DOE-ET-53088-140 IFSR #140

COMPACT FORM FOR THE RELATIVISTIC PONDEROMOTIVE HAMILTONIAN

John R. Cary
Institute for Fusion Studies
University of Texas
Austin, Texas 78712

and

Barry S. Newberger
Los Alamos National Laboratory
Los Alamos, New Mexico 87545

May 1984

Compact Form for the Relativistic Ponderomotive Hamiltonian

John R. Cary
Institute for Fusion Studies
University of Texas
Austin, Texas 78712

and

Barry S. Newberger
Los Alamos National Laboratory
Los Alamos, New Mexico 87545

ABSTRACT

The infinite series of functions present in the ponderomotive Hamiltonian are explicitly summed using Newberger's sum rule. The result is a compact and easily evaluated expression for the ponderomotive Hamiltonian. Application of the K-χ theorem yields the linear susceptibility of relativistic magnetized plasma in agreement with, but generalizing, the previous result of Weiss and Weitzner.

Ponderomotive force is the average or beat force felt by a particle in an electromagnetic wave with slowly varying amplitude and wave vector. For slow particles, this force is given by the gradient of the ponderomotive potential.[1] The kinetic generalization of the ponderomotive potential is the ponderomotive Hamiltonian. The ponderomotive Hamiltonian was originally derived for unmagnetized particles in electrostatic waves.[2] It has since been generalized to describe the motion in electromagnetic waves of relativistic unmagnetized particles,[3–5] nonrelativistic particles in a uniform magnetic field,[3–5] nonrelativistic particles in a nonuniform magnetic field,[6] and relativistic particles in a nonuniform magnetic field.[7]

The ponderomotive Hamiltonian is extremely versatile, since it completely describes the motion of the oscillation center. From the ponderomotive Hamiltonian one readily obtains the ponderomotive parallel force, drifts, and gyrofrequency shift. However, in its previously published form the ponderomotive Hamiltonian can be difficult to use because it is represented as an infinite series of Bessel functions. This is especially true in the intermediate regime where the gyrofrequency is comparable to the wave frequency and the product of the gyroradius and the perpendicular wave vector is comparable to unity. In the present note we show how to obtain a compact form for the ponderomotive Hamiltonian by summing the Bessel function series using Newberger's recently derived sum rule,[8]

$$\sum_{\ell=-\infty}^{\infty} \frac{J_\ell(z)J_{\ell-m}(z)}{\alpha-\ell} = \frac{(-1)^m\pi}{\sin\pi\alpha} J_{m-\alpha}(z)J_\alpha(z) \;, \quad (m\geq 0) \;. \tag{1}$$

We apply this sum rule to the ponderomotive Hamiltonian for a particle moving in a nonuniform magnetic field. The electric field of the wave is represented in the form,

$$\underset{\sim}{E} = \underset{\sim}{\mathscr{E}}(\underset{\sim}{x},t)\exp(i\psi(\underset{\sim}{x},t)) + \text{c.c.} \quad . \tag{2}$$

We define $\underset{\sim}{k} \equiv \nabla\psi$ and $\omega \equiv -\partial\psi/\partial t$. The approximations needed for deriving the ponderomotive Hamiltonian are described in detail in Ref. 5, Sec. V B. The form for the relativistic ponderomotive Hamiltonian is[7]

$$K_{2\nu} = \frac{e^2}{m\omega^2\Gamma}\left[|\underset{\sim}{\mathscr{E}}|^2 - \frac{1}{mc^2\Gamma^2}\left(\bar{\mu}B|\mathscr{E}_\perp|^2 + m\bar{U}_\parallel^2|\mathscr{E}_\parallel|^2\right)\right]$$

$$+ \frac{e^2}{\omega^2}\underset{\sim}{\mathscr{E}}^* \cdot \sum_\ell \left(\frac{k_\parallel}{m}\frac{\partial}{\partial\bar{U}_\parallel} + \frac{e\ell}{mc}\frac{\partial}{\partial\bar{\mu}}\right)\frac{\underset{\sim}{U}_\ell\underset{\sim}{U}_\ell^*}{\omega - k_\parallel\bar{V}_\parallel - \ell\Omega} \cdot \underset{\sim}{\mathscr{E}} \; , \tag{3}$$

where

$$\underset{\sim}{U}_\ell = \bar{V}_\parallel J_\ell(k_\perp\rho)\hat{b} + \frac{\Omega\rho}{\sqrt{2}} J_{\ell-1}(k_\perp\rho)\hat{u}_- + \frac{\Omega\rho}{\sqrt{2}} J_{\ell+1}(k_\perp\rho)\hat{u}_+ \; .$$

The symbols $\Gamma, \bar{\mu}, \bar{U}_\parallel, \bar{V}_\parallel, \rho$, and Ω are defined in Ref. 7. The unit vectors $\hat{u}_\pm \equiv (\hat{k}_\perp \pm i\hat{b}\times\hat{k}_\perp)/\sqrt{2}$ are the circularly polarized unit vectors.

All series present in Eq. (3) can be put into the form of Eq. (1). After considerable algebra we find

$$K_{2\nu} = \frac{e^2(\Gamma^2-\bar{U}_\parallel^2/c^2)}{m\Gamma^3(\omega-k_\parallel\bar{V}_\parallel)^2} \; |\mathcal{E}_\parallel|^2 + \frac{e^2}{m\omega^2} \; \underset{\sim}{\mathcal{E}}^* \cdot \left[k_\parallel \frac{\partial}{\partial\bar{U}_\parallel} \underset{\approx}{T} + \frac{e}{c}\frac{\partial}{\partial\bar{\mu}} (\alpha\underset{\approx}{T})\right] \cdot \underset{\sim}{\mathcal{E}} \, , \qquad (4)$$

where $\alpha \equiv (\omega-k_\parallel\bar{V}_\parallel)/\Omega$, and

$$\underset{\approx}{T} \equiv \frac{\pi}{\sin\pi\alpha} \Big\{\frac{\bar{V}_\parallel^2}{\Omega} \left(J_{-\alpha}J_\alpha - \frac{\sin\pi\alpha}{\pi\alpha}\right)\hat{b}\hat{b} + \frac{\rho\bar{V}_\parallel}{\sqrt{2}} J_{1+\alpha}J_{-\alpha}(\hat{b}\hat{u}_- + \hat{u}_+\hat{b})$$

$$- \frac{\rho\bar{V}_\parallel}{\sqrt{2}} J_{1-\alpha}J_\alpha(\hat{b}\hat{u}_+ + \hat{u}_-\hat{b})$$

$$- \frac{1}{2}\,\Omega\rho^2\left[J_{1-\alpha}J_{\alpha-1}\hat{u}_-\hat{u}_+ + J_{1+\alpha}J_{-1-\alpha}\hat{u}_+\hat{u}_- + J_{1-\alpha}J_{1+\alpha}(\hat{u}_-\hat{u}_- + \hat{u}_+\hat{u}_+)\right] \, . \qquad (5)$$

Finally, we note that this expression for $K_{2\nu}$ can be used to deduce the local linear susceptibility of relativistic magnetized plasma via the K-χ theorem of Refs. 3-5. We find

$$\underset{\approx}{\chi} = -\frac{4\pi e^2}{m\omega} \int d^3U \, \frac{\partial f}{\partial\bar{U}_\parallel} \, \frac{\bar{V}_\parallel}{\omega-k_\parallel\bar{V}_\parallel} \, \hat{b}\hat{b} + \frac{4\pi e^2}{m\omega^2} \int d^3U \left(k_\parallel \frac{\partial f}{\partial\bar{U}_\parallel} + \frac{e\alpha}{c}\frac{\partial f}{\partial\bar{\mu}}\right)\underset{\approx}{T} \, . \qquad (6)$$

This formula generalizes the recently derived result of Weiss and Weitzner[9] to allow for arbitrary momentum distributions.

Acknowledgments

This work was supported by the U.S. Department of Energy under grant no. DE-FG05-80ET-53088.

References

1. H. Motz and C. J. H. Watson, in <u>Advances in Electron and Electronic Physics</u> (Academic, New York, 1967), Vol. 23, p. 168.

2. R. L. Dewar, Phys. Fluids <u>16</u>, 1102(1973).

3. J. R. Cary and A. N. Kaufman, Phys. Rev. Lett. <u>39</u>, 402(1977).

4. J. R. Cary, Ph.D thesis, University of California at Berkeley (1979).

5. J. R. Cary and A. N. Kaufman, Phys. Fluids <u>24</u>, 1238(1981).

6. C. Grebogi, A. N. Kaufman, and R. G. Littlejohn, Phys. Rev. Lett. <u>43</u>, 1668(1979).

7. C. Grebogi and R. G. Littlejohn, Phys. Fluids, (to be published).

8. B. S. Newberger, J. Math. Phys. <u>23</u>, 1278(1982); <u>24</u>, 2250(E)(1983).

9. I. Weiss and H. Weitzner, private communication.