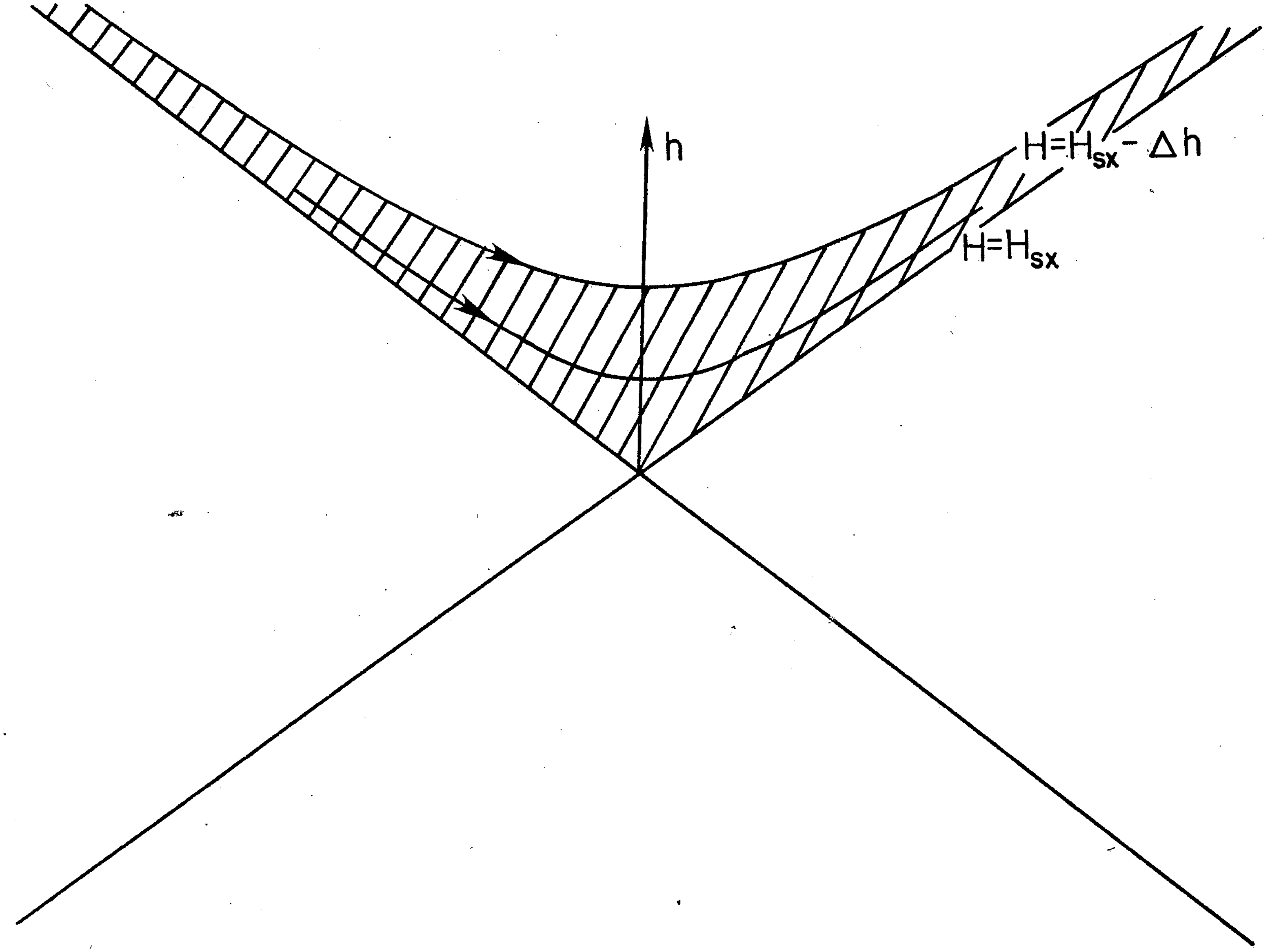


Fig. 1

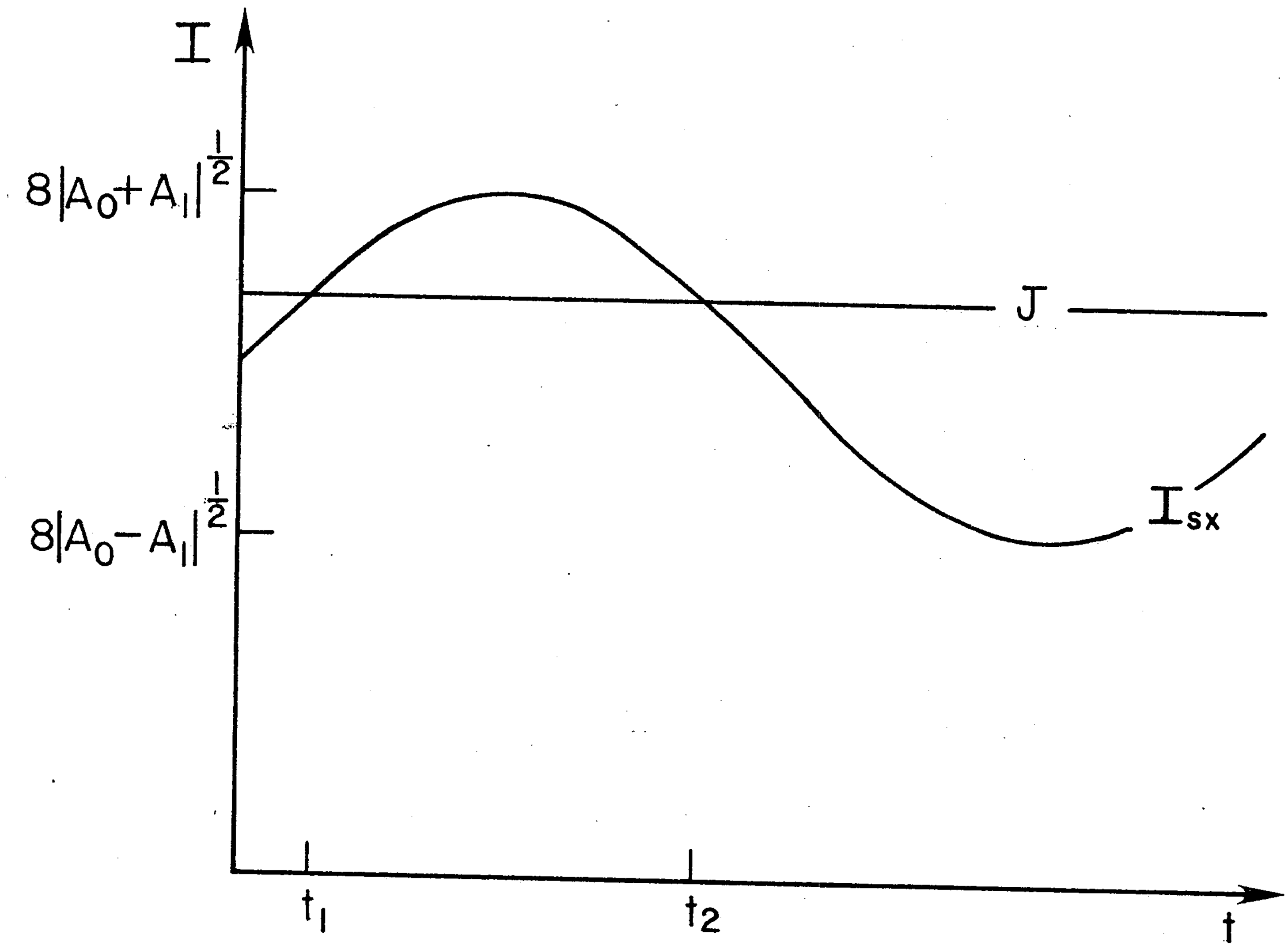


Fig. 2

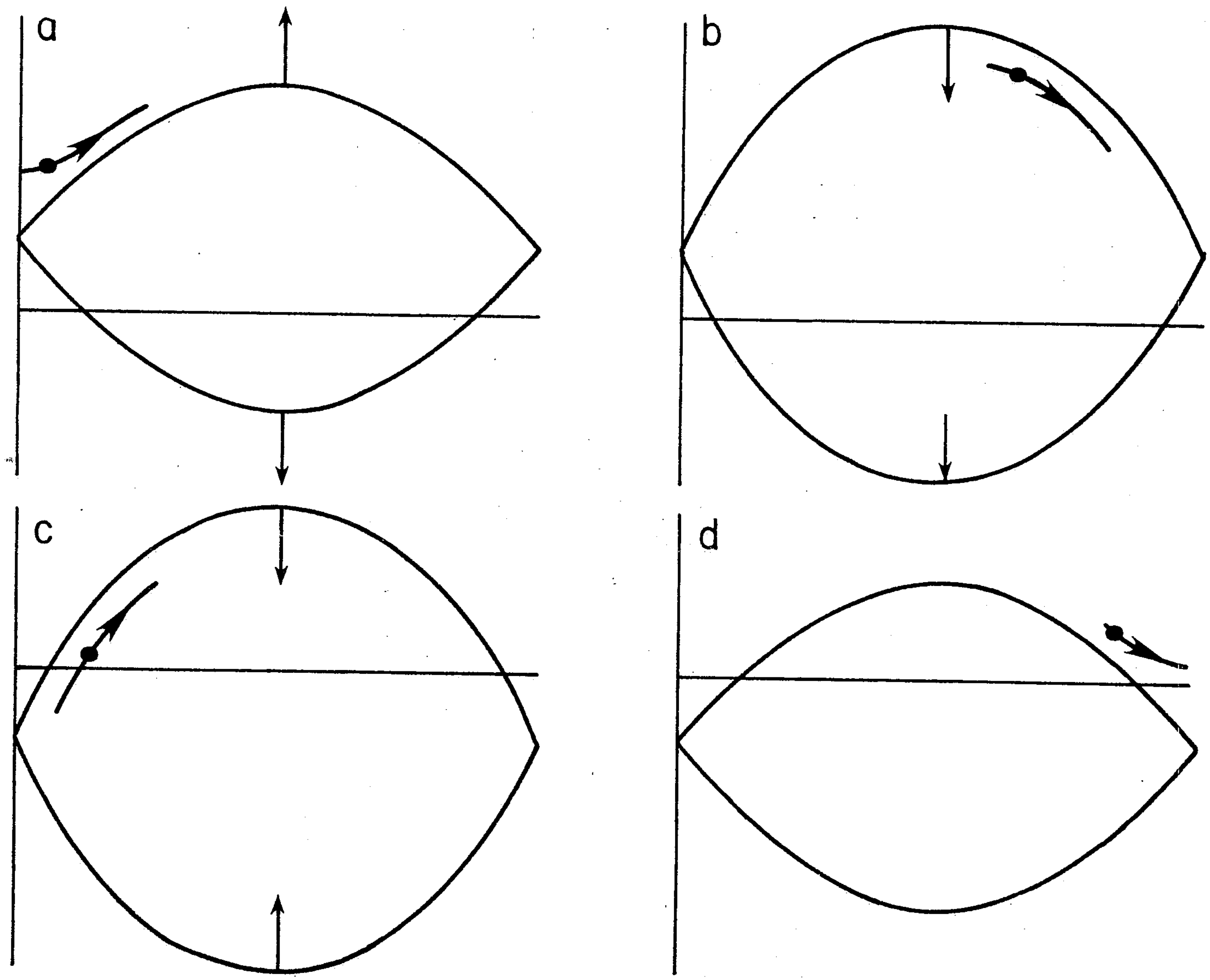
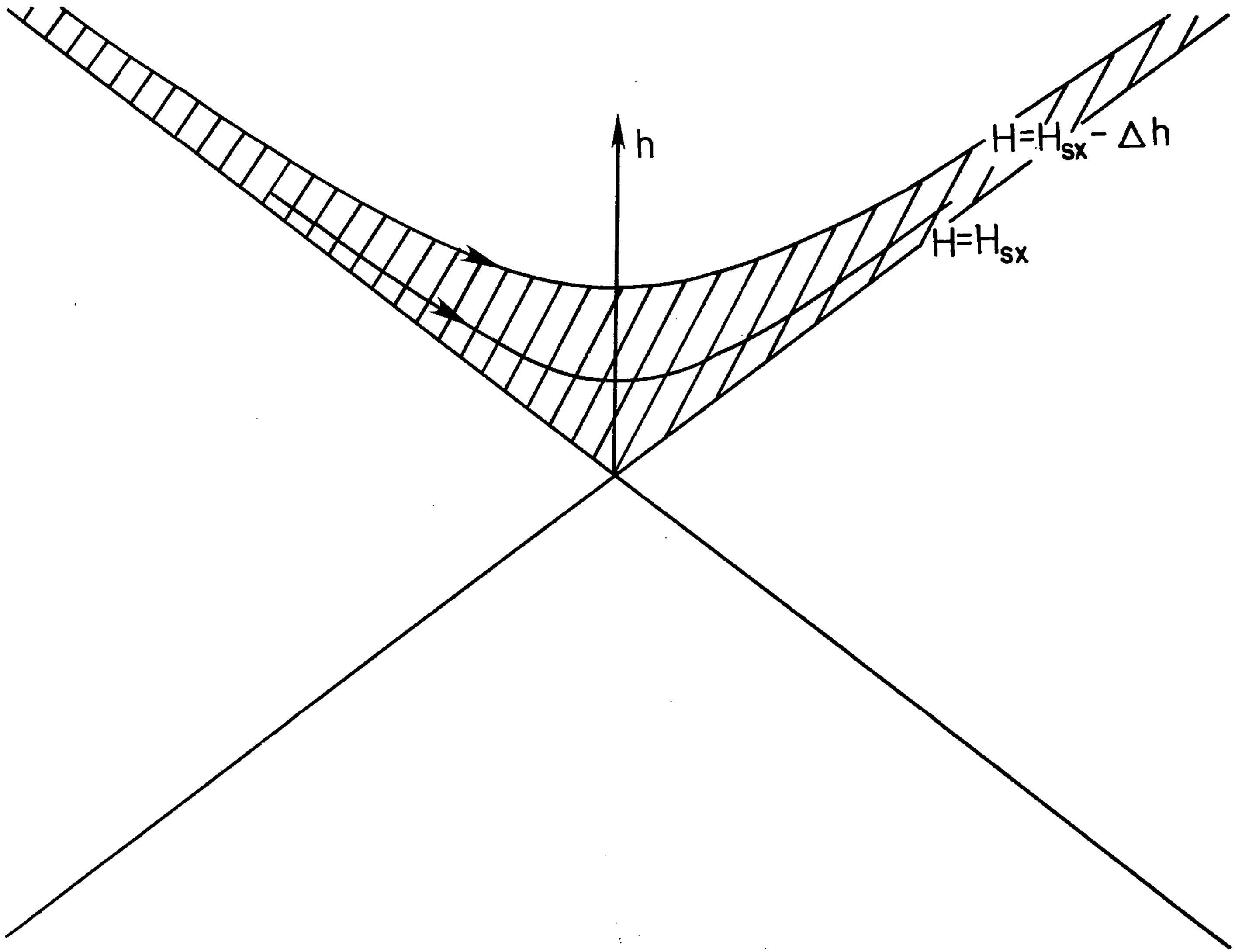
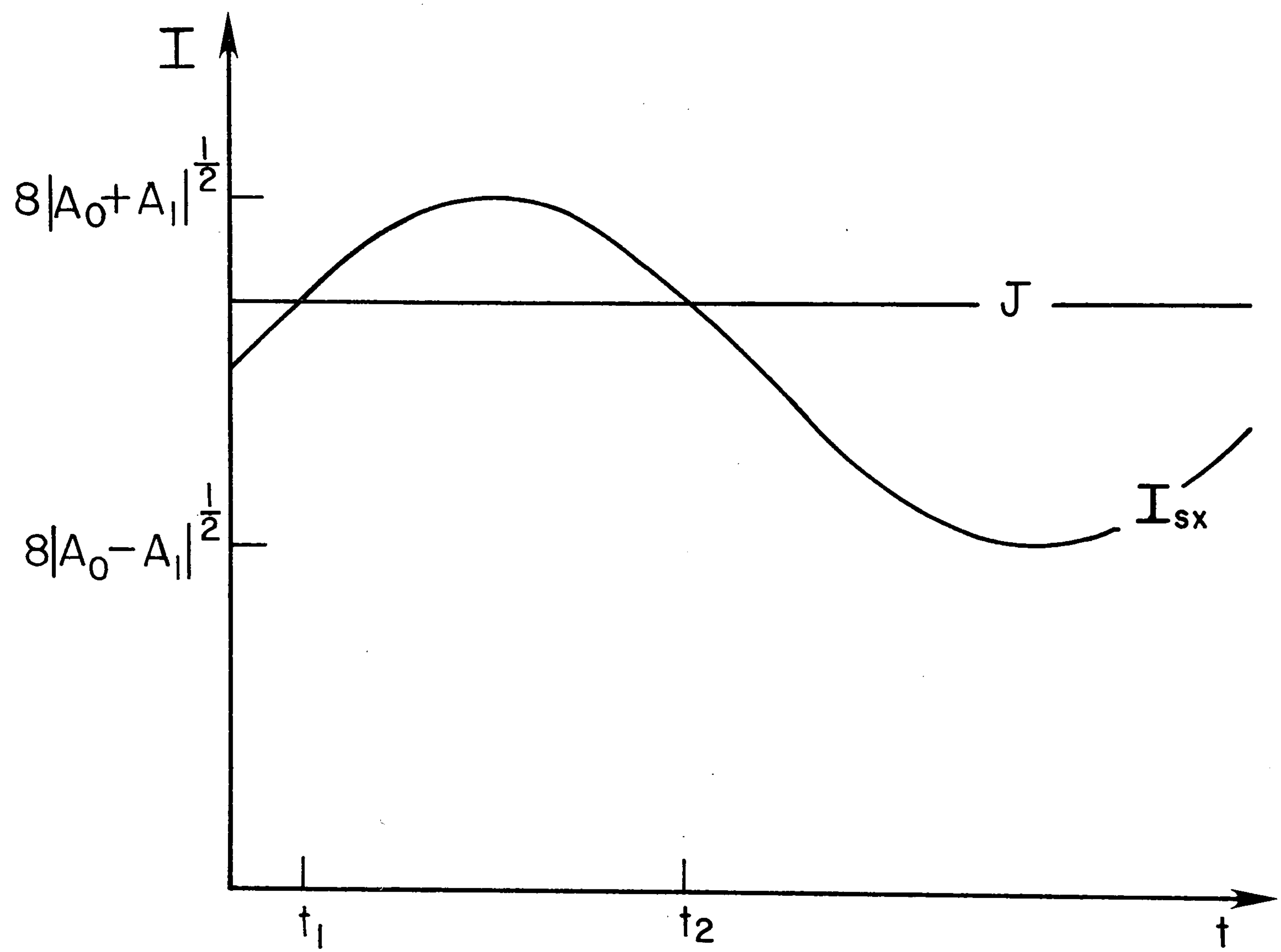


Fig. 3


h
$H=H_{sx}-\Delta h$
$H=H_{sx}$


I
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8|A0−A1|^1/2
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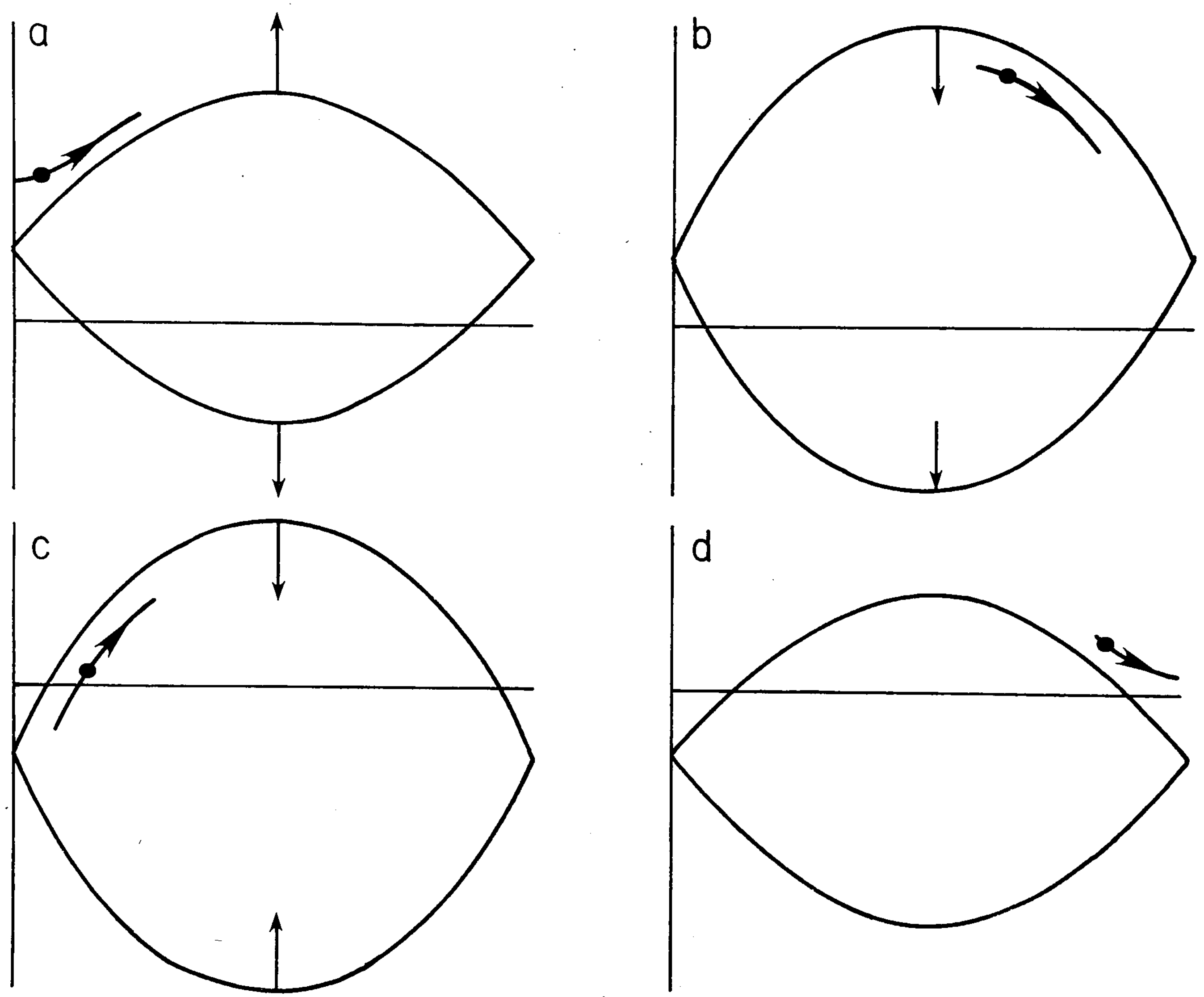
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