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DRIFT AND TEARING MODES IN A SHEARED CYLINDER

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Abstract

Drift and tearing modes in a sheared cylindrical collisionless plasma column are studied. A set of differential equations in the radial coordinate are derived with small gyroradius and low-β expansion. The finite-β effects include curvature drifts, gradient-B drifts, and the parallel magnetic field perturbation. Algebraic elimination reduces the resulting set of equations to a fourth-order system. Analysis shows that bad curvature does not drive the collisionless modes unstable.

I. Introduction

Collisionless drift and tearing modes are low frequency perturbations that occur in inhomogeneous systems. These modes have been extensively studied in slab geometry. However, real systems have curvature which may be stabilizing or destabilizing. To study the effect of curvature, we consider collisionless drift and tearing modes in a sheared cylinder, which is the simplest system having intrinsic curvature. Using small gyroradius and low-β expansions, we derive differential equations governing these modes, and we study their stability by analytic techniques. Numerical analysis of these equations will be carried out in a companion paper.[1]

The development of the theory of collisionless drift waves indicates a need to understand the effect of curvature on these modes. Collisionless electrostatic drift modes were studied by Krall and Rosenbluth in slab geometry.[2] A series of papers[3-7] eventually led to the conclusion that electromagnetic collisionless drift waves are stable. This conclusion has since been strengthened by more correctly handling the finite-ion-gyroradius effect.[8] While more recent work[9] indicates the presence of modes induced by toroidicity, the basic slab branch remains stable. However, since resistive MHD indicates[10] the presence of localized modes driven by bad curvature, it is important to ask whether bad curvature can drive the slab mode unstable.

Similarly, the development of tearing mode theory indicates a need to understand the effects of curvature on collisionless tearing modes. Tearing modes are driven by external free energy[11] represented by the parameter Δ'. An early theory of collisionless tearing modes was presented by Laval et al.[12] This theory was later modified to include arbitrary collisionality.[13,14] More recently it was noted[15,16] that the stabilizing effects of ion dynamics and electron temperature gradients yield the requirement that Δ' exceed a certain critical value, $\Delta' > \Delta_{cK}$, for kinetic tearing modes to be unstable. Of course, it has long been known[10,17] that in systems with good curvature, resistive MHD theory predicts a threshold Δ_{cMHD}. Thus, we are motivated to develop a theory capable of calculating the threshold Δ_c due to kinetic effects and curvature in combination.

The paper proceeds as follows. Section II contains a brief presentation of cylindrical guiding center theory. Such theory is needed to transform to action-angle variables, in which the motion is simple, and the orbit integrals of linear theory are easily carried out. The theory is simple and useful because it is Hamiltonian and canonical; it does not use noncanonical coordinates like the more complicated theory[18] needed for more general configurations.

In Sec. III, the charge and current induced by the linear perturbations is calculated. This calculation involves gyroaveraging, which is carried out via Taylor expansion. For this reason, these equations are strictly valid only for $T_e \gg T_i$. However, recent work[19] indicates that this technique gives reasonable results for tearing modes even for T_e/T_i near unity. The charge and current responses are used in Maxwell's equations to obtain a variational[20] sixth-order set of differential equations. This set reduces algebraically to a fourth-order system which looks much like previously derived equations, except for the presence of new terms due to the magnetic drifts and parallel magnetic field perturbations. Boundary conditions are imposed by requiring either that the perturbation vanish far from the resonant surface (high-$k_\perp$) or that the fields match asymptotically to the exterior solution.

Section IV contains a discussion of analytic stability proofs for this model. The method of Ref. 7 is extended to the present set of equations. It is shown that finite-β effects do not modify the stability results significantly: one can prove the absence of marginal modes with $\omega/\omega_{*e} > 0$. However, it is also found that when the full ion Z-function is retained, this stability proof fails because nothing can be said about the existence of marginal modes in the range $0 < \omega/\omega_{*i} < 1 + d\ell n\, T_i/d\ell n\, n_i$.

II. Unperturbed Systems

In linear kinetic theory the perturbation of the distribution function is obtained by integrating the perturbing field along an unperturbed orbit. The charge density and current are found by taking moments of the distribution. In this section the quantities needed for these calculations are obtained. A simple, cylindrical version of guiding center theory of the unperturbed motion is developed. The appropriate velocity operators, needed for taking moments, are written in terms of the convenient guiding center variables.

A. Cylindrical guiding center theory

The unperturbed motion in this cylindrically symmetric system is detemined by the unperturbed Hamiltonian,

$$h_0 = \frac{1}{2} p_r^2 + \frac{1}{2} (p_\theta - rA_\theta)^2/r^2 + \frac{1}{2} (p_z - A_z)^2 \ , \tag{1}$$

where A_θ and A_z are functions of r alone. As mentioned in the introduction, units satisfying e=m=c=1 have been chosen for this part of the analysis. The Hamiltonian (1) does not depend on θ nor z, so one need find only the radial motion, which is that of a particle in an effective potential,

$$W(r, p_\theta, p_z) = \frac{1}{2} (p_\theta - rA_\theta)^2/r^2 + \frac{1}{2} (p_z - A_z)^2 \ . \tag{2}$$

In this section we analyze the Hamiltonian (1) in the limit of small energy (gyroradius/scale length). We introduce cylindrical guiding center coordinates and find the unperturbed motion.

Low energy particles are trapped near and oscillate about the minimum of the effective potential. This leads to the definition of the "cylindrical guiding-center radius" R as the position of the minimum. That is, R is a function of the canonical invariants p_θ and p_z that satisfies

$$\frac{\partial W}{\partial r}\left(R(p_\theta,p_z),\ p_\theta,\ p_z\right) = 0 \ . \tag{3}$$

Continuing with the low energy expansion, we write

$$h_0 = \frac{1}{2}\,p_r^2 + W(R,p_\theta,p_z) + \frac{1}{2}\,(r-R)^2\,\frac{\partial^2 W}{\partial r^2}\,(R,p_\theta,p_z)$$

$$+ \frac{1}{2}\,(r-R)^3\,\left(B\,\frac{\partial B}{\partial r} - \frac{B_z^2}{r}\right)\Big|_R + \ldots \ . \tag{4}$$

As one can see from the divergence of the last coefficient of Eq. (4), this expansion technique breaks down near the axis. Thus, we assume the region of interest to be at least a few gyroradii from the center.

The lowest order part of the Hamiltonian (4) consists of three terms. The first and third terms yield the gyration and, hence, must comprise the perpendicular energy. The second turn must, therefore, be the parallel energy,

$$\frac{1}{2}\,u^2 = W(R,p_\theta,p_z) \ . \tag{5}$$

To relate u to the velocity, we define the quantities

$$\bar{v}_\theta \equiv \frac{\partial h}{\partial p_\theta}(R,p_\theta,p_z) = p_\theta/R - A_\theta(R) \ , \tag{6a}$$

and

$$\bar{v}_z \equiv \frac{\partial h}{\partial p_z}(R,p_\theta,p_z) = p_z - A_z(R) \ , \tag{6b}$$

which are the velocities of particles whose amplitude of radial oscillation vanishes. In terms of these variables, Eq. (3) becomes

$$\bar{v}_\theta\left[B_z(R) + \bar{v}_\theta/R\right] - \bar{v}_z B_\theta(R) = 0 \ . \tag{7}$$

This result allows us to deduce that u defined to be

$$u \equiv \frac{\bar{v}_\theta B_\theta(R) + \bar{v}_z\left[B_z(R) + \bar{v}_\theta/R\right]}{\left\{B_\theta^2(R) + \left[B_z(R) + \bar{v}_\theta/R\right]^2\right\}^{1/2}} \tag{8}$$

satisfies Eq. (5). We note that u is primarily $\underline{v}(R)\cdot\underline{B}(R)/B(R)$, with corrections due to the centrifugal terms $\bar{v}_\theta/R$, which are small because of the small gyroradius assumption. Furthermore, this approximation allows us to deduce

$$\frac{\partial^2 W}{\partial r^2}(R,p_\theta,p_z) = B^2(R) \equiv \Omega^2 \ . \tag{9}$$

For later purposes, we need the partial derivative of the guiding-center variables R and u with respect to the canonical invariants p_θ and p_z. We find

$$\frac{\partial R}{\partial p_z} = -B_\theta(R)/B^2(R) \ ,$$

$$\frac{\partial R}{\partial p_\theta} = B_z/RB^2(R) \ ,$$

$$\frac{\partial u}{\partial p_z} = \frac{B_z}{B} + \frac{uB_\theta^3}{RB^4} \ ,$$

and

$$\frac{\partial u}{\partial p_\theta} = \frac{B_\theta(R)}{RB(R)} - \frac{uB_\theta^2 B_z}{R^2B^4} \ . \qquad (10)$$

This allows us to compute the Jacobian,

$$dp_\theta dp_z = RB(R)dRdu \ , \qquad (11)$$

to lowest order.

One can put the unperturbed Hamiltonian completely into action-angle form with the generating function,

$$F = -\frac{1}{2}\,(p_r^2/\Omega)\ \cot\!an\psi + \Theta p_\theta + Zp_z + Rp_r \ , \qquad (12)$$

which yields the transformation to new variables $(\psi, J, \Theta, P_\theta, Z, P_z)$ via the usual rules,

$$r = \frac{\partial F}{\partial p_r}, \qquad J = \frac{\partial F}{\partial \psi},$$

etc. These rules yield

$$\begin{aligned}
r &= R + (2J/\Omega)^{1/2} \cos\psi \\
p_r &= -(2\Omega J)^{1/2} \sin\psi \\
\theta &= \Theta - (2\Omega J)^{1/2} (B_z/RB^2) \sin\psi + \text{higher order terms} \\
p_\theta &= P_\theta \\
z &= Z + (2\Omega J)^{1/2}(B_\theta/B^2) \sin\psi + \text{higher order terms} \\
p_z &= P_z \,.
\end{aligned} \tag{13}$$

In terms of the new variables, the unperturbed Hamiltoian is

$$h_0 = \Omega J + \frac{1}{2} u^2 + \text{higher order terms} \,. \tag{14}$$

As the Hamiltonian is now in action-angle form, the motion of the new variables is particularly simple. From Hamilton's equations we obtain

$$\dot{\psi} = \frac{\partial h_0}{\partial J} = \Omega \tag{15a}$$

$$\dot{\theta} = \frac{uB_\theta}{RB} - \frac{u^2 B_\theta^2 B_z}{R^2 B^4} + J \frac{\partial B}{\partial R} \frac{B_z}{RB^2} \tag{15b}$$

$$\dot{z} = \frac{uB_z}{B} + \frac{u^2 B_\theta^3}{RB^4} - J \frac{\partial B}{\partial R} \frac{B_\theta}{B^2} \; . \tag{15c}$$

In the last two equations we recognize the terms due to the parallel velocity, the curvature drift, and the ∇B drift.

Two useful auxilary variables are the perpendicular energy $w_\perp$ and the gyroradius ρ. We define them as follows:

$$w_\perp \equiv \Omega J \tag{16a}$$

$$\rho \equiv (2J/\Omega)^{1/2} = (2w_\perp)^{1/2}/\Omega \; . \tag{16b}$$

Finally, for later purposes we will need the functions $v_\parallel \equiv \hat{b}\cdot\partial h_0/\partial \underset{\sim}{p}$ and $v_\perp \equiv \hat{b}\times\hat{r}\cdot\partial h_0/\partial \underset{\sim}{p}$. To lowest nonvanishing order we obtain

$$v_\parallel = u \; , \tag{17a}$$

and

$$v_\perp = -\rho \cos\psi \; . \tag{17b}$$

B. Unperturbed distribution

The unperturbed distribution may be any function of the three invariants (h_0, p_θ, p_z) or, alternatively, (h_0, u, R). For present purposes we want a distribution that is locally Maxwellian. Thus, we choose

$$f_0 = n_0(R)[2\pi T(R)]^{-3/2} \exp[-h_0/T(R)] \ . \qquad (18)$$

By taking moments of this distribution one can easily verify that it yields a density, $n_0(r)$, no parallel current, and a local temperature $T(r)$ through first order in gyroradius to scale length.

For the purpose of taking moments one must note that the number of particles in an element of phase space is $f dr d\theta dz dp_r dp_\theta dp_z$. The number of particles in an element of configuration space is $n(r) r dr d\theta dz$, where $n(r)$ is the usual density. Together these facts yield the formula,

$$n(r) = \frac{1}{r} \int dp_r dp_\theta dp_z \ f \ , \qquad (19)$$

for the density.

For later purposes we define the quantities,

$$\kappa_n \equiv \partial \ell n \ n_0 / \partial R \qquad (20a)$$

and

$$\kappa_T \equiv \partial \ell n \ T_0 / \partial R \ , \qquad (20b)$$

whose inverses are the radial scale lengths associated with the density in the temperature. As defined, κ_n and κ_T are typically negative. We also define

$$\eta \equiv \kappa_T/\kappa_n \ . \tag{20c}$$

III. Linear Equations

Linear theory for the cylindircal model is facilitated by the existence of two ignorable coordinates, θ and z. This ignorability allows the use of the ansätz,

$$\Phi_1 = \phi(r)\ \exp(ikz + im\theta - i\omega t)$$

$$\underset{\sim}{A}_1 = \underset{\sim}{\mathscr{A}}(r)\ \exp(ikz + im\theta - i\omega t)$$

for the electromagnetic perturbations. The perturbation of the distribution function due to these fields is found by integrating the linearized Vlasov equation along the unperturbed orbits. Taking moments yields the perturbed charges and currents. Finally, a closed set of equations is obtained by substituting these charges and currents into Maxwell's equations.

This procedure is carried out in this section. The main assumption made is that the radial scale length of the perturbations is intermediate between the ion gyroradius and the background scale length. The result is a set of two coupled second-order differential equations with independent variables ϕ and $\mathscr{A}_\parallel \equiv \hat{b}\cdot\underset{\sim}{\mathscr{A}}$. The component $\mathscr{A}_r$ is eliminated by a choice of gauge, and the remaining component $\mathscr{A}_\perp \equiv \hat{b}\times\hat{r}\cdot\underset{\sim}{\mathscr{A}}$, is algebraically eliminated.

A. Perturbed distribution function

The perturbation of the distribution function is found from the linearized Vlasov equation, which, in Hamiltonian form, is

$$\frac{\partial f_1}{\partial t} + \{f_1, h_0\} = -\{f_0, h_1\} \ , \tag{21}$$

where the braces indicate Poisson brackets, and the perturbaton of the Hamiltonian is given by

$$h_1 = \Phi_1 - \frac{\partial h_0}{\partial p_\theta} rA_{1\theta} - \frac{\partial h_0}{\partial p_z} A_{1z} \ . \tag{22}$$

As the unperturbed distribution f_0 depends upon only h_0, p_θ, and p_z, and given the form of the perturbation fields, one can separate f_1 into an "adiabatic" piece and a remainder.

$$f_1 = \frac{\partial f_0}{\partial h_0} h_1 + g_1 \ . \tag{23}$$

The remainder satisfied the equation,

$$\frac{\partial g_1}{\partial t} + \{g_1, h_0\} = i(\omega \frac{\partial f_0}{\partial h_0} + k \frac{\partial f_0}{\partial p_z} + m \frac{\partial f_0}{\partial p_\theta})h_1 \ . \tag{24}$$

To solve this equation it is useful to write g_1 in the form

$$g_1 = \left(\omega \frac{\partial f_0}{\partial h_0} + k \frac{\partial f_0}{\partial p_z} + m \frac{\partial f_0}{\partial p_\theta}\right) w_1 \ . \tag{25}$$

It is then easy to show that w_1 satisfies

$$\frac{\partial w_1}{\partial t} + \{w_1, h_0\} = i h_1 \ . \tag{26}$$

This divides the calculation of g_1 into two parts. The first is the explicit calculation of the factor in parentheses in Eq. (24). The second is finding the solution of Eq. (26) for w_1 by integrating along the unperturbed orbits.

The first task is accomplished by applying the differential operator $(k\ \partial/\partial p_z + m\ \partial/\partial p_\theta)$ to the distribution (18). The chain rule for differentiation, together with Eqs. (10), yields

$$k \frac{\partial}{\partial p_z} + m \frac{\partial}{\partial p_\theta} = k_\parallel(R) \frac{\partial}{\partial u} + \frac{k_\perp(R)}{B(R)} \frac{\partial}{\partial R} \ , \tag{27}$$

through lowest order in the ratio of gyroradius to equilibrium scale length, where

$$k_\parallel(R) \equiv \left[k B_z(R) + m B_\theta(R)/R\right]/B(R) \tag{28a}$$

and

$$k_\perp(R) \equiv \hat{b}\times\hat{r}\cdot\underset{\sim}{k}\big|_R = \left[k B_\theta(R) - m B_z(R)/R\right]/B(R) \ . \tag{28b}$$

These results allow one to calculate the derivative of f_0:

$$\omega \frac{\partial f_0}{\partial h_0} + k \frac{\partial f_0}{\partial p_z} + m \frac{\partial f_0}{\partial p_\theta} = -T^{-1}\{\omega-(k_\perp/B)\left[\kappa_n T+\kappa_T(h_0-\frac{3}{2}T)\right]\}f_0 \ . \tag{29}$$

Here we see the prominence of the drift frequency,

$$\omega_* \equiv k_\perp \kappa_n T/B \ . \tag{30}$$

Solving for w_1 involves considerably more work. First, it is convenient to divide h_1 into three parts, $h_1 = h_{1\phi} + h_{1\parallel} + h_{1\perp}$, with

$$h_{1\phi} \equiv \phi(r) \exp(im\theta + ikz - i\omega t) \ , \tag{31a}$$

$$h_{1\parallel} \equiv -v_\parallel \mathscr{A}_\parallel \exp(im\theta + ikz - i\omega t) \cong -u\mathscr{A}_\parallel \exp(im\theta + ikz - i\omega t) \ , \tag{31b}$$

and

$$h_{1\perp} = -v_\perp \mathscr{A}_\perp \exp(im\theta + ikz - i\omega t) \cong \rho B \cos\psi \ \mathscr{A}_\perp \exp(im\theta + ikz - i\omega t) \ . \tag{31c}$$

Correspondingly, w_1 and g_1 are so divided. As the solution for each part of w_1 is identical, we present the calculation only for $w_{1\phi}$. Just the results for $w_{1\parallel}$ and $w_{1\perp}$ are presented.

To integrate Eq. (26) for $w_{1\phi}$ it is convenient to use the coordinates of Sec. II.

$$-i\omega w_{1\phi} + \Omega \frac{\partial w_{1\phi}}{\partial \psi} + \dot{\Theta} \frac{\partial w_{1\phi}}{\partial \Theta} + \dot{Z} \frac{\partial w_{1\phi}}{\partial Z}$$

$$= i\phi(R+\rho \cos\psi) \exp(im\Theta + ikZ - ik_\perp\rho \sin\psi - i\omega t) \ . \tag{32}$$

To solve this equation, one can expand both sides in harmonics of the gyroangle ψ and then divide through by the resonance denominator. The ordering $\omega \ll \Omega$ then allows one to discard all harmonics save the average term,

$$w_{1\phi} = - \frac{\exp(im\Theta+ikZ-i\omega t)}{\omega-m\dot{\Theta}-k\dot{Z}} \int_0^{2\pi} \frac{d\psi}{2\pi} \phi(R+\rho\cos\psi) \exp(-ik_\perp\rho\sin\psi) . \quad (33)$$

The averaging of Eq. (33) is facilitated by the small gyroradius approximation mentioned in the introduction. Thus we use $\rho\partial\ell n\phi/\partial r$, $k_\perp\rho \ll 1$ to obtain

$$\int_0^{2\pi} \frac{d\psi}{2\pi} \phi(R+\rho\cos\psi) \exp(-ik_\perp\rho\sin\psi) = \left(1 - \frac{1}{4} k_\perp^2\rho^2\right)\phi(R) + \frac{1}{4}\rho^2 \frac{\partial^2\phi}{\partial r^2}(R) . \quad (34)$$

The resonance denominator of Eq. (33) is evaluated using Eqs. (15) and (28). The result is

$$\omega - m\dot{\Theta} - k\dot{Z} = \omega - k_\parallel(R)u - \omega_c - \omega_B , \quad (35a)$$

where

$$\omega_c \equiv - \frac{k_\perp u^2 B_\theta^2}{RB^3} \quad (35b)$$

and

$$\omega_B \equiv \frac{k_\perp J}{B} \frac{\partial B}{\partial R} \tag{35c}$$

are the contributions due to the curvature and ∇B drifts.

Finally, Eqs. (33-35) are combined to yield

$$w_{1\phi} = - \frac{\left[(1 - \frac{1}{4} k_\perp^2 \rho^2) \phi(R) + \frac{1}{4} \rho^2 \phi''(R) \right] \exp(im\Theta + ikZ - i\omega t)}{\omega - k_\parallel(R) u - \omega_c - \omega_B} . \tag{36a}$$

Similarly, one can find the other pieces of w_1:

$$w_{1\parallel} = + \frac{\left[(1 - \frac{1}{4} k_\perp^2 \rho^2) u \mathscr{A}_\parallel(R) + \frac{1}{4} \rho^2 u \mathscr{A}''_\parallel(R) \right] \exp(im\Theta + ikZ - i\omega t)}{\omega - k_\parallel(R) u - \omega_c - \omega_B} \tag{36b}$$

$$w_{1\perp} = - \frac{\frac{1}{2} \rho^2 B \mathscr{A}'_\perp(R) \exp(im\Theta + ikZ - i\omega t)}{\omega - k_\parallel(R) u - \omega_c - \omega_B} . \tag{36c}$$

These results, together with Eqs. (23), (25), and (29) specify the perturbed distribution function.

B. Plasma response

1. Density perturbation

The density perturbation is obtained by taking the lowest moment of the distribution as in Eq. (19). In doing so we divide the density perturbation into several pieces, an adiabatic piece

$$n_{1a} = \frac{1}{r} \int dp_r dp_\theta dp_z \frac{\partial f_0}{\partial h_0} h_1 ,$$

and the pieces due to $g_{1\phi}$, $g_{1\phi}$, $g_{1\parallel}$, and $g_{1\perp}$. With Eqs. (18) and (22) one finds

$$n_{1a} = -\frac{1}{T} \left(n_0 \phi + j_{0\parallel} \mathscr{A}_\parallel + j_{0\perp} \mathscr{A}_\perp\right) ,$$

in which n_0, $j_{0\parallel}$, and $j_{0\perp}$ are the equilibrium density and currents in the $\hat{b}$ and $\hat{b}\times\hat{r}$ directions. For the assumed distribution (18), $j_{0\parallel} = 0$. Pressure balance does dictate the existence of $j_{0\perp} = B^{-1}\partial(n_0 T)/\partial r$. However, the corresponding contribution to the density perturbation is small compared with the result (44) in the ratio of perturbation scale length to equilibrium scale length. Thus, the adiabatic part of the density perturbation is

$$n_{1a} = -\frac{n_0 \phi}{T} \tag{37}$$

The calculation of the remaining part of the denisty is more difficult because g_1 is expressed in terms of the transformed variable, not the physical variable. This difficulty is easily overcome by first writing n_1 in the form

$$n_{1g}(r,\theta,z,t) = \frac{1}{r} \int dp_r dp_\theta dp_z dr' d\theta' dz' \;\; \delta(r-r')\delta(\theta-\theta')\delta(z-z') g_1 \; .$$

Then the transformation to cylindrical guiding center coordinates is introduced.

$$n_{1g}(r,\theta,z,t) = \frac{1}{r} \int dRdudJd\psi d\Theta dzRB(R)\ \delta(r-R-\rho\cos\psi)\ \delta\big(\theta-\Theta+\rho(B_z/RB)\sin\psi\big)$$

$$\delta\big(z-Z-\rho(B_\theta/B)\sin\psi\big)\ g(u,J,R,\Theta,Z)\ .$$

The δ-functions allow one to collapse the integral to obtain

$$n_{1g}(r,\theta,z,t) = \int dud\psi dJB(r)\ g_1\big(u,J,r-\rho\cos\psi\ ,$$

$$\theta+\rho(B_z/rB)\sin\psi,\ z-\rho(B_\theta/B)\sin\psi\big) \qquad (38)$$

to lowest order in the ratio of gyroradius to equilibrium scale length.

The integration over ψ in Eq. (38) proceeds via the expansion technique introduced in Section IIIA. To highlight one further difficulty encountered, we show some of the details of the calculation of $n_{1\phi}$, the density perturbation due to ϕ. With the results of Sec. IIIA and Eq. (38), one can derive the following expression:

$$n_{1\phi}-n_{1a} = \frac{n_0}{T} \int \frac{du}{(2\pi T)^{1/2}} \frac{d\psi}{2\pi} \frac{dw_\perp}{T} \left[\omega-\omega_* + \eta\omega_*\left(\frac{3}{2} - \frac{w_\perp}{T} - \frac{u^2}{2T}\right)\right] \exp\left(-\frac{u^2}{2T} - \frac{w_\perp}{T}\right)$$

$$\times \frac{\left[\left(1 - \frac{1}{4} k_\perp^2\rho^2\right)\phi + \frac{1}{4}\rho^2\phi''\right] \exp(im\theta+ikz+ik_\perp\rho\sin\psi-i\omega t)}{\omega - k_\parallel u - \omega_c - \omega_B}\ , \qquad (39)$$

where $w_\perp \equiv \Omega J$. In this expression, the radial argument, i.e. r or $R = r-\rho\cos\psi$, has been deliberately left vague. Clearly one can use r nearly everywhere in the integrand of Eq. (39) because the gyroradius is small compared with the equilibrium scale length. However, this distinction must be maintained in two places, the functions $\phi(R)$ and $k_\parallel(R)$. The reason for using $\phi(R-\rho\cos\psi)$ in Eq. (39) is to obtain the entire FLR connection to n_1. The reason for using $k_\parallel(R=r-\rho\cos\psi)$ is that the scale of the resonant factor vanishes at resonance. We will see, momentarily, what this implies for n_1. For the moment, let us expand, e.g.,

$$[\omega - k_\parallel(r-\rho\cos\psi)u - \omega_c - \omega_B]^{-1} \cong [\omega - k_\parallel(r)u - \omega_c - \omega_B]^{-1}$$
$$- \rho\cos\psi \frac{\partial}{\partial r}\left[\omega - k_\parallel(r)u - \omega_c - \omega_B\right]^{-1} . \qquad (40)$$

We thus obtain

$$n_{1\phi} = -\frac{n_0}{T}\left(\left[(1-\frac{\omega_*}{\omega})(1+\xi W_{00}) + \frac{\eta\omega_*}{\omega}\xi(\frac{3}{2}W_{00} - \frac{1}{2}W_{20} - W_{01})\right]\phi\right.$$
$$- k_\perp^2\rho_T^2\left[(1-\frac{\omega_*}{\omega})\xi W_{01} + \frac{\eta\omega_*}{\omega}\xi(\frac{3}{2}W_{01} - \frac{1}{2}W_{21} - W_{02})\right]\phi$$
$$\left.+ \rho_T^2\frac{\partial}{\partial r}\left\{\left[(1-\frac{\omega_*}{\omega})\xi W_{01} + \frac{\eta\omega_*}{\omega}\xi(\frac{3}{2}W_{01} - \frac{1}{2}W_{21} - W_{02})\right]\frac{\partial\phi}{\partial r}\right\}\right) , \qquad (41)$$

in which we have used the finite-β resonant integrals of App. A, the thermal gyroradius is given by

$$\rho_T^2 \equiv T/B^2 , \qquad (42)$$

and $\xi \equiv \omega/|k_\parallel|v_T$, where $v_T = T^{1/2}$.

Similarly, we calculate the density perturbation due to $\mathscr{A}_\parallel$ and $\mathscr{A}_\perp$. We find

$$n_{1\parallel} = \frac{n_0}{T}\left[\left(1 - \frac{\omega_*}{\omega}\right)W_{10} + \frac{\eta\omega_*}{\omega}\left(\frac{3}{2}W_{10} - \frac{1}{2}W_{30} - W_{11}\right)\right]\omega\mathscr{A}_\parallel k_\parallel , \tag{43}$$

and

$$n_{1\perp} = -\frac{n_0}{T}\rho_T^2 B\left[\left(1 - \frac{\omega_*}{\omega}\right)\xi W_{01} + \frac{\eta\omega_*}{\omega}\xi\left(\frac{3}{2}W_{01} - \frac{1}{2}W_{21} - W_{02}\right)\right]\mathscr{A}_\perp' . \tag{44}$$

The total current is found from

$$n_1 = n_{1\phi} + n_{1\parallel} + n_{1\perp} . \tag{45}$$

At this point it is apparent that we have kept the FLR correction to $n_{1\phi}$ but neglected them in $n_{\perp\parallel}$ and $n_{1\perp}$. The reason for keeping the FLR terms in $n_{1\phi}$ is that they yield the dominant contribution for $(k_\parallel v_e/\omega,\ \omega_c/\omega,\ \omega_B/\omega) \to 0$. In fact, if these terms are neglected, the resulting quasineutrality equation, which may then be algebraically solved for ϕ, predicts a singularity in ϕ at x=0. In contrast, the FLR terms in $n_{1\parallel}$ and $n_{1\perp}$ are merely small corrections and may be ignored.

2. Parallel current perturbation

The parallel current perturbation is obtained from the integral,

$$j_{\parallel} = \frac{1}{r} \int dp_r dp_\theta dp_z \left(\hat{b} \cdot \frac{\partial h_0}{\partial \underset{\sim}{p}} f_1 + \hat{b} \cdot \frac{\partial h_1}{\partial \underset{\sim}{p}} f_0\right) . \qquad (46)$$

The second term of this equation is due to the change in the velocity function of canonical variables by the perturbation. The adiabatic part of this current is defined to be

$$j_{\parallel a} = \frac{1}{r} \int dp_r dp_\theta dp_z \left(\hat{b} \cdot \frac{\partial h_0}{\partial \underset{\sim}{p}} \frac{\partial f_0}{\partial h_0} h_1 + \hat{b} \cdot \frac{\partial h_1}{\partial \underset{\sim}{p}} f_0\right) .$$

With the formulae (17), (22), and (51) we find $j_{\parallel a} = 0$. Thus, we need find only

$$j_{\parallel} = \frac{1}{r} \int dp_r dp_\theta dp_z \, u g_1 .$$

This proceeds exactly parallel to the corresponding density calculation. The result is

$$j_{\parallel} = j_{\parallel\phi} + j_{\parallel\parallel} + j_{\parallel\perp} , \qquad (47a)$$

with

$$j_{\parallel\phi} = -\frac{n_0}{T} \left[\left(1 - \frac{\omega_*}{\omega}\right)W_{10} - \frac{\eta\omega_*}{\omega} \left(\frac{3}{2} W_{10} - \frac{1}{2} W_{30} - W_{11}\right)\right]\omega\phi/k_{\parallel} , \qquad (47b)$$

$$j_{\parallel\parallel} = n_0\Big[\Big(1 - \frac{\omega_*}{\omega}\Big)\xi W_{20} + \frac{\eta\omega_*}{\omega}\,\xi\Big(\frac{3}{2}\,W_{20} - \frac{1}{2}\,W_{40} - W_{21}\Big)\Big]\mathscr{A}_{\parallel}\ , \qquad (47c)$$

and

$$j_{\parallel\perp} = -\frac{n_0}{T}\,\frac{\omega v_T}{k_{\parallel}}\,\Big[\Big(1 - \frac{\omega_*}{\omega}\Big)W_{11} + \frac{\eta\omega_*}{\omega}\,\Big(\frac{3}{2}\,W_{11} - \frac{1}{2}\,W_{31} - W_{22}\Big)\Big]\rho_T\mathscr{A}_{\perp}'\ . \qquad (47d)$$

3. Perpendicular current perturbation

The perpendicular current perturbation is obtained from the expression

$$j_{\perp} = \frac{1}{r}\int dp_r dp_\theta dp_z\Big(\hat{b}\times\hat{r}\cdot\frac{\partial h_0}{\partial \underset{\sim}{p}}\,f_1 + \hat{b}\times\hat{r}\cdot\frac{\partial h_1}{\partial \underset{\sim}{p}}\,f_0\Big)\ . \qquad (48)$$

The adiabatic part of this current is given by the definition

$$j_{\perp a} = \frac{1}{r}\int dp_r dp_\theta dp_z\Big(v_{0\perp}\,\frac{\partial f_0}{\partial h_0}\,h_1 + v_{1\perp}f_0\Big)\ .$$

Straightforward calculation yields

$$j_{\perp} = -\frac{\phi}{B}\,\frac{\partial n_0}{\partial r}\ .$$

However, this contribution to $j_{\perp}$ may be neglected because it is small compared with the main contribution [cf. Eqs. (49)] by the ratio of the perturbation scale length to the equilibrium scale length. The calculation of the remaining part of the current,

$$j_\perp = \frac{1}{r}\int dp_r dp_\theta dp_z\ \hat{b}\times\hat{r}\cdot\frac{\partial h_0}{\partial \underset{\sim}{p}}\, g_1\ ,$$

proceeds in parallel with the calculation of n_1. The result is

$$j_\perp = j_{\perp\phi} + j_{\perp\parallel} + j_{\perp\perp}\ , \qquad (49a)$$

where

$$j_{\perp\phi} = -\frac{n_0}{T}\,\rho_T^2 B\,\frac{\partial}{\partial r}\,\{[(1-\frac{\omega_*}{\omega})\xi W_{01} + \frac{\eta\omega_*}{\omega}\,\xi(\frac{3}{2}\,W_{01} - \frac{1}{2}\,W_{21} - W_{02})]\phi\}\ , \qquad (49b)$$

$$j_{\perp\parallel} = \frac{n_0}{T}\,\rho_T^2 B\,\frac{\partial}{\partial r}\,\{[(1-\frac{\omega_*}{\omega})W_{11} + \frac{\eta\omega_*}{\omega}\,(\frac{3}{2}\,W_{11} - \frac{1}{2}\,W_{31} - W_{12})]\omega\mathcal{A}_\parallel/k_\parallel\}\ , \qquad (49c)$$

and

$$j_{\perp\perp} = -\frac{n_0}{T}\,\rho_T^4 B^2\,\frac{\partial}{\partial r}\,\{[(1-\frac{\omega_*}{\omega})\xi W_{02} + \frac{\eta\omega_*}{\omega}\,\xi(\frac{3}{2}\,W_{02} - \frac{1}{2}\,W_{22} - W_{03})]\,\frac{\partial\mathcal{A}_\perp}{\partial r}\}\ . \qquad (49d)$$

C. Differential equations for the perturbations

In this section we obtained a closed set of differential equations for ϕ, $\mathscr{A}_{\parallel}$, and $\mathscr{A}_{\perp}$ by summing the expressions for charge density and current over species and inserting these sums into the quasineutrality equations and Ampere's law. As we shall see, the perpendicular component of Ampere's law can be solved algebraically for $\mathscr{A}_{\perp}$. Thus, one is left with two coupled second-order differential equations for ϕ and $\mathscr{A}_{\parallel}$.

In order to sum the charge and current densities over species we need to restore ordinary units. This involves appending subscripts on the results of the last section and inserting factors of e_j, m_j, and c to make the units come out right. Our notation uses $v_j \equiv (T_j/m_j)^{1/2}$ for the thermal velocity, $\omega_j^2 \equiv 4\pi e_j^2 n_j/m_j$ for the plasma frequency, $\Omega_j \equiv e_j B/m_j c$ for the signed gyrofrequency, $\lambda_j \equiv v_j/\omega_j$ for the Debye length, $\xi_j \equiv \omega/|k_{\parallel}|v_j$, and other quantities as will be obvious.

The perpendicular component of Ampere's law is simply

$$- \frac{\partial^2 \mathscr{A}_{\perp}}{\partial r^2} = \frac{4\pi}{c} j_{\perp} , \tag{50}$$

with the assumption that the scale length of the perturbation is small compared with the equilibrium scale length. From Eqs. (49) we see that the perpendicular current has the form of a perfect radial derivative. Thus we obtain the particular solution of Eq. (50):

$$\frac{\partial \mathscr{A}_{\perp}}{\partial r} = \Gamma^{-1}(C_{\phi}\phi + C_a \mathscr{A}_{\parallel}) , \tag{51}$$

where

$$\Gamma \equiv 1 - \sum_j \frac{\omega_j^2 v_j^2}{\Omega_j^2 c^2} \left[\left(1 - \frac{\omega_{*j}}{\omega}\right) \xi_j W_{02}^j + \frac{\eta_j \omega_{*j}}{\omega} \xi_j \left(\frac{3}{2} W_{02}^j - \frac{1}{2} W_{22}^j - W_{03}^j\right)\right] , \quad (52a)$$

$$C_\phi \equiv \sum_j \frac{\omega_j^2}{\Omega_j c} \left[\left(1 - \frac{\omega_{*j}}{\omega}\right) \xi_j W_{01}^j + \frac{\eta_j \omega_{*j}}{\omega} \xi_j \left(\frac{3}{2} W_{01}^j - \frac{1}{2} W_{21}^j - W_{02}^j\right)\right] , \quad (52b)$$

and

$$C_a \equiv - \sum_j \frac{\omega_j^2 \omega}{\Omega_j k_\parallel c^2} \left[\left(1 - \frac{\omega_{*j}}{\omega}\right) W_{11}^j + \frac{\eta_j \omega_{*j}}{\omega} \left(\frac{3}{2} W_{11}^j - \frac{1}{2} W_{31}^j - W_{12}^j\right)\right] . \quad (52c)$$

The general solution of Eq. (50) would have an additional constant of integration in Eq. (51). However, this additional constant would prevent $\mathcal{A}_\perp$ from satisfying the boundary conditions, which will be discussed in Sec. IIID.

We note also that the solution (51) is singular if Γ vanishes. This happens if the frequency ω is low enough such that $\omega_j^2 v_j^2 \omega_{*j}/(\Omega_j^2 c^2 \omega)$ is of order unity. This occurs at $(\omega/\omega_{*j}) = \mathcal{O}(\beta_j)$, where we have used $\omega_j^2 v_j^2/\Omega_j^2 c^2 = \frac{1}{2}\beta_j$. To properly treat this singularity one should include FLR corrections, which would resolve the singularity by adding a term of order $\rho_i^2 \partial^4 \mathcal{A}_\perp/\partial r^4$ to Eq. (50). For present purposes, we assume $\omega/\omega_{*j} = \mathcal{O}(1)$ and ignore this effect.

Upon substituting this result for $\partial \mathcal{A}_\perp/\partial r$ into the quasineutrality condition, $\sum_j n_j e_j = 0$, and the parallel component of Ampere's law,

$-\partial^2 \mathscr{A}_\parallel / \partial r^2 + k_\perp^2 \mathscr{A}_\parallel = 4\pi j_\parallel / c$, one obtains a pair of second-order differential equations,

$$\frac{\partial}{\partial r} M(r) \frac{\partial \phi}{\partial r} = T_{\phi\phi}\phi + T_{\phi a}\mathscr{A}_\parallel \tag{53a}$$

and

$$\frac{\partial^2 \mathscr{A}_\parallel}{\partial r^2} = T_{\phi a}\phi + T_{aa}\mathscr{A}_\parallel \ , \tag{53b}$$

where

$$M \equiv (\rho_i^2/\lambda_i^2)\left[\left(1 - \frac{\omega_{*i}}{\omega}\right)\xi_i W_{01}^i + \frac{\eta_i \omega_{*i}}{\omega}\ \xi_i\left(\frac{3}{2}\ W_{01}^i - \frac{1}{2}\ W_{21}^i - W_{02}^i\right)\right] \ , \tag{54a}$$

$$T_{\phi\phi} \equiv k_\perp^2 M - \sum_j \lambda_j^{-2}\left[\left(1 - \frac{\omega_{*j}}{\omega}\right)\left(1 + \xi_j W_{00}^j\right)\right.$$
$$\left. + \frac{\eta_j \omega_{*j}}{\omega}\ \xi_j\left(\frac{3}{2}\ W_{00}^j - \frac{1}{2}\ W_{20}^j - W_{01}^j\right)\right] - C_\phi^2/\Gamma \ , \tag{54b}$$

$$T_{\phi a} \equiv \sum_j \frac{1}{\lambda_j^2}\ \frac{\omega}{k_\parallel c}\ \left[\left(1 - \frac{\omega_{*j}}{\omega}\right)W_{10}^j\right.$$
$$\left. + \frac{\eta_j \omega_{*j}}{\omega}\ \left(\frac{3}{2}\ W_{10}^j - \frac{1}{2}\ W_{30}^j - W_{11}^j\right)\right] - C_\phi C_a/\Gamma \ , \tag{54c}$$

and

$$T_{aa} \equiv k_\perp^2 - \sum_j \frac{\omega_j^2}{c^2} \left[\left(1 - \frac{\omega_{*j}}{\omega}\right)\xi_j W_{20}^j\right.$$

$$\left. + \frac{\eta_j \omega_{*j}}{w} \xi_j \left(\frac{3}{2} W_{20}^j - \frac{1}{2} W_{40}^j - W_{21}^j\right)\right] - C_a^2/\Gamma \ . \qquad (54d)$$

As much research[20] in this field is based on variational principles, it is useful to note that Eqs. (53) can be derived from a variational principle,

$$\delta \int dr L(r) = 0 \ ,$$

with

$$L(r) = \frac{1}{2} M\left(\frac{\partial \phi}{\partial r}\right)^2 + \frac{1}{2} \left(\frac{\partial \mathscr{A}_\parallel}{\partial r}\right)^2 + \frac{1}{2} T_{\phi\phi}\phi^2 + T_{\phi a}\phi \mathscr{A}_\parallel + \frac{1}{2} T_{aa}\mathscr{A}_\parallel^2 \ . \qquad (55)$$

D. Boundary conditions

To complete our specification of the perturbation, we need to determine the boundary conditions to be imposed on the Eqs. (53-54). In the absence of finite-β effects, i.e. $\mathscr{A}_\perp$ and magnetic drifts, these boundary conditions have been discussed in detail.[3,10,15] Thus, here we discuss only the finite-β modifications.

The Eqs. (53-54) are seen to have a parity symmetry with ϕ and $\mathscr{A}_\parallel$ having opposite parity. Hence, we may work on the half-line $x>0$ and specify the boundary conditions at the origin according to whether we are looking for modes with drift symmetry,

$$\phi'(0) = \mathscr{A}_{\parallel}(0) = 0 \ , \tag{56a}$$

or tearing symmetry,

$$\phi(0) = \mathscr{A}'_{\parallel}(0) = 0 \ . \tag{56b}$$

The remaining two boundary conditions for these equations are found by considering their large x ($x \equiv r-r_0$, where $k_{\parallel}(r_0) = 0$) behavior. With the WKB ansätz, $\phi = \tilde{\phi}(x)\exp[S(x)]$ and $\mathscr{A}_{\parallel}(x) = \tilde{\mathscr{A}}(x)\exp[S(x)]$, where $(S'\tilde{\phi}, S'\tilde{\mathscr{A}}) >> (\tilde{\phi}', \tilde{\mathscr{A}}')$, we find four possible solutions for the phase: $S(x) = \pm S_{\phi}(x)$ and $S(x) = \pm S_a(x)$, where

$$S_{\phi}(x) = \frac{2}{3}\left[\frac{ik_{\perp}v_i(1+T_e/T_i)}{\omega-\omega_{*i}-\frac{1}{2}\eta_i\omega_{*i}}\right]^{1/2}\left(\frac{x^3}{\rho_s^2 L_s}\right)^{1/2}$$

and

$$S_a(x) = k_{\perp}x \ ,$$

and the branch in $S_{\phi}(x)$ is chosen to have $\mathrm{Re}[S_{\phi}(x)] \to +\infty$ as $x \to \infty$. In these equations we have introduced the usual ion sound radius, $\rho_s = \rho_i(T_e/T_i)^{1/2}$ and shear length L_s defined such that $k_{\parallel}(x) \cong k_{\perp}x/L_s$. A WKB analysis of the amplitude shows that any asymptotic solution of Eqs. (53) has the form

$$\phi = \phi_+\left(\frac{x}{\rho_s}\right)^{1/4} e^{S_\phi} + \phi_-\left(\frac{x}{\rho_s}\right)^{1/4} e^{-S_\phi} + \frac{\omega L_s}{k_\perp xc}\left[\mathscr{A}_+ e^{S_a} + \mathscr{A}_- e^{-S_a}\right] \tag{57a}$$

and

$$\mathscr{A}_\parallel = -i\sqrt{\pi/2}\,\frac{\omega_i^2\omega(\omega-\omega_{*i} - \frac{1}{2}\eta\omega_{*i})}{\Omega_i^2 k_\parallel^2 v_i c}\left[\phi_+\left(\frac{x}{\rho_s}\right)^{1/4} e^{S_\phi} + \phi_-\left(\frac{x}{\rho_s}\right)^{1/4} e^{-S_\phi}\right]$$

$$+ \mathscr{A}_+ e^{S_a} + \mathscr{A}_- e^{-S_a} , \tag{57b}$$

where ϕ_+, ϕ_-, $\mathscr{A}_+$, and $\mathscr{A}_-$ are constants for any given solution. To obtain an acceptable bounded solution to Eqs. (53) we must therefore apply the boundary conditions $\phi_+ = \mathscr{A}_+ = 0$.

However, in the case where $k_\perp$ is small, e.g., of the order of the inverse of the machine radius, as it is in tearing modes, then the solutions appear to be degenerate on the interior scale length. In this case one matches onto the exterior solutions.[10] To find the matching parameters we examine th Eqs. (53) with the assumptions $(M\phi')' \ll T_{\phi\phi}\phi$. This yields the equation

$$\frac{\partial^2 \mathscr{A}_\parallel}{\partial x^2} = D_s \mathscr{A}_\parallel / x^2 , \tag{58}$$

where

$$D_s \equiv -\frac{L_s^2}{r}\frac{B_\theta^2}{B^3}\frac{dP}{dr} \tag{59}$$

is the usual coefficient of resistive MHD tearing theory (cf. Ref. 10). Thus, the asymptotic solution for $\mathcal{A}_\parallel$ is now

$$\mathcal{A}_\parallel = \mathcal{A}_0\left[(x/\rho_s)^{\nu_-} + \Delta\rho_s(x/\rho_s)^{\nu_+}\right] , \tag{60}$$

plus the pieces proportional to $\exp(\pm S_\phi)$, which are unchanged, where $\mathcal{A}_0$ and Δ are constants and

$$\nu_\pm = \frac{1}{2} \pm \sqrt{1/4+D_s} \; . \tag{61}$$

Therefore, for $k_\perp$ small the boundary conditions are $\phi_+=0$ and that Δ must match the parameter Δ' obtained from solving the exterior equations.

E. Low-β approximation

In writing Eqs. (53-55) no approximation has been made concerning the size of the curvature and gradient-B drifts. However, for the modes to be considered, ω is of the order of ω_{*e}, which is much greater than $\bar{\omega}_c$ and $\bar{\omega}_B$. Thus, we can expand the resonant denominators in the finite-β resonant integrals to obtain the lowest order correction due to β. This procedure reduces the coefficients of the differential equation to combinations of generalized Z-functions (see App. B), which are easily evaluated numerically.

In this expansion procedure one can use $\Gamma=1$, since the terms such as $C_\phi C_a$ in Eq. (54c) are already of order β. The resulting coefficients through first order in β are

$$\hat{M} = \frac{\rho_i^2}{\lambda_i^2}\{(1 - \frac{\omega_{*i}}{\omega})\zeta_i(Z^i_{0,1} + \frac{2\bar{\omega}_{ci}}{\omega}\,\zeta_i Z^i_{2,2} - \frac{2\bar{\omega}_{Bi}}{\omega}\,\zeta_i Z^i_{0,2})$$

$$+ \frac{\eta_i\omega_{*i}}{\omega}\,\zeta_i[-\frac{1}{2}\,Z^i_{0,1} - Z^i_{2,1} + \frac{\bar{\omega}_{ci}}{\omega}\,\zeta_i(-Z^i_{2,2} + 2Z^i_{4,2})$$

$$+ \frac{\bar{\omega}_{Bi}}{\omega}\,\zeta_i(3Z^i_{0,2} - 2Z^i_{2,2})]\} + \mathcal{O}(\beta^2)\ , \qquad (62)$$

$$\hat{T}_{\phi\phi} = k_\perp^2\hat{M} - \hat{C}_\phi^2 - \sum_j \lambda_j^{-2}\{(1 - \frac{\omega_{*i}}{\omega})Z^j_{1,1} - \frac{\eta_j\omega_{*j}}{\omega}\,(Z^j_{3,1} - \frac{1}{2}\,Z^j_{1,1})$$

$$- \frac{2\bar{\omega}_{cj}}{\omega}\,[(1 - \frac{\omega_{*j}}{\omega})\zeta_j^2 Z^j_{2,2} - \frac{\eta_j\omega_{*j}}{\omega}\,\zeta_j^2(Z^j_{4,2} - \frac{1}{2}\,Z^j_{2,2})]$$

$$+ \frac{\bar{\omega}_{Bj}}{\omega}\,[(1 - \frac{\omega_{*j}}{\omega})\zeta_j^2 Z^j_{0,2} - \frac{\eta_i\omega_{*j}}{\omega}\,\zeta_j^2(Z^j_{2,2} + \frac{1}{2}\,Z^j_{0,2})]\} + \mathcal{O}(\beta^2)\ , \qquad (63)$$

$$\hat{T}_{\phi a} = -\hat{C}_\phi\hat{C}_a + \sum_j \frac{1}{\lambda_j^2}\,\frac{\omega}{k_\parallel c}\,\{(1 - \frac{\omega_{*j}}{\omega})Z^j_{1,1} - \frac{\eta_j\omega_{*j}}{\omega}\,(Z^j_{3,1} - \frac{1}{2}\,Z^j_{1,1})$$

$$+ \frac{2\bar{\omega}_{cj}}{\omega}\,[(1 - \frac{\omega_{*j}}{\omega})\,{}^{\sim}{}_j Z^j_{3,2} - \frac{\eta_j\omega_{*j}}{\omega}\,\zeta_j(Z^j_{5,2} - \frac{1}{2}\,Z^j_{3,2})]$$

$$- \frac{\bar{\omega}_{Bj}}{\omega}\,[(1 - \frac{\omega_{*j}}{\omega})\zeta_j Z^j_{1,2} - \frac{\eta_j\omega_{*j}}{\omega}\,\zeta_j(Z^j_{3,2} + \frac{1}{2}\,Z^j_{1,2})]\} + \mathcal{O}(\beta^2)\ , \qquad (64)$$

and

$$\hat{T}_{aa} = k_\perp^2 - \hat{C}_a^2 - \sum_j \frac{\omega^2}{\lambda_j^2 k_\parallel^2 c^2} \{(1 - \frac{\omega_{*j}}{\omega})Z_{1,1}^j - \frac{\eta_j \omega_{*j}}{\omega} (Z_{3,1}^j - \frac{1}{2} Z_{1,1}^j)$$

$$+ \frac{2\bar{\omega}_{cj}}{\omega} [(1 - \frac{\omega_{*j}}{\omega})Z_{4,2}^j - \frac{\eta_j \omega_{*j}}{\omega} (Z_{6,2}^j - \frac{1}{2} Z_{4,2}^j)]$$

$$- \frac{\bar{\omega}_{Bj}}{\omega} [(1 - \frac{\omega_{*j}}{\omega})Z_{2,2}^j - \frac{\eta_j \omega_{*j}}{\omega} (Z_{4,2}^j + \frac{1}{2} Z_{2,2}^j)]\} + \mathcal{O}(\beta^2) \; , \qquad (65)$$

where

$$\hat{C}_\phi = \sum_j \frac{\omega_j^2}{\Omega_j c} [(1 - \frac{\omega_{*j}}{\omega}) \zeta_j Z_{0,1}^j - \frac{\eta_j \omega_{*j}}{\omega} \zeta_j (\frac{1}{2} Z_{0,1}^j + Z_{2,1}^j)] \qquad (66)$$

and

$$\hat{C}_a = \sum_j \frac{\omega_j^2}{\Omega_j c} \frac{\omega}{k_\parallel c^2} [(1 - \frac{\omega_{*j}}{\omega})Z_{1,1}^j - \frac{\eta_j \omega_{*j}}{\omega} (\frac{1}{2} Z_{1,1}^j + Z_{3,1}^j)] \; . \qquad (67)$$

The argument ζ in these expressions is given by $\zeta_j = \xi_j/\sqrt{2}$.

It is useful to note that this low-β expansion does not modify the asymptotics of Sec. IIID.

IV. Stability Proofs

Following the numerical results of Ross and Mahajan[4] and Tsang, et al.,[5] Antonsen[6] proved that the equation governing electrostatic, cold-ion drift waves had no unstable solutions. Later, Lee and Chen[7] showed that the inclusion of $\mathcal{A}_{\parallel}$ and some warm ion effects did not modify this reuslt. Here we examine how these results are modified by the inclusion of finite-β effects: $\mathcal{A}_{\perp}$, curvature drifts, and ∇B drifts. We find that the stability results are not modified by these finite-β effects. However, we also find that in contrast with previous work, these stability proofs say nothing about the existence of marginally stable modes in the range $\omega_{*i}[1+ \mathcal{O}(\beta)] < \omega < \mathcal{O}(\beta)\omega_{*i}$.

Without loss of generality we take $k_{\perp} > 0$ throughout this section. This convention implies $\omega_{*e} \equiv (k_{\perp}v_e^2/\Omega_e)\partial \ell n\ n/\partial r > 0$, and $\omega_{*i} < 0$.

We analyze the marginally stable modes, $\mathrm{Im}(\omega) = 0$, of Eqs. (53) by analytic continuation of these equations to the complex-x plane. In particular we wish to consider the ray on which the argument ζ of the Z_{mn}'s is a positive imaginary number. Since $\zeta_j = \omega L_s/k_{\perp}xv_j2^{1/2}$, we introduce a new variable y such that $x = -iy$ when $\omega>0$ and $x = +iy$ when $\omega<0$. Along the ray Z_{mn} is pure real (imaginary) if m is odd (even). Therefore, along this ray $\hat{M}$, $\hat{T}_{\phi\phi}$, and $\hat{T}_{aa}$ are real, while $\hat{T}_{\phi a}$ is imaginary. Thus one deduces that

$$\frac{\partial Q}{\partial y} = 0 \qquad (68)$$

along this ray where

$$Q \equiv \mathrm{Im}\left[\mathcal{A}_{\parallel}^{*}(y)\,\frac{\partial \mathcal{A}_{\parallel}}{\partial y} - \phi^{*}(y)\hat{M}(y)\,\frac{\partial \phi}{\partial y}\right] . \tag{69}$$

Furthermore, the boundary conditions (56) imply that $Q(0) = 0$. Therefore, $Q(y) = 0$ for all real values of y.

We note that it is not possible to come to the same conclusion on the ray where ζ is negative imaginary. The reason lies in the essential singularity of the Z_{mn}'s at x=0. This essential singularity prevents one from concluding $\lim_{x\to 0} Q(x) = 0$ for ζ being negative imaginary even though $\lim_{x\to 0} Q(x) = 0$ for x real.

Next we wish to evaluate Q using the asymptotic forms of Eqs. (57) and (60) with the coefficients $\phi_{\pm}$, $\mathcal{A}_{\pm}$ having the same value as they have on the real axis, e.g., $\phi_{+} = \mathcal{A}_{+} = 0$. For this procedure to be valid one must verify (1) that in transforming from the real-x axis to the real-y axis one does not cross two antistokes rays,[21] and (2) that the set of differential equations has no singularities in the inclusive quadrant containing the real x-axis and the real-y axis. The first requirement is easily verified. The second requirement holds provided the coefficient M has no roots in this quadrant.

To locate the roots of M, we first locate the roots of

$$M_0 \equiv \frac{\rho_i^2}{\lambda_i^2}\left[\left(1 - \frac{\omega_{*i}}{\omega}\right)\zeta_i Z_{0,1}^{i} - \frac{\eta_i \omega_{*i}}{\omega}\left(Z_{2,1}^{i} + \frac{1}{2} Z_{0,1}^{i}\right)\right] ,$$

which is M when magnetic drifts are neglected, i.e., the $\mathcal{O}(\beta^0)$ part of M. We see that the vanishing of M_0 gives a relation between the frequency, ω, and the value of $\zeta^i = \omega/(k'_\parallel v^i x\sqrt{2})$ at the location of the root:

$$\frac{1}{\eta_i}\left(\frac{\omega}{\omega_{*i}} - 1\right) = \frac{1}{2} + Z^i_{2,1}/Z^i_{0,1} \ . \tag{70}$$

The left side of this equation is real; thus, the right side must be real for some ζ_i satisfying $\mathrm{Im}(\zeta_i) > 0$. In the domain the right side of Eq. (70) has real values between one-half and unity when $\mathrm{real}(\zeta_i) = 0$. We therefore conclude that the evaluation of Q as discussed above is valid provided ω is not in the range

$$1 + \frac{1}{2}\eta_i < \frac{\omega}{\omega_{*i}} < 1 + \eta_i \ . \tag{71}$$

The analysis of the vanishing of $\hat{M}$ proceeds similarly with the exception that a quadratic equation for ω is obtained. As a result one can deduce that the range (71) of the lack of validity enlarges to

$$\left(1 + \frac{1}{2}\eta_i\right)[1 + \mathcal{O}(\beta)] < \frac{\omega}{\omega_{*i}} < (1 + \eta_i)[1 + \mathcal{O}(\beta)] \ ,$$

and a new range near zero, $\omega/\omega_{*i} = \mathcal{O}(\beta)$, comes into existence.

With these caveats in mind we proceed to evaluate Q using the asymptotic forms of Eqs. (57) and (60) with $\phi_+ = \mathscr{A}_+ = 0$. For $k_\perp \neq 0$, with the definitions

$$Q_a \equiv k_\perp^2 |\mathscr{A}_-|^2$$

and

$$Q_\phi \equiv \frac{1}{\lambda_e^2} \left[\frac{\pi L_s \rho_e |\omega-\omega_{*i}-\frac{1}{2}\eta_i\omega_{*i}|}{2k_\perp v_i}\left(1+\frac{T_e}{T_i}\right)\right]^{1/2} |\phi_-|^2 ,$$

one finds

$$Q = (Q_a+Q_\phi)\ \mathrm{sign}(\omega) \quad \text{for} \quad \omega>0 \quad \text{or} \quad \omega < \omega_{*i}\left(1+\frac{1}{2}\eta_i\right) \tag{72}$$

and

$$Q = -Q_a \qquad \text{for} \quad 0 < \frac{\omega}{\omega_{*i}} < 1 + \frac{1}{2}\eta_i . \tag{73}$$

From the earlier results we know that Q vanishes. We see that in the domain of Eq. (72) this implies that ϕ_- and $\mathcal{A}_-$ vanish, which implies that there is no mode. However, in the frequency domain of Eq. (73) there may exist a mode with $\mathcal{A}_- = 0$ but $\phi_- \neq 0$.

Next we consider the case where $k_\perp = 0$. We consider only the Suydam stable case, since otherwise the column is MHD unstable. With the definition

$$\hat{Q}_a \equiv \frac{\Delta'}{\rho_s} \frac{2(1-2\nu_-)}{\beta_e} \left(\frac{L_n}{L_s}\frac{\omega_{*e}}{\omega}\right)^2 \cos(\nu_-\pi)|\mathcal{A}_-|^2 ,$$

we obtain the results of Eqs. (72) and (73) with the replacement

$Q_a \to \hat{Q}_a$. Since $0 < \nu_- < \frac{1}{2}$, we deduce the absence of marginal modes provided $\Delta' > 0$.

In combination the results of this section indicate that marginal modes with $k_\perp \neq 0$ may be present only in the frequency domain,

$$\mathcal{O}(\beta) < \frac{\omega}{\omega_{*i}} < (1+\eta_i)[1 + \mathcal{O}(\beta)] \ . \qquad (74)$$

Therefore, electron drift wave eigenmodes, which have $\omega/\omega_{*e} \to \mathcal{O}(1)$, cannot be driven unstable unless their frequency changes dramatically. Similarly we deduce that modes with $k_\perp = 0$ cannot be driven unstable by bad curvature unless they have frequencies in the domain (74).

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Appendix A

Finite-β resonant integrals

The results of Sec. III are expressed in terms of integrals of the form

$$I_{mn} \equiv \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{du}{T^{1/2}} \int_{0}^{\infty} \frac{dw_\perp}{T} \left(\frac{u}{T^{1/2}}\right)^m \left(\frac{w_\perp}{T}\right)^n \frac{\exp(-\,u^2/2T - w_\perp/T)}{\omega - k_\parallel u - \omega_c - \omega_B} , \qquad (A1)$$

where ω_c and ω_B are defined in Eqs. (35). As usual, this expression is valid for $\mathrm{Im}(\omega) > 0$. Values for $\mathrm{Im}(\omega) \leqslant 0$ are found by analytic continuation. To remove scales from this integral, one can define the variables

$$\xi \equiv \frac{\omega}{|k_\parallel| T^{1/2}} , \qquad (A2)$$

$$K \equiv -\frac{k_\perp T^{1/2} B_\theta^2}{|k_\parallel| R B^3} , \qquad (A3)$$

and

$$G \equiv \frac{k_\perp T^{1/2}}{|k_\parallel| B^2} \frac{\partial B}{\partial R} . \qquad (A4)$$

Then, with the definition of the finite-β resonant integral

$$W_{mn}(\xi,K,G) \equiv \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy \frac{x^m y^n \exp(-\frac{1}{2}x^2 - y)}{x + Kx^2 + Gy - \zeta} , \quad (A5)$$

one finds

$$I_{mn} = - \frac{\text{sign}(k_{\parallel}^{m+1})}{k_{\parallel}T^{1/2}} W_{mn}(\xi,K,G) . \quad (A6)$$

To make the variables ξ, K, and G more easily understood, we provide their form in dimensional variables:

$$\xi = \frac{\omega}{|k_{\parallel}|v_T} , \quad (A7)$$

$$K = - \frac{k_{\perp}}{|k_{\parallel}|} \frac{\rho_T B_{\theta}^2}{RB^2} , \quad (A8)$$

and

$$G = \frac{k_{\perp}}{|k_{\parallel}|} \frac{\rho_T}{B} \frac{\partial B}{\partial R} , \quad (A9)$$

where $v_T \equiv (T/m)^{1/2}$ is the thermal velocity, and $\rho_T = mcv_T/eB$ is the signed gyroradius.

Appendix B

Generalized Z-functions

The generalized Z-function is defined by

$$Z_{m,n}(\zeta) = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{\pi}} \frac{z^m e^{-z^2}}{(z-\zeta)^n} , \tag{B1}$$

for $\mathrm{Im}(\zeta) > 0$. The usual plasma dispersion function[22] is $Z_{0,0}$. The generalized Z-function occurs naturally in the expansion of the finite-β resonant integrals to first order in β. We find

$$w_{m,n}(\xi,K,G) = n!2^{\frac{m-1}{2}} Z_{m,1}(\zeta) + n!2^{\frac{m+1}{2}} \frac{\bar{\omega}_c}{\omega} \zeta Z_{m+2,2}(\zeta)$$

$$- (n+1)!2^{\frac{m-1}{2}} \frac{\bar{\omega}_B}{\omega} \zeta Z_{m,2}(\zeta) , \tag{B2}$$

where

$$\zeta \equiv \xi/\sqrt{2} , \tag{B3}$$

$$\bar{\omega}_c \equiv \frac{k_\perp v_T^2 B_\vartheta^2}{\Omega RB^2} , \tag{B4}$$

$$\bar{\omega}_B \equiv \frac{k_\perp v_T^2}{\Omega B} \frac{\partial B}{\partial r} , \qquad (B5)$$

and $\Omega = eB/mc$ is the signed gyrofrequency.

Footnotes and References

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