RELAXATION TO THE STEADY STATE IN NEUTRAL-BEAM INJECTED MIRRORS

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ABSTRACT

The nonlinear stability of ion velocity distributions in simple beam-injected mirror machines is investigated using a one-dimensional kinetic equation. The steady states and their linear response functions are calculated, and the nonlinear behavior is studied both analytically and numerically. It is shown that when charge exchange effects are negligible, the oscillatory instabilities found in a previous work [1] are suppressed. When unstable steady states are present, the eventual equilibrium of the system is shown to depend not only on the initial density, but on the initial velocity distribution as well.

I. Introduction

A steady state density can be maintained in a mirror machine using neutral beam injection. The ionization of fast neutrals compensates for the loss of ions and electrons from the system ends. The ionization rate increases as the first power of the density while the loss rate usually grows faster than this (for example, the loss rate caused by Coulomb collisions is proportional to the density squared). As a result, the plasma density, after an initial growth period, usually comes to a stable steady state. Unstable steady states may exist when the ionization rate increases more rapidly with density than the loss rate (negative density perturbations in this case can result in the loss of all particles from the machine, positive perturbations in the saturation of the injected beam).

Besides this "rough" mechanism of stabilization, there exist kinetic effects that can also effect the stability. These result from the fact that the ions entering the system have a much higher energy than those leaving the system. If the density increases due to an increase in the ionization rate, the corresponding increase in the loss rate will be delayed by the time necessary for a particle to diffuse from the injection velocity to the loss velocity (all velocities in this work refer to the transverse velocity $v_\perp$). As was shown in reference [1], this delay can intensify the density perturbation and promote instability. The instability produced by the time delay effect is oscillatory in nature and is characterized by a frequency which is close to the inverse of the particle lifetime (the delay time).

In reference [1], the stability analysis was based on the solution of a

linearized kinetic equation. The present paper uses another approach in which an exact nonlinear equation for the density evolution is derived. This allows for the investigation of the long-time nonlinear behavior far from the steady state. The analytic model is described in section two. In section three, the density evolution equation is derived. The steady states and linear stability are rederived in section four. The nonlinear evolution of the system is investigated analytically in section five, and numerically in section six.

2. DESCRIPTION OF THE MODEL

A simple mirror machine is considered in which a monoenergetic neutral beam with velocity v_1 is injected at the midplane, perpendicular to the magnetic field. It is assumed that only a small fraction of the beam is ionized as it passes through the plasma column. The beam density N may thus be treated as a constant, independent of position and the plasma parameters. When the neutral beam is turned on at time t=0, the plasma is characterized by an initial ion velocity distribution $h(\overline{v})$. The behavior of the system at later times is studied here using a one dimensional model for the evolution of the ion velocity distribution $f(\overline{v},t)$ in a uniform volume of the plasma.

There are two parameter regimes that can be analyzed relatively easily. The first corresponds to the case in which ions drag on the electrons but do not undergo significant angular scattering. The second is the opposite case in which electron drag is negligable and the main relaxation mechanism is ion-ion angular scattering at constant energy (the small mirror ratio

regime R-1 << 1). In this paper, only the first regime is considered. The electron temperature T_e is assumed to be small enough ($T_e << T^* = (m_e/m_i)^{1/3} m_i v_1^2/2$) that the ion velocity is damped by electron collisions while ion-ion collisions play no role.

Neglecting the small terms that describe ion-ion angular scattering and the small second order diffusive term (which is of higher order in T_e/T^*), the ion kinetic equation can be written

$$\frac{\partial F}{\partial t} = \upsilon' n \frac{\partial}{\partial v}(vF) + n(\upsilon_i+\upsilon_{CX})\, q(v) - \upsilon_{CX} F \qquad (1)$$

where F(v,t) is the velocity distribution function averaged over the angles and multiplied by the factor v^2:

$$F(v,t) = 2\pi v^2 \int f(\theta,v,t) \sin\theta \, d\theta$$

The first term on the RHS of (1) describes the drag on the ions due to the electrons. The ion damping rate is given by $\upsilon \propto n T_e^{-3/2}$, while its derivative $\upsilon' \equiv d\upsilon/dn \propto T_e^{-3/2}$ is introduced to separate the dependencies of υ on the density n and T_e. The time-dependent density is defined in terms of the distribution function by

$$n(t) = \int_0^\infty F(v,t)\, dv$$

The second term in (1) is a source representing the ionization and charge exchange of the beam neutrals. Here, $\upsilon_i = n_b \langle \sigma_i v \rangle$ is the frequency of electron impact, $\upsilon_{CX} = n_b \langle \sigma_{CX} v \rangle$ is the charge exchange frequency, and q(v) is proportional to the velocity distribution of the neutral beam. It is assumed to

be localized near v_1 and normalized to unity;

$$\int_0^\infty q(v)\ dv = 1 \qquad (2)$$

The electron impact frequency υ_i depends on the electron temperature via the factor $\langle\sigma_i v\rangle$, while υ_{CX} is considered to be a constant. The third term on the RHS of (1) represents the removal of ions from the system due to charge exchange.

The presence of a nonzero source in eq. (1) leads to the accumulation of ions near v=0. Since the angular scattering of ions is assumed to be small, it does not contribute significantly to the ion detrapping rate. The primary ion loss mechanism is related instead to the ambipolar potential which always exists in mirrors with finite electron temperatures. Ions whose velocities fall below a certain threshold $v_0 << v_1$ are detrapped and rapidly (during a bounce time) lost from the system. Consequently, F(v,t) should be zero for $v < v_0$

$$F(v,t) = 0 \qquad v < v_0 \qquad (3)$$

Formally, eq. (1) is a first order equation and cannot admit nontrivial solutions which are equal to zero at both v=0 and v=∞. Physically, this is due to the neglect of the diffusive effects of ion-electron scattering, which would be represented by a small coefficient times the second derivative of F with respect to v. If this term were taken into account in eq.(1), the

detrapping effect, eq.(3), would be an appropriate boundary condition and $F(v,t)$ would drop smoothly to zero at $v=v_0$. Since this diffusive term is proportional to the small parameter T_e/T^*, it affects the form of $F(v,t)$ only in a thin boundary layer near $v=v_0$. As the width of this layer tends to zero, the distribution function profile tends to a limit which is discontinuous at $v=v_0$. Thus, for $v>v_0$, the distribution function satisfies eq.(1) with the single boundary condition

$$F(\infty,t) = 0 \qquad (4)$$

For $v<v_0$, the limiting profile is given by eq.(3). The detrapping effect is included by taking v_0 for the lower integration limit in the expression for the plasma density

$$n(t) = \int_{v_0}^{\infty} F(v,t)\, dv \qquad (5)$$

where $F(v,t)$ satisfies eq.(1) with the boundary condition (4).

Equations (1) and (5) are a system of equations for the time evolution of the density. However, the ionization rate υ_i and the density derivative of the ion damping rate υ' both depend on the electron temperature T_e. The value of T_e is determined by the electron energy balance. The energy is transmitted to the electrons from the ions, and can be lost via various mechanisms: mirror end losses, contact with a cold plasma outside the mirror, contact with warm streaming plasma, etc. Since the electron transit time is short, the electrons are close to equilibrium so that T_e depends only on the

instantaneous plasma density n. The specific form of the function $T_e(n)$ depends sensitively on the mechanism of electron loss. The following analysis is not restricted, therefore, to a particular function $T_e(n)$. It investigates instead the dependence of the density evolution on certain general properties of $\upsilon_i(n) \equiv \upsilon_i[T_e(n)]$ and $\upsilon'(n) \equiv \upsilon'[T_e(n)]$.

3. Derivation of the Density Evolution Equation

An analysis of the linear stability of the system (1) has already been presented in reference [1]. There the steady state solution of eqs. (1),(5) were found, the functions F(v,t), $\upsilon_i(n)$, and $\upsilon'(n)$ were linearized near the steady state point, and the dispersion relation was derived. The approach here is to examine the properties of the solutions of the exact nonlinear eqs.(1),(5) with the boundary condition (4) and initial condition

$$F(v,0) = h(v) \tag{6}$$

The steady state solution is easily found upon setting the time derivative equal to zero:

$$F_0(v) = \frac{\upsilon_i + \upsilon_{CX}}{\upsilon' v} \int_v^\infty \left[\frac{v}{v'}\right]^P q(v')\, dv' \tag{7}$$

where $P(n) = \upsilon_{CX}/\upsilon' n$ expresses the relative importance of charge exchange with respect to damping.

The steady state density is obtained by substituting (7) into (5)

$$n = \frac{v_i + v_{CX}}{v' P} \int_{V_0}^{\infty} \left[1 - \left[\frac{V_0}{v'}\right]^P \right] q(v')\, dv' \qquad (8)$$

Since q(v) is highly peaked and normalized, it may be approximated with a delta function

$$q(v) = \delta(v - v_1) \qquad (9)$$

Substituting (9) into (8) gives

$$n = \frac{v_i + v_{CX}}{v' P} \left(1 - e^{-P\Lambda}\right) \qquad (10)$$

where $\Lambda = \ln (v_1/v_0)$.

As mentioned above, v_i and v' are functions of n, so that n appears on both sides of (10). The steady state density n_0 is therefore found by solving (10) for n. It is convenient to represent the solutions of (10) as the intersections of two functions G(n) and N(n), where

$$G(n) = \Lambda (v_i + v_{CX}) / v'$$
$$N(n) = \Lambda\, n\, P(n) / \left(1 - e^{-\Lambda P(n)}\right) \qquad (11)$$

In the limit $v_{CX}=0$, (11) simplifies to

$$G(n) \rightarrow g(n) = \Lambda v_i / v'$$
$$N(n) \rightarrow n \qquad (12)$$

Depending on the physical situation (the specific dependence of T_e on n), a wide range of functions G(n) are possible. In Fig. 1, three realizations of G(n) are shown. The first, curve "A", intersects N(n) at one point; the others, curves "B" and "C", intersect N(n) at three and no points respectively.

The first step in studying the time evolution is to find the formal solution of (1), given an arbitrary function n(t) (i.e., disregarding, for the time being, the dependence of n on F given by (5)). Equation (1) can be solved by the method of characteristics. Defining

$$\psi(v,t) = \Lambda v\, F(v,t) \tag{13}$$

equation (1) can be written

$$\frac{\partial \psi}{\partial t} - n v' v \frac{\partial \psi}{\partial v} = \Lambda n (v_i + v_{cx})\, v\, q(v) - v_{cx}\, \psi \tag{14}$$

The characteristic of (14) is given by

$$v = u \exp[-\Lambda \tau(t)] \tag{15}$$

where u is the velocity at t=0 and $\tau(t)$ is the dimensionless time

$$\tau = \frac{1}{\Lambda} \int_0^t v'\, n\, dt' \tag{16}$$

Replacing the variables (v,t) with (u,t), equation (14) reduces to

$$\frac{d\psi}{dt} = \Lambda\, n(v_i + v_{cx})\, v\, q(v) - v_{cx}\, \psi \qquad (17)$$

where the full derivative indicates the derivative taken with u fixed. The solution of (17) is

$$\psi(u,t) = \Lambda \exp[-v_{cx}t] \left[\int_0^t dt' \exp[\, v_{cx}t'] \, n(v_i+v_{cx})\, v'q(v') + uh(u) \right] \qquad (18)$$

where v' is v at time t', from eq. (15)

$$v' = v(t',u) = u \exp\,[-\Lambda\tau(t')] \qquad (19)$$

Changing the integration variables from t' to v' in accordance with (19), and using (15) to express u as a function of (v,t), (18) can finally be written in the form

$$\psi(v,t) = \exp[-v_{cx}t]\left[\int_v^{ve^{\Lambda\tau(t)}} dv' \exp[v_{cx}t']\, G(t')q(v') + \Lambda v\, e^{\Lambda\tau}\, h(v\, e^{\Lambda\tau})\right] \qquad (20)$$

where G(t') = G[n(t')]. The intermediate time t'(v',v,t) is related to the current time by

$$\tau(t) - \tau(t') = (1/\Lambda)\ \ln\,(v'/v) \qquad (21)$$

Note that t'(v',v,t) is the time at which a particle, moving along a

characteristic, must leave v' in order to reach v at time t.

The differential equation for the evolution of the density may be obtained by integrating (1) over v from v_0 to ∞. This results in

$$\frac{dn}{d\tau} = g[n(t)] - \psi(v_0,t) \tag{22}$$

in terms of the dimensionless variable, τ, eq.(16). With a localized source function (9), the value of ψ at v_0 is, from (20),

$$\psi(v_0,t) = \begin{cases} \Lambda v_0 e^{\Lambda\tau} h(v_0 e^{\Lambda\tau}) \exp[-v_{cx}t] & \tau<1 \\ \exp[v_{cx}(\tilde{t}-t)]\, G(\tilde{t}) & \tau>1 \end{cases} \tag{23}$$

The retarded time $\tilde{t}\equiv t'(v_1,v_0,t)$ can be written in the form

$$\tau(t) = \tau(\tilde{t}) + 1 \tag{24}$$

The two expressions (23) for $\tau<1$ and $\tau>1$ are not equal at $\tau=1$ due to the use of the delta function source. If the source had a finite velocity spread, a smooth transition between the two solutions would appear. Substituting (23) into (22) gives

$$\frac{dn}{d\tau} = g[n(\tau)] - \begin{cases} \Lambda\, v_0\, e^{\Lambda\tau}\, h(v_0 e^{\Lambda\tau})\, \exp[-v_{cx}t(\tau)] & \tau<1 \\ G[n(\tau-1)]\, \exp[v_{cx}\{\tilde{t}(\tau)-t(\tau)\}] & \tau>1 \end{cases} \tag{25}$$

On the RHS of (25), t has been replaced by the dimensionless time τ. For the

sake of simplicity, the function h(v) was assumed to be zero for $v>v_1$. If $v_{CX}=0$, (25) reduces to:

$$\frac{dn}{d\tau} = g[n(\tau)] - \begin{cases} \Lambda\, v_0\, e^{\Lambda\tau}\, h(v_0 e^{\Lambda\tau}) & \tau<1 \\ g[n(\tau-1)] & \tau>1 \end{cases} \qquad (26)$$

This delay-differential equation describes the long time behavior of n. The first term on the RHS is the rate at which particles are being injected. The second term is the rate at which particles are being lost due to detrapping. The specification of two distinct time regimes results from the way in which the initial conditions are defined: when $\tau<1$, the rate at which particles are lost is determined only by the initial distribution h(v), since particles injected before $\tau=1$ have not yet had time to reach v_0. At later times, $\tau>1$, all of the particles initially confined in the system have been lost, and the rate of detrapping is determined by the rate of injection at $\tau-1$.

Note that eq.(26) can be easily derived from a simple qualitative analysis. When $\tau>1$, the influx of ions during some short period Δt at time t is

$$\Delta n_+ = n v_i \Delta t$$

During the same period, the particles leaving the system are those that were injected at the retarded time $\tilde{t}$.

$$\Delta n_- = \tilde{n}\, \tilde{v}_i\, \Delta \tilde{t}$$

where $\tilde{n}$ and $\tilde{\upsilon}_i$ are the density and ionization rate at $\tilde{t}$. Using the characteristic eq.(15), one can relate Δt to $\Delta\tilde{t}$.

$$\upsilon' n \, \Delta t = \tilde{\upsilon}' \tilde{n} \, \Delta \tilde{t}$$

Combining the above three expressions, the final equation for the evolution of the density can be presented in the form

$$\frac{\Delta n}{\Delta t} \frac{\Lambda}{\upsilon' n} = \left[\frac{\Lambda \upsilon_i}{\upsilon'} - \frac{\Lambda \tilde{\upsilon}_i}{\tilde{\upsilon}'} \right]$$

which coincides with (26) when $\tau > 1$.

Besides the differential forms (25) and (26), the density equation can be presented in an integral form (which will be useful in the nonlinear analysis of section five). To derive this form, the solution (20) is substituted into (5), using (13). Introducing new variables of integration, $\tau' \equiv \tau(t')$ instead of v' (see eq. (21)) and u instead of v (see eq. (15)), and changing the order of integration, (5) becomes

$$n(\tau) = \int_0^{\tau} d\tau' \, \exp[\upsilon_{CX}\{t(\tau')-t(\tau)\}] \; G(\tau') \int_{v_0 e^{\Lambda(\tau-\tau')}}^{\infty} q(v) \, dv \; +$$

$$+ \; \exp[-\upsilon_{CX} t(\tau)] \int_{v_0 e^{\Lambda\tau}}^{\infty} h(u) \, du \qquad (27)$$

Assuming a delta function source (9), and a time long enough that the second term on the RHS of (27) becomes negligibly small, then the density given by

(27) reduces to

$$n(\tau) = \int_{\tau-1}^{\tau} d\tau' \exp[\upsilon_{CX}\{t(\tau') - t(\tau)\}]\, G[n(\tau')] \qquad \tau>1 \qquad (28)$$

In the limit $\upsilon_{CX}=0$, (28) further reduces to

$$n(\tau) = \int_{\tau-1}^{\tau} g[n(\tau')]\, d\tau' \qquad (29)$$

eq. (28) expresses the fact that, at the moment τ, the value of the density is determined by the ions that entered the system between $\tau-1$ and τ ($\Delta\tau = 1$ is the ion lifetime, i.e. the time taken for an ion to damp from v_1 to v_0).

Note that the steady state (10) satisfies the evolution equation (28). To see this, put n=constant, υ'=constant, and υ_i=constant. Then $t(\tau)$ can be easily found from (16): $t=\tau\Lambda/\upsilon' n$. Substituting this into (28) and integrating over τ' results in (10) .

The following analysis is concerned with the relaxation of the system (26) to a steady state from far-from-steady-state initial conditions (without linearization and for arbitrary dependencies $\upsilon_i(n)$, $\upsilon'(n)$). Because of the difficulties of this analysis, only the particular case $\upsilon_{CX}=0$ is considered. For review, and to illustrate the present approach, the linear results of [1] are rederived in the next section using the density equation (29).

4. Linear Stability of Steady State Solutions

Equation (29) shows that in the case $v_{CX}=0$, the long time behavior of the system is governed only by the function g(n) (in the general case, $v_{CX}\neq 0$, it is determined by both G(n) and P(n)). The steady state values of the density satisfy the equation $n_0=g(n_0)$, which follows from (12). These values are represented by the intersections of the two functions graphed in Fig. 1. Linearization of the system is accomplished by linearizing g(n) about the steady state n_0, to obtain from eq. (26):

$$\frac{d\delta n}{d\tau} = g'\,\delta n(\tau) - \begin{cases} \Lambda v_0\, e^{\Lambda\tau}\,\delta h(v_0 e^{\Lambda\tau}) & \tau<1 \\ g'\delta n(\tau-1) & \tau>1 \end{cases} \qquad (30)$$

where g' is dg/dn evaluated at the steady state point n_0, and $\delta n=n-n_0$ and $\delta h=h-F_0$ are the deviations of the density and the initial distribution function from their steady state values (10) and (7), taken with $v_{CX}=0$. During the initial period $0<\tau<1$, the solution of Eq. (30) has the form

$$\delta n(\tau) = \exp(g'\tau)\left[\,\delta n(0) - \int_{v_0}^{v_0 e^{\Lambda\tau}} \delta h(v)\,(v/v_0)^{-g'/\Lambda}\,dv\,\right] \qquad (31)$$

When $\tau>1$, its solution can be found from its Laplace transformation. If we define the Laplace transform as

$$\delta n_S = \int_1^\infty d\tau\, e^{-s\tau}\,\delta n(\tau) \qquad (32)$$

then the Laplace transform of (30) when $\tau>1$ is

$$\Omega(s)\,\delta n_s = e^{-s}\left[\,\delta n(1) - g'\int_0^1 d\tau\, e^{-s\tau}\,\delta n(\tau)\,\right]$$

The inverse response function Ω is defined by

$$\Omega(s) = s - g'\,(1 - e^{-s}) \tag{33}$$

Note that since the integral (32) starts at τ=1, (33) can only be used to describe behavior at times later than τ=1.

Equation (33) depends on $n(\tau)$ for τ<1, which is obtained from (31). Substituting (31) into (33) gives (after some manipulation)

$$\begin{aligned}\delta n_s &= \frac{H(s)}{\Omega(s)} + \frac{H(g')\,e^{(g'-s)} - H(s)}{s-g'} \\ H(s) &= \int_{v_0}^{v_1} \delta h(v)\,\left|\,1 - (v/v_0)^{-s/\Lambda}\,\right|\,dv\end{aligned} \tag{34}$$

The long time behavior of the solution is governed by the poles in the expression for δn_s. Equation (34) appears to have poles at the zeros of Ω and at s=g'. However, two of these zeros, s=g' and s=0, have zero residues and therefore do not correspond to eigenmodes of the system. From (34), the other zeros of Ω can be found by solving for $s = \gamma + i\omega$. Separating $\Omega(s) = 0$ into real and imaginary parts gives

$$1 - \frac{\gamma}{g'} = e^{-\gamma}\cos\omega \;; \qquad \frac{\omega}{g'} = e^{-\gamma}\sin\omega \tag{35}$$

Solutions with ω=0 satisfy (35) when γ satisfies the condition

$$\frac{1}{g'} = \frac{1-e^{-\gamma}}{\gamma} \tag{36}$$

One can see from (36) that this particular solution describes a purely growing mode when $g'>1$, and a purely damped mode when $g'<1$.

When $\omega \neq 0$, we obtain

$$g' - \ln|g'| = \ln\left|\frac{\sin\omega}{\omega}\right| + \omega\cot\omega = k(\omega)$$

$$\gamma = \ln\left[g'\frac{\sin\omega}{\omega}\right] \tag{37}$$

where the values of g' and ω are required to satisfy the condition

$$g' \sin\omega / \omega \geq 0 \tag{38}$$

Equation (37) can be solved graphically. In Fig. 2, the function $k(\omega)$, which is an even function of ω, is plotted for $\omega>0$. The roots of (37) correspond to the intersections of the $k(\omega)$ graph with the horizontal line defined by the function $g'-\ln|g'|$. One can see that there exist an infinite number of roots $\omega_i(g')$ and $\gamma_i(g')$ $(i=0,1,2,...)$, each of which belongs to a certain interval $\pi i < \omega < \pi(i+1)$. From condition (38), the even numbers $i=0,2,...$ correspond to positive values of g', the odd numbers $i=1,3,...$ to negative values.

In the first interval, corresponding to $i=0$, the graph in Fig. 2 gives no solution with $\omega \neq 0$ because for any $g'>0$ the value of $g'-\ln|g'|$ is greater than one (see Fig. 3). Since the particular solution (36) with $\omega=0$ formally belongs to this interval, we will associate it with $i=0$ and call it the "zero"

mode.

Within the interval i=1, the solution exists only if $g'<0$. When g' varies from $-\infty$ to 0, the function $g'-\ln|g'|$ changes from $-\infty$ to $+\infty$ and correspondingly, the frequency $\omega_1(g')$ varies across the whole interval i=1. The sign of $\gamma_1(g')$ given by eq.(37) is clearly negative for $|g'|<\pi$. It is also negative when $|g'|>\pi$, as can be seen by the following argument. In order for $\gamma_1(g')$ to change sign, it must be zero at some value g'_0. If $\gamma_1(g'_0)=0$, then from (37) $g'_0 \sin\omega_0 = \omega_0$. But this also implies, when substituted into the first part of eq.(37), that $g'_0 \cos\omega_0 = g'_0$. This is not possible since $\cos\omega \neq 1$ everywhere. The contradiction implies that $\gamma_1(g')$ is always negative.

The interval i=2 is different from i=1 in that the solutions for ω_2 and γ_2 exist only if $g'>0$. When g' varies from 0 to $+\infty$, the function $g'-\ln|g'|$ changes from $+\infty$ to $+\infty$ and reaches the minimum value, unity, at $g'=1$. Correspondingly, $\omega_2(g')$ does not pass through the entire interval i=2, but goes up and down within some part of it, while $\gamma_2(g')$ is always negative for the same reasons that $\gamma_1(g')$ is negative. The dependencies of ω_i and γ_i on g' are plotted for i=1-4 in Fig. 4.

In summary, if $g'<1$, all modes are linearly stable. The relaxation to the steady state is dominated by the zero-mode (36), which is purely damped, but decays more slowly than the oscillatory modes. All oscillatory modes come in complex-conjugate pairs, are infinite in number, and are always damped. The high frequency oscillations decay faster than the low frequency, and the decay rates increase with ω approximately as $\gamma \approx \ln(g'/\omega)$.

If $g'>1$, the zero-mode becomes unstable and the steady state is unstable. For example, the points "A" and "C" in Fig. 1 correspond to stable

steady states, while "B" corresponds to an unstable one. These results show that the oscillatory instabilities described in section one do not exist when $\upsilon_{CX}=0$ (they were shown to exist for $\upsilon_{CX}\neq 0$ in reference [1]). Since we allow for arbitrary functions $\upsilon_i(n)$, $\upsilon'(n)$, and $g(n)$, the condition for instability $g'>1$ corresponds to the physical situation in which the ionization rate grows faster with density n than the loss rate.

5. Nonlinear Analysis

For the linear theory to be valid, the density deviations from the steady state must be small in comparison to the characteristic variations of $g'(n)$ with n.

$$\delta n \frac{d}{dn} (\ln g') \ll 1 \qquad (39)$$

It should be pointed out that the linear equation (30) is exact, even far from a steady state, when g(n) is a purely linear function of n. Consider, for example, a linear g(n) with $g'>1$, for which the steady state is unstable. Since all oscillatory modes decay rapidly, the long time behavior is principally determined by the zero-mode (36). Using the inverse Laplace transformation, one finds that $\delta n(\tau)\propto H(\gamma)e^{\gamma\tau}$, where the factor $H(\gamma)$ is given in (34). This shows that whether the density drops to zero or grows to infinity does not depend on the sign of the initial density perturbation $\delta n(0)$, but is instead determined by some moment of the initial distribution function h(v). In particular, situations are possible in which the initial density

perturbation n_1 is substantially smaller than the steady state density n_0, but for which, nevertheless, n eventually grows to infinity. Its possible to find the minimum value of n_1 for which n can still reach infinity. The (exceptional) limiting situation in which the system reaches the unstable steady state and stops there is characterized by the condition $H(\gamma)=0$. When $v_{CX}=0$ and $q(v')=\delta(v'-v_1)$ in (7), $F_0(v) = n_0/\Lambda v$. If $\delta h(v)$ is replaced by $h(v)-F_0(v)$ in (36), the condition $H(\gamma)=0$ can be written

$$\int_{v_0}^{v_1} h(v)\ \{1 - \exp[-(\gamma/\Lambda)\ \ln\ (v/v_0)]\}\ dv = n_0\ (1 - 1/g') \qquad (40)$$

where use has been made of (36) and the identity $\Lambda=\ln v_1/v_0$.
In order to minimize $n_1= \int h\ dv$ under the constraint (45), h(v) should be localized near the upper limit of integration $v=v_1$. This is because the weighting factor (in brackets on the RHS of (40)) has a minimum at v_1. When $h(v)=n_1\delta(v-v_1+0)$, equations (40) and (35) give for the minimum value of n_1

$$n_{1min} = n_0 \frac{(g'-1)}{\gamma} \qquad (41)$$

where γ is the growth rate of the zero mode, eq.(36). The graph of $n_{1min}(g')/n_0$ is plotted in Fig. 5. It shows that in the case where $g'-1<<1$, an initial density less than n_0 can still grow eventually to infinity; but that if $n_1<n_0/2$, the density always drops eventually to zero.

It is possible to make certain rigorous statements about the general evolution of the system, even when g(n) is not a linear function of n. It can be shown, for example, that if g(n) satisfies the rather weak restriction

$$| g(n) - g(n_0) | < |n - n_0| \qquad (42)$$

(which implies there is only one steady-state solution, and it is stable), then regardless of the specific profile of g(n), any initial perturbation of the density will always tend to the steady state. To see this, note that the exact nonlinear equation (29) can be used to obtain the inequalities

$$|n - n_0| = \left| \int_{\tau-1}^{\tau} (g - g_0)\, d\tau \right| \leq \int_{\tau-1}^{\tau} |g-g_0|\, d\tau < \int_{\tau-1}^{\tau} |n-n_0|\, d\tau \qquad (43)$$

Where the last inequality follows from (42). Eq. (43) says that the value of the positive function $|n(\tau)-n_0|$ at any moment τ is always smaller than the average of the same function over the previous unit time interval. Thus $|n-n_0|$ necessarily approaches zero. These results show that the nonlinearity does not affect the stability of steady states when they are linearly stable.

6. Numerical Calculations

The actual nonlinear behavior of the system can be computed numerically. In this section, equations (1) and (5), with $v_{CX}=0$, are integrated numerically for several different functions G(n) and initial conditions h(v). Using the definitions from section three, (11) and (16), and assuming q(v) to be a delta function at $v=v_1$, eq. (1) can be written in dimensionless form

$$\frac{\partial f}{\partial \tau} = \Lambda \frac{\partial (vf)}{\partial v} + G(n)\ \delta\ (v-v_1) - \Lambda\ P(n)\ f \qquad (44)$$

In the numerical model, velocity space is divided into 100 cells of width $\Delta V = 0.01\ (v_1 - v_0)$ where $v_1 = 1$ and $v_0 = 0.1$. Each cell is labeled with an integer i, where $1 \leq i \leq 100$. Each cell is characterized by an average velocity, $V_n = v_0 + \Delta V\ (i - 1/2)$, and an occupation number N_i. The evolution of each N_i is obtained by integrating (44) over the ith cell:

$$\frac{dN_i}{dt} = \frac{\Lambda}{\Delta V}\ (\ V_{i+1}\ N_{i+1} - V_i\ N_i\ ) - A\ N_i \qquad i \neq 100$$

$$\frac{dN_{100}}{dt} = \frac{-\Lambda}{\Delta V}\ V_{100}\ N_{100} - A\ N_{100} + G \qquad i = 100 \qquad (45)$$

where $A = \Lambda\ P(N)$ and $G = G(N)$ are given functions of the total density N. The 100 coupled ODEs are integrated numerically using a 5th order Runge Kutta routine.

The results for A=0, shown in Figs. 7-9, show that there are no unexpected nonlinear effects. The system is attracted to stable equilibria and repelled from unstable equilibria. The asymptotic behavior is consistent with the evolution of the zero and one eigenmodes described in section four (the zero-mode dominates when $G(N_0)>0$, the one-mode when $G(N_0)<0$). The two-mode can be seen when $G(N_0)>0$, providing the zero-mode is supressed.

The evolution of the system is followed in two ways:

1. By plotting the total density N as a function of time.

2. By tracing the system's trajectory in a two dimensional space, the dimensions of which correspond, roughly, to the two dominant modes of the system.

The first of these diagnostics is self-explanatory. The second requires a more detailed description.

The trajectory of the second diagnostic is plotted in N,N' space, where N and N' are the total density and partial density (density of particles between v(24) and v(100)) respectively.

$$N' = \sum_{i=24}^{100} N_i \qquad (46)$$

The number i=24 was chosen as the dividing point because, in a steady state, about half the particles have velocities above V_{24} and about half have velocities below.

The initial conditions are specified by giving initial values for N and N' and calculating f(v,t=0) in the following way:

$$N_i = \frac{(N-N')\,\Delta V}{\Lambda_3\, V_i} \qquad 1 \leq i \leq 24$$
$$N_i = \frac{N'\,\Delta V}{\Lambda_2\, V_i} \qquad 24 \leq i \leq 100 \qquad (47)$$

The constants Λ_1, Λ_2, and Λ_3 are defined as

$$\Lambda_1 = \ln \frac{V_{100}}{V_1} \quad \Lambda_2 = \ln \frac{V_{100}}{V_{24}} \quad \Lambda_3 = \ln \frac{V_{24}}{V_1} \tag{48}$$

The initial distribution f(v,0) then has the appearance shown in Fig. 6. Initial distributions of this type are used because they form a surface of section of the infinite dimensional state space which intersects the steady state solutions (the steady states are themselves acceptable initial states). Two reference functions are also plotted on the N, N' plane

$$\begin{aligned} a(N) &= G(N) \frac{\Lambda_2}{\Lambda_1} \\ b(N) &= N \frac{\Lambda_2}{\Lambda_1} \end{aligned} \tag{49}$$

These are proportional to the functions G(n) and n plotted in Fig. 1. The function a(N) is the partial density N' that would exist, according to (45), were N a constant in G(N). The function b(N) is the partial density that would exist if the distribution f(v) (with total density N) had the equilibrium velocity dependence $f(v) \propto 1/v$. As was shown in section three, the equilibria are given by the intersections of a(N) and b(N).

The numerical analysis consists of two studies, each examining a different function G(N).

In the first of these studies,

$$G(N) = .5 \Lambda (1. + 2.5 \ N^3 \ \exp[-N^2]) \tag{50}$$

This function intersects N at three points (see Fig. 7). At the first point, $0 <$

$G' < -1$, at the second $1 < G'$, and at the third $G' < 0$. According to the criterion derived in section 4, points a and c are stable, their basins of attraction being separated by a boundary intersecting the unstable point b. By numerically integrating (45) for a number of initial conditions, the location of the basin boundary was established and is shown in Fig. 7. The unstable point b is a saddle point in the state space. If a trajectory is initialized precisely on the basin boundary, it falls into b. This is illustrated in Figs. 7 and 8. Note that the basin boundary shown in Fig. 7 intersects a one dimensional section of an infinite dimensional boundary, while the trajectory shown in the same plot is the projection of the actual trajectory. This accounts for the fact that the trajectory appears to leave the basin boundary as it spirals into b. In fact, it does not.

The density evolution corresponding to the trajectory of Fig. 7 is shown in Fig.8. As the system spirals into the unstable steady state, N undergoes damped oscillations. This behavior results from the fact that the boundary trajectories (trajectories that move along the ridge that separates the two basins of attraction) do not excite the zero-mode. The next most unstable mode is the two-mode, which is damped and oscillatory. That the oscillations seen in Fig. 8 correspond to the two-mode is supported by Fig. 4 which shows the eigenmode frequencies for the value of G' at point b ($G'(b) = 1.72$). From Fig. 4, the period and damping time of the two-mode are $P_2 = 2\pi/\omega_2 = .83$ and $T_2 = 1/\gamma_2 = -.71$, which compares to about .91 and -.52 in Fig. 8. Since this trajectory could not follow the boundary exactly, a small initial excitation of the zero-mode resulted in the eventual departure of the system from steady state b and its ultimate settling at steady state c. From

Fig. 4, the zero-mode growth time for point b is T_a =.80, compared to about .95 from Fig. 8.

Figure 9 illustrates the case in which a stable steady state is flanked by two unstable steady states. The plane of initial conditions is divided into three parts. Trajectories that begin in region 1 are attracted to the origin with zero density, trajectories that begin in region 2 are attracted to the center steady state, and trajectories that begin in region 3 go out to infinity. (This corresponds to the plasma becoming so dense that the entire beam is stopped; under these conditions, the model (1) is no longer valid).

Note that in both figures 7 and 9, the boundaries intersecting the unstable steady state points have negative slopes. This means, as described in section five, that initial densities can indeed grow to excede n_0 when the initial distribution h(v) is concentrated at high velocities.

The numerical studies support the analytic predictions of sections four and five. In particular, the linear modes described in section four are observed, as is the qualitative nonlinear behavior described in section five.

7. Summary and Conclusions

The integral equation (27) in section three gives a general description of the time evolution of the ion density in a simple beam-injected mirror machine. The solutions of (27) were investigated for the special case in which the charge exchange rate υ_{CX} is zero. Linear analysis showed that the behavior of the system close to a steady state is characterized by an infinite set of eigenmodes. The frequencies ω_i and growth rates γ_i were shown to

depend on the value of a single dimensionless parameter g'. The growth rates γ_i of the oscillatory modes $\omega \neq 0$ were shown to be damped, and the high frequency modes were shown to decay faster than those with low frequency. When g'>0, the temporal behavior of the density was shown to be dominated by a zero frequency mode which is unstable when g'>1, and stable when 0<g'<1.

An important feature of (27) and the special case (29) is that they allow one to make very general rigorous statements concerning the nonlinear stability of linearly stable states. Specifically, it was shown that if $|g'|<1$ everywhere, a linearly stable steady-state solution is also nonlinearly stable. Numerical experiments were performed that confirmed the predictions of both the linear and nonlinear analysis.

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FIGURE CAPTIONS

Fig. 1 The intersections of the graph of G(n) with the graph of N(n) give the steady state solutions. Three possible realizations of G(n) are shown. Function A gives a single solution, while B and C give three and none, respectively. In section four it is shown that steady states a, c, and d are stable, while b is unstable.

Fig. 2 The frequency ω_i of the ith eigenmode is given by the intersection of the function $K(\omega)$ in the ith interval $i\pi<\omega<(i+1)\pi$ with the function $g'-\ln|g'|$. If $g'<0$, the even-numbered modes are suppressed while the value of $g'-\ln|g'|$ ranges from $-\infty$ to $+\infty$. If $g'>0$, the odd-numbered modes are suppressed and $g'-\ln|g'|$ ranges one to $+\infty$.

Fig. 3 A graph of the function $g'-\ln|g'|$.

Fig. 4 The frequencies and growth rates of the first four eigenmodes are plotted as a function of g'. The zero-mode has zero frequency and is the only eigenmode with a positive growth rate. Higher frequency are more strongly damped.

Fig. 5 A graph of the function n_{1min}/n_0 , eq. (41), as a function of g'.

Fig. 6 A typical initial velocity distribution for a numerical run. The velocity space is divided into a fast part (above V_{24}) and a slow part (below

V_{24}). The initial conditions N_0 and N'_0 are such that the total density is N_0, the density of particles with velocities above V_{24} is N'_0, and within each part, N_i is proportional to $1/V_i$. The two distributions join smoothly when $N'_0 = N_0/\Lambda_1$, which corresponds to the line b(N) in N, N' space.

Fig. 7 The basin boundary and a trajectory in N, N' space for the model defined in the text by equation (50). The steady states are given by the intersections of the two curves a(N) and b(N). The trajectory projection shown begins on (or very close to) the basin boundary, spirals into unstable point b with two-mode behavior, leaves b with zero-mode behavior, and then spirals into stable point c with one-mode behavior.

Fig. 8 The density evolution of the trajectory shown in Fig. 7 displays explicitly, at different times, the oscillations and growth rates characterizing the first three modes. For times less than τ=8, the damped, oscillatory two-mode dominates. After τ=8, the growing zero-mode takes over. Finally, when the trajectory approaches point c, the damped oscillations of the one-mode begin to appear.

Fig. 9 The boundaries of the basin of attraction in a system with a stable steady state flanked by two unstable steady states. The system is defined by the function $g(N)= a(N-1) + b(N-1)^3 + 1/\Lambda_1$, with a=-.2 and b=2; so that g'=1 at the steady state N=1. Trajectories originating in region I go to N=0, those originating in region II go to the stable steady state, and those originating in region III go to infinite density.