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THE EFFECT OF NOISE ON TIME-DEPENDENT QUANTUM STOCHASTICITY

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The dynamics of a time-dependent quantum system can be qualitatively different from those of its classical counterpart when the latter is chaotic. It is shown that small noise can strongly alter this situation.

What is the nature of a quantum system whose classical counterpart exhibits chaotic dynamics? The subfield dealing with this question has been called quantum stochasticity. A very striking result in quantum stochasticity has been obtained by Casati et al.[1] These authors considered a particular Hamiltonian with one space dimension and a potential representing periodic impulses kicking the system. If the strength of the impulse kicks is large enough, then, in the classical description the motion is chaotic, and the momentum variable, p, behaves diffusively. That is, the average value of p^2 apparently increases linearly with time. Casati et al. considered numerically the quantum mechanical version of the same problem with $\hbar$ small. They found that for early times, the average value of p^2 increased linearly with time at roughly the classical diffusive rate, but that for long time this linear increase slowed and eventually appeared to cease. Thus, there is apparently no diffusion in the quantum case. The observed saturation of the growth of $\langle p^2 \rangle$ is understandable if the Schroedinger operator for this problem has a discrete quasi-energy level spectrum.[1-4] Recently, Fishman, Grempel, and Prange[4] have presented strong arguments supporting the idea that the quasi-energy spectra for systems of the type studied by Casati et al. are discrete. These arguments are based on an analogy with Anderson localization of an electron in a solid with a random lattice. Furthermore, it has been pointed out that these results have implications for other physical systems[5-7] and experiments have been proposed. In this paper we consider the effect of adding a small amount of noise to the quantum kicked rotator problem (see also Shepelyanski[3]). We find that the quantum interference leading to localization of p^2 is a delicate effect that is strongly effected by small noise. For finite noise, there is always diffusion.[8] It is the goal of this paper to investigate the mechanisms by which small noise leads to diffusion, as well as the regimes of dependence of the quantum momentum diffusion coefficient on the noise and kicking strength.

We consider a Hamiltonian

$$H = P^2/(2I) + [\bar{\varepsilon}R\cos\theta + \bar{\nu}\phi(\theta,t)] \sum_{n=-\infty}^{+\infty} \delta(t-nT) , \tag{1}$$

where θ is an angle with period 2π, P is the angular momentum, I is the moment of inertia, $\bar{\varepsilon}$ is the strength of a periodically applied (period T) horizontal impulsive force, R is the radius at which the force is applied, and the term $\phi(\theta,t)$ is a random function of time representing a noise component in the kicking with $\bar{\nu}$ a parameter governing the strength of this noise.

The classical problem corresponding to the Hamiltonian (1) yields the well-known standard mapping[9] (including noise),

$p_{n+1} = p_n + \varepsilon\sin\theta_{n+1} - \nu\phi'_{n+1}(\theta_{n+1})$, $\theta_{n+1} = \theta_n + p_n$, where $\phi_n(\theta) = \phi(\theta,nT)$, $\phi'_n = d\phi_n/d\theta$, (p_n,θ_n) denote the values of $(p(t),\theta(t))$ just after the n^{th} kick (at $t = nT$), and we have introduced normalizations $\varepsilon = \bar{\varepsilon}RT/I$, $p = PT/I$, and $\nu = \bar{\nu}T/I$. One possible choice for ϕ_n that we will use in all of our subsequent calculations is $\phi_n(\theta) = \sqrt{2}\,\Delta_n\cos(\theta + \alpha_n)$, where Δ_n is a Gaussian random variable $\langle\Delta_n\Delta_{n'}\rangle = \delta_{nn'}$, and α_n is random with a uniform distribution in $[0,2\pi]$. For the case where ε is large most initial conditions for the classical map generate orbits which are diffusive with a momentum diffusion coefficient given approximately by[9,10] $D_{cl} \cong \varepsilon^2/4 + \nu^2/2$. Thus, if $\nu^2 \ll \varepsilon^2$ (which applies to all of our subsequent considerations), the noise has little effect on the value of D_{cl}.

Turning now to the quantum version of the problem, we impose periodic boundary conditions, $\psi(\theta,t) = \psi(\theta+2\pi,t)$. Thus momenta are quantized at $p = \ell\hbar$, where ℓ is an integer. Integration of Schroedinger's equation with the Hamiltonian (1) through one time period[1,11] yields

$$\psi_{n+1}(\theta) = \exp[(i\nu/\hbar)\phi_{n+1}(\theta)]L[\psi_n(\theta)] \; , \tag{2}$$

$$L[\psi(\theta)] \equiv \sum_{\ell}\int_0^{2\pi}\frac{d\theta'}{2\pi}\,[-i\hbar\ell^2/2 + i\ell(\theta-\theta') + i\varepsilon\cos\theta/\hbar]\,\psi(\theta') \; ,$$

where ψ_n denotes the value of $\psi(\theta,t)$ just after the kick at $t = nT$ and $\hbar$ has been normalized to I/T.

In what follows we shall consider $\varepsilon^2 \gg \nu^2$ and discuss the parameter dependence of D_q on ν, ε, and $\hbar$. We distinguish three regimes in terms of which we can state our main results as follows: (a) $(\varepsilon/\hbar)^2 \ll 1$ (large $\hbar$) for which we find $D_q \cong \nu^2/2$; (b) $(\varepsilon/\hbar)^2 \gg 1$ and $(\nu/h)^2(\varepsilon/\hbar)^2 \ll 1$ (moderate $\hbar$) for which we find $D_q \sim \nu^2(\varepsilon/\hbar)^4$; and (c) $(\nu/\hbar)^2(\varepsilon/\hbar)^2 \gg 1$ (small $\hbar$) for which we find $D_q \cong D_{cl}$.

Thus, from our regime (c) result, we have that in the "classical limit" (the limit $\hbar \to 0$) $D_q \to D_{cl}$ when $\nu > 0$ (see also Ref. 3). This is not so for $\nu = 0$, since then the quantum diffusion coefficient is apparently zero for any $\hbar > 0$ (hence, with $\nu = 0$, $\text{Lim}\, D_q = 0$ as $\hbar \to 0$). Thus we may say that, in the presence of noise, however small, the classical limit is restored.

Regimes (a) and (b) may be treated by random phase approximation perturbation theory considering the effect of finite noise ($\nu > 0$) as the perturbation. For $\nu = 0$, we assume that (2) has an essentially discrete quasi-energy spectrum.[2,4] Thus $\psi_n(\theta)$ may be expanded as $\psi_n(\theta) = \sum A_m \exp(-\, i\omega_m n) u_m(\theta)$, where from Eq. (2) the $u_m(\theta)$ and $\exp(i\omega_m)$ are the eigenfunctions and eigenvalues of the unitary operator L, $L[u_m] = \exp[-i\omega_m]u_m$.

Since $\nu/\hbar \ll 1$ for both regimes (a) and (b), the factor $\exp[i\nu\phi_n'(\theta)/\hbar] \cong 1 + i\nu\phi_n'(\theta)/\hbar$ in Eq. (2), and, assuming perturbation theory is valid, the probability per kick of a transition from u_m to $u_{m'}$ is $\alpha_{mm'} = (\nu/\hbar)^2 |\langle u_{m'}|\phi_n'|u_m\rangle|^2_{\text{ave.}}$, where the subscript ave. indicates an average over the ensemble of random ϕ_n. Using the transition probability $\alpha_{mm'}$, the diffusion coefficient is

$$D_q(m) = \frac{(\nu/\hbar)^2}{2} \sum_{m'} |\langle u_{m'}|\phi_n'|u_m\rangle|^2_{\text{ave.}} (p_{m'} - p_m)^2 \ , \qquad (3)$$

where p_m is the momentum expectation value for the state u_m. Note that, whenever Eq. (3) applies, we can immediately conclude that D_q is proportional to ν^2.

We now consider regime (a). In this case the term $\exp[i(\varepsilon/\hbar)\cos\theta]$ in L may be neglected to lowest order; thus the $u_m(x)$ are as in the freely rotating (unkicked) rotator, $u_m(x) \cong (2\pi)^{-1/2}\exp(im\theta)$. For $\phi_n = \sqrt{2}\,\Delta_n\cos(\theta+\alpha_n)$, we obtain $\alpha_{mm'} = (\nu/\hbar)^2[\delta_{m,m'+1} + \delta_{m,m'-1}]/2$. Since, in this approximation, u_m is an eigenfunction of the momentum operator corresponding to a momentum $p = m\hbar$, we may readily evaluate the diffusion to obtain from (3) $D_q \cong \nu^2/2$. This result is the same as the diffusion one would obtain for the classical map with noise if ε were set equal to zero.

We now consider regimes (b) and (c). In these cases, $(\varepsilon/\hbar)^2 \gg 1$, and the eigenvalue problem for $u_m(\theta)$ is not analytically soluable. Thus we shall only be able to obtain estimates for D_q. To do this we require some qualitative information concerning the u_m. First, we note that, on the basis of Anderson localization, Fishman et al.[4] have shown that, in the momentum representation, the eigenfunctions are exponentially localized about the "lattice points" $p = \ell\hbar$. The localization length in p (which we denote Δ) is large compared to $\hbar$. Furthermore, one can readily argue from the momentum representation version of the eigenvalue problem for u_m that, for $(\varepsilon/\hbar)^2 \gg 1$, the momentum eigenfunctions $[\hat{u}_m(\ell) = (2\pi)^{-1}\int_0^{2\pi}\exp(-i\ell\theta)u_m(\theta)d\theta]$ are not smoothly varying on the lattice. That is, although on average there is a slow exponential decrease of $|\hat{u}_m(\ell)|$ with ℓ away from the center of localization of $\hat{u}_m(\ell)$, there are also typically ~100% variations of $\hat{u}_m(\ell)$ on the lattice spacing scale (i.e., typically $|\hat{u}_m(\ell) - \hat{u}_m(\ell\pm1)| \sim |\hat{u}_m(\ell)|$).

We now obtain an estimate of Δ. To do this we utilize the arguments of Chirikov, Izrailev, and Shepelyanski.[2] We observe numerically, for the case with no noise, that $\langle p^2 \rangle$ increases with time initially at roughly the classical rate, but then turns over at some time $n \sim n_*$. This is interpreted as being due to the excitation of many Anderson localized modes by the initial condition (which is localized near $p = 0$). Furthermore, those modes most strongly excited are those which are localized around momenta within Δ of $p = 0$. Hence the effective number of modes excited by an initial condition with $p = 0$ is of the order of $\Delta/\hbar$. Each mode has an associated eigenvalue $\exp(-i\omega_m)$. Thus the ω_m may be taken to lie in the interval $[0,2\pi]$. Since there are $\Delta/\hbar$ modes, the typical spacing between modes with adjacent frequencies is $\delta\omega \sim 2\pi/(\Delta/\hbar)$. For $n \lesssim 1/\delta\omega$, the system does not yet "know" that the quasi-energy spectrum is discrete. Thus we expect that $\langle p^2 \rangle$ increases with time (n) until $1/\delta\omega$, at which time the turnover in $\langle p^2 \rangle$ should occur. Thus $n_* \sim 1/\delta\omega \sim \Delta/\hbar$. In addition, at the turnover the characteristic spread in momentum will be the localization width of the modes, i.e., $\langle p^2 \rangle \sim \Delta^2$. Let n_d denote the time to classically diffuse the distance Δ, $n_d \sim \Delta^2/D_{cl} \sim \Delta^2/\varepsilon^2$. Since the initial increase is at the classical rate, we have $n_* \sim n_d$ or $\Delta/\hbar \sim \Delta^2/\varepsilon^2$, which yields the result[2] $\Delta \sim \varepsilon^2/\hbar$.

Before calculating the diffusion coefficient in regime (b), we ask what is the limit for the validity of perturbation theory, Eq. (3). The localization of the modes, u_m, is dependent on the maintenance of phase coherence of a wavepacket as it diffuses over the distance Δ in p. Thus, if noise destroys this phase coherence in the time n_d, then the localized modes will also be destroyed. With localization no longer operable we expect a return to the classical result $D_q \cong D_{cl}$. To see how much noise is needed to do this, we recall that an eigenstate in the momentum representation has ~100% variations down to momentum separations of $\hbar$ (the lattice spacing). Thus, if the cumulative effect of the noise scatters p by an amount equal to $\hbar$, then the phases have been randomized. Noting that $\nu^2/2$ is the component of momentum diffusion due to the noise, the time n_c for the noise to scatter p by $\hbar$ is $n_c(\nu^2/2) \sim \hbar^2$ or $n_c \sim \hbar^2/\nu^2$. Thus, if $n_c < n_d$, or $(\nu/\hbar)^2(\varepsilon/\hbar)^2 > 1$, then we expect that $D_q \cong D_{cl}$. This defines the boundary between regimes (b) and (c).

To estimate D_q when $(\nu/\hbar)^2(\varepsilon/\hbar)^2 < 1$ and $(\varepsilon/\hbar)^2 > 1$ (i.e., regime (b)), we note that the phase coherence of the waves is maintained for a time n_c. Thus we expect transitions between localized modes on this time scale. Since transitions are appreciable only for modes within a localization length of each other, $D_q \sim \Delta^2/n_c$, or $D_q \sim \nu^2(\varepsilon/\hbar)^4$.

The above arguments are similar to those of Thouless[12] who considered the effect of finite temperature on localization in a solid. Thus our numerical experiments testing the above arguments (described below) may also be viewed as a test of Thouless's heuristic treatment of the low temperature conductivity of disordered solids. To our knowledge no other numerical experiments testing Thouless's arguments presently exist.

The estimate $D_q \sim \nu^2(\varepsilon/\hbar)^4$ can also be obtained directly from (3) as follows. $u_m(\theta) = \Sigma\, \hat{u}_m(\ell)\exp(i\ell\theta)$. From the fact that the $\hat{u}_m$ are localized, there are effectively of the order of $\Delta/\hbar$ appreciable terms in the sum over ℓ. Thus, using the $\hat{u}_m(\ell)$ representation, the quantity $\langle u_{m'}|\phi_n'|u_m\rangle$, with $\phi_n = \sqrt{2}\,\Delta_n\cos(\theta + \alpha_n)$, will involve a sum over roughly $\Delta/\hbar$ appreciable terms. Since $\langle u_m|u_m\rangle = 1$, $|\hat{u}_m(\ell)|^2 \sim (\Delta/\hbar)^{-1}$. Now assuming that the $\hat{u}_m(\ell)$ are pseudorandom in ℓ, we see that the sum involved in calculating $\langle u_{m'}|\phi_n'|u_m\rangle$ will be of the order of $(\Delta/\hbar)^{-1/2}$.

Thus (3) yields $D_q \sim (\nu/\hbar)^2\Delta^2$ which again gives $D_q \sim \nu^2(\varepsilon/\hbar)^4$.

As a test of these arguments, Fig. 1 shows numerical results obtained from long time evolutions of Eq. (2). (Values of ε chosen avoid accelerator modes,[9] while values of $\hbar/4\pi$ are irrational to avoid quantum resonances.[13]) The dots show results for D_q versus $\varepsilon/\hbar$ with $\varepsilon = 5.0$, $\nu = 0.0354$, and $\hbar$ varying (horizontal axis). For $(\varepsilon/\hbar)^2 \ll 1$ (regime a) there is good agreement with $D_q \cong \nu^2/2$, and D_q apparently asymptotes to D_{cl} for large $(\varepsilon/\hbar)$ appropriate to regime (c).

Figure 1 also shows other data (circles and crosses) for regime (b). The circles and dots have ν and ε fixed and $\hbar$ varying, while the crosses correspond to ν and $\hbar$ fixed and ε varying. The three sets of data fall close to each other and are consistent with an approximate proportionality of D_q to the fourth power of $(\varepsilon/\hbar)$ in regime (b), as predicted theoretically (solid line on Fig. 1).

In addition, we have obtained extensive data on the variation of D_q with ν (ε and $\hbar$ held fixed). Excellent agreement is found with the theoretically predicted proportionality to ν^2 in regimes (a) and (b) (cf. Eq. (3)).

In conclusion, we have found that the presence of a small amount of noise can greatly modify the behavior of a quantum mechanical system which is classically chaotic, particularly for systems in the semiclassical regime.

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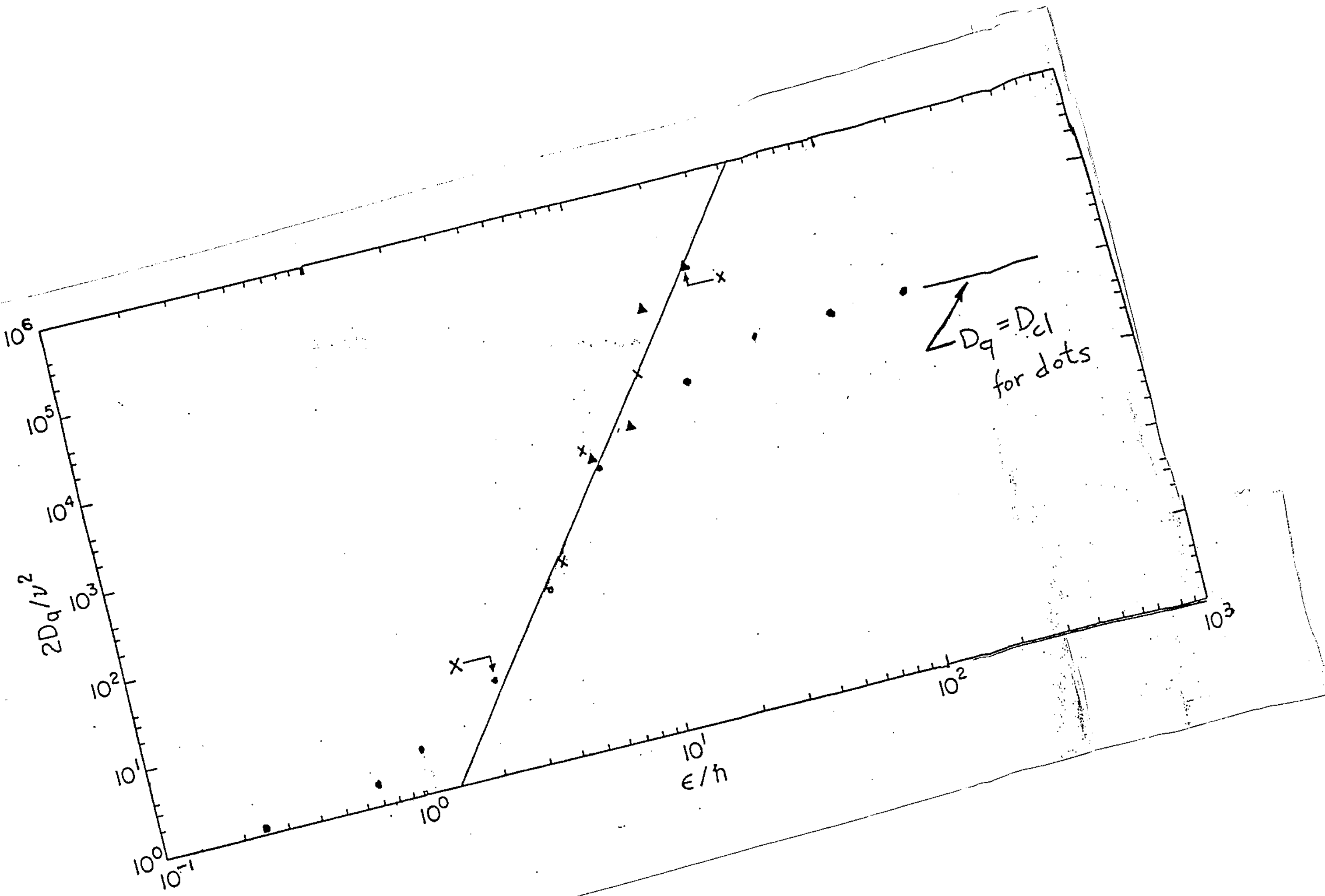

10^6
10^5
10^4
10^3
10^2
10^1
10^0
$2D_q/\nu^2$
10^{-1}
10^0
10^1
10^2
10^3
$\epsilon/\hbar$
$D_q = D_{cl}$
for dots

Figure 1

$D_q/(1/2\ \nu^2)$ versus $\varepsilon/\hbar$ with $\nu = 0.0354$ in regime (b). Solid line corresponds to $D_q \propto (\varepsilon/\hbar)^4$. Dots: $\varepsilon = 5.0$, $\hbar$ varies. Crosses: $\hbar = 5.0$, ε varies. Triangles: $\varepsilon = 55.26$, $\hbar$ varies. The iteration method is discussed in Hanson et al.[11]