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ABSTRACT

The variational structure of the Vlasov-Maxwell integral equations is derived for a plasma equilibrium having two ignorable coordinates. It is shown that the kernel of the Maxwell equations is a self-adjoint integral operator. This operator may also be represented as a differential equation of arbitrary order. This representation is useful when the differential operator is truncated to finite order, yielding a system of intrinsically self-adjoint differential equations.

I. INTRODUCTION

Recently there has been increasing interest in applying variational methods to formulate and solve the linearized Vlasov-Maxwell equations.[1-7] The essence of the technique is to express the integral operator determining the system in a form which shows explicitly the property of (generalized) self-adjointness defined below. The object of the present paper is to derive this property for plasma equilibria having two ignorable coordinates. The demonstration is based on a symmetry relation satisfied by the unperturbed particle trajectories in the case of two ignorable coordinates. Thus the rather surprising self-adjointness property that has been previously found after rather detailed orbit calculations, as for example performed by Seylor[8], can be explained from general considerations.

We have been able to show that self-adjointness applies a restricted class of systems with one ignorable coordinate. The condition is that there be no field component in the ignorable direction. However, we leave the discussion of this case to subsequent work.

We also show that the integral operator of a Vlasov-Maxwell system can be represented in the form of a differential operator of arbitrarily high order. This representation is useful for problems where truncation of the order of the differential operator can be justified. In particular, truncating an electrostatic system with two ignorable coordinates, yields a system of differential equations which are self-adjoint, in contrast to the equations usually found in the literature. Some of these results have been discussed by Harry Mynick in his Ph.D. dissertation.[9]

In Sec. II we formulate the variational method corresponding to the problem of solving an integral equation with a self-adjoint kernel. In Sec. III we discuss the orbit symmetry that leads to self-adjointness. This property is then demonstrated in Sec. IV for the electrostatic problem and in Sec. V for the electromagnetic kernel. Section VI discusses the differential operator representation of the kernel mentioned above.

II. VARIATIONAL METHOD FOR SELF-ADJOINT PROBLEMS

In the case of an equilibrium depending on a single coordinate x, the linearized Vlasov-Maxwell equations lead to a general integral equation of the form

$$\sum_j \int dx' \; L_{ij}(x,x',\omega) \; \phi_j(x') = 0 \quad , \tag{1}$$

where $\phi_j(x)$ denotes the Fourier amplitudes (with respect to t,y,z) of the perturbed magnetic and electric potentials. Associated with the vector function $\phi_j(x)$ is an adjoint vector function $\phi_j^+(x)$ which is defined here to satisfy the equation,

$$\sum_j \int dx' \; L_{ji}(x,x',\omega) \; \phi_j^+(x') = 0 \quad . \tag{2}$$

Multiplying Eq. (1) by $\phi_i^+(x)$, integrated over x and summing over i, we form the following quadratic functional,

$$\langle\phi|L(\omega)|\phi\rangle \equiv \sum_{i,j} \int dx \, dx' \; \phi_i^+(x) \; L_{ij}(x,x',\omega) \; \phi_j(x') = 0 \quad . \tag{3}$$

If test functions,

$$\phi_j(x) = \phi_{oj}(x) + \varepsilon\delta\phi_j(x)$$

$$\phi_j^+(x) = \phi_{oj}^+(x) + \varepsilon\delta\phi_j^+(x)$$

are substituted into Eq. (3) and ω is sought, the error in ω is second order in ε. Specifically, if ω_o denotes the exact solution corresponding to the eigenvector $\phi_{jo}(x)$ then $\delta\omega = \omega - \omega_0$ is given by,

$$\delta\omega = -\frac{\varepsilon^2 \langle\delta\phi|L(\omega)|\delta\phi\rangle}{\langle\phi_o|\frac{\partial L}{\partial\omega}|\phi_0\rangle} \quad . \tag{4}$$

In order to apply this method one needs a prescription for the adjoint function $\phi^+(x)$ given the test function $\phi(x)$. In general there is not a general prescription for $\phi^+(x)$ given $\phi(x)$. For a general Vlasov-Maxwell system, Schwarzmeir and Symon[10] have given a prescription in terms of the analytic structure of $\phi(\underset{\sim}{r},\omega)$. However, in general the knowledge of the desired analytic properties is difficult to determine. For a more limited class of problems an operator can have the structure.

$$L_{ij}(x,x',\omega) = \sigma_j L_{ji}(x',x,\omega)\sigma_i \quad , \tag{5}$$

where $\sigma_i = \pm 1$. In this case the components of the adjoint function $\phi_j^+(x)$ satisfy the same equation as $\sigma_j\phi_j(x)$. The adjoint function is essentially identical to the original function, and hence the correct prescription for a test adjoint function $\phi_j^+(x)$ is $\sigma_j\phi_j(x)$. We shall define an operator with the property of Eq. (5) as self-adjoint.

Thus in applications of complicated systems of integral equations such as appears in the work of Johner and Maschke,[11] where eigenfunctions and eigenvalues are complex, second order accuracy in determining the eigenvalue for a given test function is guaranteed for the self-adjoint forms we have defined. We note here that one frequently finds in the literature quadratic forms where the complex conjugate of ϕ, ϕ^*, is used instead of the adjoint function ϕ^+. This in general is an incorrect second order form. It only applies when ϕ^* happens to be the adjoint function which is usually the case for systems where ω is real and particle resonances can be ignored.

We now wish to show why the kernel of the linearized Vlasov-Maxwell system should be self-adjoint. The important symmetry that allows this is a property of particle orbits that requires two ignorable coordinates and we limit our analysis to this case.

III. ORBIT SYMMETRY

We first consider orbits in a slab model with an external field $\underset{\sim}{B}(x) = B_y(x)\hat{\underset{\sim}{y}} + B_z(x)\hat{\underset{\sim}{z}}$ in which z and y are ignorable coordinates so that P_z and P_y are constants of motion. We define two trajectories as $[\underset{\sim}{r}^+(t), \underset{\sim}{v}^+(t)]$ and $[\underset{\sim}{r}^-(t), \underset{\sim}{v}^-(t)]$ as conjugate when their initial conditions are respectively,

$$\underset{\sim}{r}_o^+ = (x_o, y_o, z_o) \quad , \quad \underset{\sim}{v}_o^+ = (v_{xo}, v_{yo}, v_{zo})$$

$$\underset{\sim}{r}_o^- = (x_o, y_o, z_o) \quad , \quad \underset{\sim}{v}_o^- = (-v_{xo}, v_{yo}, v_{zo}) \quad .$$

These two trajectories characterize orbits with the same constants of motion, E, P_y, and P_z. The equations of motion are of the form

$$\frac{M}{q}\frac{dv_x}{dt} = \frac{v_y}{c} B_z(x_o + \delta x) - \frac{v_z}{c} B_y(x_o + \delta x) - \frac{\partial}{\partial x}\Phi(x_o + \delta x) \quad ,$$

$$\frac{M}{q}\frac{dv_y}{dt} = -\frac{v_x}{c} B_z(x_o + \delta x) \quad ,$$

$$\frac{M}{q}\frac{dv_z}{dt} = \frac{v_x}{c} B_y(x_o + \delta x) \quad ,$$

$$\frac{d}{dt}\delta x = v_x \quad , \quad \frac{d}{dt}\delta y = v_y \quad , \quad \frac{d}{dt}\delta z = v_z \quad , \qquad (6)$$

where $\tau = t - t_o$, $\delta x = x - x_o$, etc. The following transformation then leaves the equations of motion invariant,

$$\delta x \to \delta x \qquad v_x \to -v_x$$

$$\delta y \to -\delta y \qquad v_y \to v_y$$

$$\delta z \to -\delta z \qquad v_z \to v_z$$

$$\tau \to -\tau$$

These operations correspond to the simultaneous application of time reversal and parity inversion in the y and z directions with respect to the initial point. From this property it readily follows that if we define

$$\underset{\sim}{r}^+(\tau) = \underset{\sim}{r}_o + \delta\underset{\sim}{r}^+(\tau) \quad ,$$

$$\underset{\sim}{r}^-(\tau) = \underset{\sim}{r}_o + \delta\underset{\sim}{r}^-(\tau) \quad ,$$

$$\delta\underset{\sim}{r}^+(\tau) = [\delta x^+(\tau),\ \delta y^+(\tau),\ \delta z^+(\tau)] \quad ,$$

$$\delta\underset{\sim}{r}^-(\tau) = [\delta x^-(\tau),\ \delta y^-(\tau),\ \delta z^-(\tau)] \quad ,$$

then

$$\delta x^-(-\tau) = \delta x^+(\tau) \quad ,$$

$$\delta y^-(-\tau) = -\delta y^+(\tau) \quad ,$$

$$\delta z^-(-\tau) = -\delta z^+(\tau) \quad , \qquad (8)$$

$$v_x^-(-\tau) = -v_x^+(\tau) \quad ,$$

$$v_y^-(-\tau) = v_y^+(\tau) \quad ,$$

$$v_z^-(-\tau) = v_z^+(\tau) \quad .$$

A similar symmetry exists in cylindrical geometry where z and θ are ignorable coordinates. The equations of motion of two conjugate particles in an external field $\underset{\sim}{B}(r) = B_\theta(r)\hat{\underset{\sim}{\theta}} + B_z(r)\hat{\underset{\sim}{z}}$ are

$$\frac{M}{q}\frac{dv_r^\pm}{d\tau} = \frac{M}{q}(r_o + \delta r^\pm)\Omega^2 + (r_o + \delta r^\pm)\frac{\Omega^\pm}{c} B_z(r_o + \delta r^\pm)$$

$$- \frac{v_z^{\,\pm}}{c} B_\theta(r_o + \delta r^\pm) - \frac{\partial}{\partial r}\Phi(r_o + \delta r^\pm) \quad ,$$

$$\frac{M}{q}\frac{d\Omega^\pm}{d\tau} = -\frac{2M}{q} v_r^\pm \Omega^\pm - \frac{v_r^\pm}{c}\frac{B_z}{r_o + \delta r^\pm} \quad ,$$

$$\frac{dv_z^\pm}{d\tau} = v_r \frac{B_\theta}{c} \quad ,$$

$$\frac{d}{d\tau}\delta z^\pm = v_z^\pm \quad , \quad \frac{d}{d\tau}\delta r^\pm = v_r^\pm \quad , \quad \frac{d}{d\tau}\delta\theta^\pm = \Omega^\pm \quad , \qquad (9)$$

where the initial conditions are $\delta r_o^\pm = \delta\theta_o^\pm = \delta z_o^\pm = 0$, $v_z(0) = v_{zo}$, $v_r^\pm(0) = \pm v_{ro}$, $\Omega^\pm(0) = \Omega_o$. These equations are invariant to the transformation

$$\delta r \to \delta r \qquad v_r \to -v_r$$

$$\delta\theta \to -\delta\theta \qquad \Omega \to \Omega$$

$$\delta z \to -\delta z \qquad v_z \to v_z$$

$\tau \rightarrow -\tau$

It then follows that the orbits of conjugate particles are related by

$$\delta r^-(-\tau) = \delta r^+(\tau) \qquad v_r^-(-\tau) = -v_r^+(\tau)$$

$$\delta z^-(-\tau) = -\delta z^+(\tau) \qquad v_z^-(-\tau) = v_z^+(\tau)$$

$$\delta\theta^-(-\tau) = -\delta\theta^+(\tau) \qquad \Omega^-(-\tau) = \Omega^+(\tau) \quad .$$

IV. SELF-ADJOINTNESS IN THE ELECTROSTATIC PROBLEM

For the slab problem with two ignorable coordinates the eigenfuntions can be written as

$$\phi(\underset{\sim}{x}) = \phi(x) \exp\left[i(k_y y + k_z z)\right] \quad . \tag{11}$$

The electrostatic problem then has the form

$$\int dx' [L_V(x,x') + L_p(x,x',\omega)] \; \phi(x') = 0 \quad , \tag{12}$$

with $L_V(x,x')$ being the vacuum part of the operator and $L_p(x,x',\omega)$ the plasma response. Specifically, the vacuum part is given by

$$L_V(x,x') = \left(k_y^2 + k_z^2 + \frac{\partial}{\partial x} \frac{\partial}{\partial x'}\right) \delta(x - x') \quad . \tag{13}$$

Note that derivatives only act on the delta function and not on $\phi(x')$. The conventional form of the operator is obtained as follows

$$\int dx' \; \phi(x') \frac{\partial}{\partial x} \frac{\partial}{\partial x'} \delta(x - x')$$

$$= - \int dx' \; \phi(x') \frac{\partial^2}{\partial x'^2} \delta(x - x')$$

$$= - \int dx' \frac{\partial^2 \phi(x')}{\partial x'^2} \delta(x - x') = - \frac{\partial^2 \phi(x)}{\partial x^2} .$$

The plasma response operator is $L_p(x,x',\omega)$, which gives rise to the perturbed charge density. From the work of Kaufman[12] or Symon et al.[6,7] one can determine that $L_p(x,x',\omega)$ can be written as

$$L_p(x,x',\omega) = \sum_s 4\pi q_s^2 \int d\underset{\sim}{v}_o \, dx_o$$
$$\times \sum_n \frac{\left(\omega_n \frac{\partial F_s}{\partial E} + k_y \frac{\partial F_s}{\partial P_y} + k_z \frac{\partial F_s}{\partial P_z}\right)}{\omega - \omega_n(E,P_y,P_z)} I_n'(x,x')$$
$$\equiv \sum_n \mathcal{L}(E,P_y,P_z,n) \, I_n'(x,x') \quad , \qquad (14)$$

where s refers to particle species, $F_s = F_s(E,P_y,P_z)$ is the particle distribution function which is a function of the particle energy E, and canonical momenta P_y, P_z, and ω_n is the particle spectral frequency which is given by

$$\omega_n = n\bar{\omega}_c + k_y\bar{v}_y + k_z\bar{v}_z \quad ,$$

$$\frac{\pi}{\bar{\omega}_c} = \frac{\tau}{2} = \int_{x_-}^{x_+} \frac{dx}{|v_x(x,E,P_y,P_z)|} \quad ,$$

$$\bar{v}_{y,z} = \frac{2}{\tau} \int_{x_-}^{x_+} dx \, \frac{v_{y,z}(x,E,P_y,P_z)}{|v_x(x,E,P_y,P_z)|} \quad .$$

Further, the function $I_n'(x,x')$ is given by

$$I_n'(x,x') = \rho_n^{+*}(x)\ \rho_n^{+}(x') \quad , \tag{15}$$

where

$$\rho_n^{+}(x) = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} d\tau'\ \exp\left[i\Omega_n^{+}(\tau')\right]\delta[x - x_o - \delta x^{+}(\tau')] \quad , \tag{16}$$

with $\Omega_n^{+}(\tau) = -\omega_n\tau + k_y\delta y^{+}(\tau) + k_z\delta z^{+}(\tau)$. The coordinates $\underset{\sim}{v}_o$, x_o are the phase space coordinates at an initial time $t = 0$. The coordinates $\delta x^{+}(\tau)$, $\delta y^{+}(\tau)$, and $\delta z^{+}(\tau)$ have been defined just before Eq. (8).

The proof of self-adjointness of the problem requires showing

$$L(x,x',\omega) = L(x',x,\omega) \quad . \tag{17}$$

From Eq. (13) this property obviously applies to $L_v(x,x')$, but it is not obvious for $L_p(x,x',\omega)$. In order to demonstrate the intrinsic self-adjointness of $L_p(x,x',\omega)$ we note that conjugate particles have the same constants of motion and the same spectrum (this is not generally the case with less than two ignorable coordinates and hence the following arguments are in general restricted to the case of two ignorable coordinates). Hence, we can group together the responses of conjugate particles in $L_p(x,x',\omega)$. The result is that $I_n'(x,x')$ in Eq. (14) can be replaced with

$$I_n(x,x') = \frac{1}{2}\,[\rho_n^{+*}(x)\ \rho_n^{+}(x') + \rho_n^{-*}(x)\ \rho_n^{-}(x')] \quad , \tag{18}$$

where

$$\rho_n^-(x) = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} d\tau' \exp\left\{i[-\omega_n\tau' + k_y\delta y^-(\tau')\right.$$

$$\left. + k_z\delta z^-(\tau')]\right\}\delta[x - x_o - \delta x^-(\tau')] \quad .$$

Now transforming $\tau' \to -\tau'$, and using Eq. (8), we find

$$\begin{aligned}\rho_n^-(x) &= \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} d\tau' \exp\left\{i[\omega_n\tau' + k_y\delta y^-(-\tau') + k_z\delta z^-(-\tau')]\right\}\delta[x - x_o - \delta x^-(-\tau')] \\ &= \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} d\tau' \exp\left\{i[\omega_n\tau' - k_y\delta y^+(\tau') - k_z\delta z^+(\tau')]\right\}\delta[x - x_o - \delta x^+(\tau')] \\ &= \rho_n^+(x)^* \quad . \qquad (19)\end{aligned}$$

Hence

$$I_n(x,x') = \frac{1}{2}\,[\rho_n^{+*}(x)\,\rho_n^+(x') + \rho_n^+(x)\,\rho_n^{+*}(x')] \quad . \qquad (20)$$

This form is then intrinsically symmetric in x and x′ and thus self-adjointness for this form is demonstrated.

Quite a similar symmetry can be established for the cylindrical problem with $\underset{\sim}{B}(r) = B_\theta(r)\hat{\underset{\sim}{\theta}} + B_z(r)\hat{\underset{\sim}{z}}$, where z is the axial coordinate, r is the radial coordinate, and θ is the azimuthal angle. The perturbed potential is

$$\phi(\underset{\sim}{r}) = \phi(r)\exp\left[i(\ell\theta + k_z z)\right] \quad , \qquad (21)$$

and $\phi(r)$ satisfies an integral equation of the form

$$\int dr' \left[M_v(r,r') + M_p(r,r',\omega)\right] \phi(r') = 0 \quad , \tag{22}$$

where

$$M_v(r,r') = \left[k_z^2 (rr')^{1/2} + \frac{\ell^2}{(rr')^{1/2}}\right] \delta(r - r') + \frac{\partial}{\partial r}\frac{\partial}{\partial r'} \left[(rr')^{1/2}\, \delta(r - r')\right] \quad , \tag{23}$$

note that

$$\int dr'\, M_v(r,r')\, \phi(r') = -\frac{\partial}{\partial r}\left(r \frac{\partial \phi(r)}{\partial r}\right) + \left(k_z^2 r + \frac{\ell^2}{r}\right) \phi(r) \quad , \tag{24}$$

$$M_p(r,r',\omega) = \sum_s 4\pi q^2{}_s \int dr_o r_o \times \sum_n \frac{\omega_n \dfrac{\partial F_s}{\partial E} + \ell \dfrac{\partial F_s}{\partial P_\theta} + k_z \dfrac{\partial F_s}{\partial P_z}}{\omega - \omega_n(E,P_\theta,P_z)} P_n'(r,r') \quad , \tag{25}$$

$$P_n'(r,r') = p_n^{+*}(r)\, p_n(r') \quad , \tag{26}$$

$$p_n^+(r) = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} d\tau' \exp i\nu_n^+(\tau')\, \delta[r - r_o - \delta r^+(\tau')] \quad , \tag{27}$$

$$\nu_n^{\pm}(\tau) = -\omega_n \tau + \ell \delta\theta^{\pm}(\tau) + k_z \delta z^{\pm}(\tau) \quad ,$$

$$\omega_n = n\bar{\omega}_c + \ell\bar{\dot{\theta}} + k_z \bar{v}_z \quad ,$$

where $\bar{\omega}_c$, $\bar{v}_z$, and $\bar{\dot{\theta}}$ are the averaged gyrofrequency, axial velocity, and angular velocity respectively, over a radial bounce (the explicit integrals are obvious generalizations of the averaged quantities in the slab case).

Now using a similar grouping of conjugate trajectories as in the slab case, we find we can replace $P_n'(r,r')$ in Eq. (25) with

$$P_n(r,r') = \frac{1}{2}\,[p_n^{+*}(r)\; p_n(r') + p_n^{-*}(r)\; p_n^{-}(r')] \quad . \tag{28}$$

From Eq. (10) it then follows that $p_n^{-}(r) = p_n^{+}(r)$, so that

$$P_n(r,r') = \frac{1}{2}\,[p_n^{+*}(r)\; p_n(r') + p_n^{+}(r)\; p_n^{+*}(r')] \quad . \tag{29}$$

Thus it is explicitly demonstrated that r and r′ enter the kernel symmetrically, and hence the kernel is self-adjoint.

V. ELECTROMAGNETIC KERNEL

We now analyze the self-adjointness structure of the electromagnetic case. We first consider a plasma slab and choose a gauge where $\phi = 0$ and the pertrubed vector is

$$\underset{\sim}{A}(\underset{\sim}{x}) = [A_1(x)\hat{\underset{\sim}{x}} + A_2(x)\hat{\underset{\sim}{y}} + A_3(x)\hat{\underset{\sim}{z}}] \exp\left[i(k_y y + k_z z)\right] \quad , \tag{30}$$

From the work of Kaufman,[12] or Symon et al.,[6,7] one can deduce that the Vlasov-Maxwell integral equations are of the form

$$\sum_{j=1}^{3} \int dx' [L_{vij}(x,x',\omega) + L_{pij}(x,x',\omega)]\, A_j(x') = 0 \quad , \tag{31}$$

where

$$\underset{\sim}{L}_v(x,x',\omega) = \begin{Bmatrix} k_y^2 + k_z^2 - \frac{\omega^2}{c^2} + \sum_s \frac{\omega_{ps}^2}{c^2} & , -ik_y \frac{\partial}{\partial x'} & , -ik_z \frac{\partial}{\partial x'} \\ ik_y \frac{\partial}{\partial x} & , k_z^2 - \frac{\omega^2}{c^2} + \frac{\partial}{\partial x}\frac{\partial}{\partial x'} + \sum_s \frac{\omega_{ps}^2}{c^2} & , -k_y k_z \\ ik_z \frac{\partial}{\partial x} & , -k_y k_z & , k_y^2 - \frac{\omega^2}{c^2} + \frac{\partial}{\partial x}\frac{\partial}{\partial x'} + \sum_s \frac{\omega_{ps}^2}{c^2} \end{Bmatrix} \delta(x-x') \quad , \tag{32}$$

with $\omega_{ps}^2 = 4\pi N_o q_s^2/m_s$ (also note that derivatives only act on the delta function),

$$L_{prs}(x,x',\omega) = -\sum_s \frac{4\pi q_s^2}{c^2} \int d\underset{\sim}{v}_o \int dx_o$$

$$\times \sum_n \frac{\left(\omega_n \frac{\partial F_s}{\partial E} + k_y \frac{\partial F_s}{\partial p_y} + k_z \frac{\partial F_s}{\partial p_z}\right)}{\omega - \omega_n} J_{rs}(x,x'|n) \quad ,(33)$$

$$J'_{rs}(x,x'|n) = j_r^{+*}(x|n)\ j_s^+(x'|n) \quad , \quad (34)$$

$$j_r^+(x|n) = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} d\tau'\ v_r(\tau')\ \exp i\Omega_n^+(\tau')$$

$$\delta[x - x_o - \delta x^+(\tau')] \quad , \quad (35)$$

where $v_1 = v_x$, $v_2 = v_y$, and $v_3 = v_z$. We note that $L_{vrs}(x,x',\omega)$ has the self-adjoint symmetry,

$$L_{vrs}(x,x',\omega) = \sigma_r\sigma_s\ L_{vrs}(x',x,\omega) \quad ,$$

with $\sigma_r = 1$, $r \neq 1$, and $\sigma_1 = -1$. We need to show that $L_{prs}(x,x',\omega)$ satisfies the same relation. This follows by pairing the response of conjugate trajectories as we did in the previous section. We then find we can replace $J'_{rs}(x,x'|n)$ in Eq. (33) by

$$J_{rs}(x,x'|n) = \frac{1}{2}\left[j_r^{+*}(x|n)\ j_s^{+}(x'|n) + j_r^{-*}(x|n)\ j_s^{-}(x'|n)\right] \quad , \qquad (36)$$

Using the symmetry of conjugate trajectories it then follows that

$$j_r^{-}(x|n) = \sigma_r[j_r^{+}(x|n)]^* \quad , \qquad (37)$$

Hence J $(x,x'|n)$ is given by

$$J_{rs}(x,x'|n) = \frac{1}{2}\left[j_r^{*}(x|n)\ j_s(x'|n) + \sigma_r\sigma_s\, j_r(x|n) j_s^{*}(x'|n)\right] \ , \qquad (38)$$

where we have suppressed the superscript "+" symbol. This is precisely the symmetry we seek.

Analogous results are obtained in cylindrical geometry. Again choosing a gauge where $\phi = 0$ and the pertured vector potential is

$$\underset{\sim}{A}(\underset{\sim}{r}) = \left[A_1(r)\hat{\underset{\sim}{r}} + A_2(r)\hat{\underset{\sim}{\theta}} + A_3(r)\hat{\underset{\sim}{z}}\right] \exp\left[i(\ell\theta + k_z z)\right] \quad , \qquad (39)$$

the integral equations are of the form

$$\sum_{j=1}^{3} \int dr' \left[M_{vij}(r,r',\omega) + M_{pij}(r,r',\omega)\right] A_j(r') = 0 \quad . \qquad (40)$$

The terms in the cylindrical kernel are defined by

$$\underset{\approx}{M}(r,r',\omega) = \left\{ \begin{matrix} \frac{\ell^2}{(rr')^{1/2}} + \left(k_z^2 - \frac{\omega^2}{c^2} + \sum_s \frac{\omega_{p_s}^2}{c^2}\right)(rr')^{1/2}, & -i\ell \frac{r'}{r} \frac{\partial}{\partial r'}, & -ik_z r \frac{\partial}{\partial r'} \\ i \frac{\ell r}{r'} \frac{\partial}{\partial r}, & rr' \frac{\partial}{\partial r} \frac{\partial}{\partial r'} \frac{1}{(rr')^{1/2}} + \left(k_z^2 - \frac{\omega^2}{c^2} + \sum_s \frac{\omega_{p_s}^2}{c^2}\right)(rr')^{1/2}, & -k_z \ell \\ ik_z r' \frac{\partial}{\partial r}, & -k_z \ell, & \frac{\ell^2}{(rr')^{1/2}} + \frac{\partial}{\partial r} \frac{\partial}{\partial r'} (rr')^{1/2} + \left(-\frac{\omega^2}{c^2} + \sum_s \frac{\omega_{p_s}^2}{c^2}\right)(rr')^{1/2} \end{matrix} \right\} \delta(r-r') \quad (41)$$

and

$$M_{pij}(r,r',\omega) = -\sum_s \frac{4\pi q_s^2}{c^2} \int dr_o r_o \, d\underset{\sim}{v}_0 \times \sum_n \frac{\left(\omega_n \frac{\partial F_s}{\partial E} + \ell \frac{\partial F_s}{\partial P_\theta} + k_z \frac{\partial F_s}{\partial P_z}\right)}{\omega - \omega_n} P_{ij}(r,r'|n) \quad , \quad (42)$$

where

$$P_{ij}(r,r'|n) = \frac{1}{2}\left[p_i^{+*}(r|n)\; p_j^{+}(r'|n) + \sigma_i\sigma_j\; p_i^{+}(r|n)\; p_j^{+*}(r'|n)\right] , \quad (43)$$

$$p_i^{+}(r|n) = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} d\tau' \; v_i(\tau') \exp i\nu_n^{+}(\tau') \times \delta[r - r_o - \delta r^{+}(\tau')] \quad , \quad (44)$$

and $v_1 = v_r$, $v_2 = v_\theta$, and $v_3 = v_z$.

Other representations of the electromagnetic Vlasov kernel may be useful in applications and it is interesting to see how other representations can maintain self-adjointness. For simplicity we consider only transformations of the slab result.

Suppose

$$A_i(x) = \sum_j U_{ij}\left(x, -\frac{\partial}{\partial x}\right) A_j'(x) \quad , \tag{45}$$

where $A_i(x)$ is the solution of

$$\sum_j \int dx' \, L_{ij}(x,x',\omega) \, A_j(x') = 0 \quad , \tag{46}$$

and $L_{ij}(x,x',\omega)$ is self-adjoint, i.e.,

$$L_{ij}(x,x',\omega) = \sigma_i \sigma_j \, L_{ji}(x',x,\omega) \quad .$$

Then, we shall prove that $A_i'(x)$ is the eigensolution of the self-adjoint operator

$$L_{ij}'(x,x',\omega) = \sum_{m,n} \sigma_i \sigma_m \, U_{m\,i}\left(x, \frac{\partial}{\partial x}\right) \times U_{n\,j}\left(x', \frac{\partial}{\partial x'}\right) L_{mn}(x,x',\omega) \quad , \tag{47}$$

where the derivatives act only on $L_{mn}(x,x',\omega)$.

First we prove that $A_i'(x)$ is the eigensolution of $L_{ij}'(x,x', \omega)$. From Eq. (45) and (46) we have

$$\sum_{n,j} \int dx' \, L_{ij}(x,x',\omega) \, U_{jn}\left(x', -\frac{\partial}{\partial x'}\right) A_n'(x') = 0 \quad .$$

Integrating by parts yields

$$\sum_{n,j} \int dx' \, U_{jn}\left(x', \frac{\partial}{\partial x'}\right) L_{ij}(x,x',\omega) \, A_n'(x') = 0 \quad .$$

Now multiplying by $\sigma_i \sigma_m \, U_{im}(x, \partial/\partial x)$ and summing over i yields

$$\sum_{n,i,j} \int dx' \, \sigma_i \sigma_m \, U_{im}\left(x, \frac{\partial}{\partial x}\right) U_{jn}\left(x', \frac{\partial}{\partial x'}\right) L_{ij}(x,x',\omega) \, A_n'(x')$$

$$= \sum_n \int dx' \, L_{mn}'(x,x',\omega) \, A_n'(x') = 0 \quad .$$

Now we prove that $L_{ij}'(x,x',\omega)$ is self-adjoint $[L_{ji}'(x',x,\omega) = \sigma_i \sigma_j \, L_{ij}'(x,x',\omega)]$. Using $\sigma_i^2 = 1$, we have,

$$L_{ji}'(x',x,\omega) = \sum_{m,n} \sigma_j \sigma_m \, U_{mj}\left(x', \frac{\partial}{\partial x'}\right) U_{ni}\left(x, \frac{\partial}{\partial x}\right) L_{mn}(x',x,\omega)$$

$$= \sigma_i \sigma_j \sum_{m,n} \sigma_i \sigma_n \sigma_m^2 \, U_{ni}\left(x, \frac{\partial}{\partial x}\right) U_{mj}\left(x', \frac{\partial}{\partial x'}\right) L_{nm}(x,x',\omega)$$

$$= \sigma_i \sigma_j \, L_{ij}'(x,x',\omega) \quad \text{QED} \quad .$$

Specific examples of alternate bases are given in the Appendix.

VI. DIFFERENTIAL OPERATOR REPRESENTATION OF PLASMA RESPONSE

In this section we show how the plasma response can be written as a differential equation of infinite order. A truncation of this equation to finite order leads to a natural finite Larmor radius expansion. We discuss the electrosatic slab case in detail and then present the results for the electromagnetic slab case.

From Eqs. (14) and (20) we have that the perturbed electrostatic charge density for the slab model is

$$\delta\rho(x) = \frac{1}{2}\int d\underset{\sim}{v}_o\, dx_o\, dx' \sum_{n,s} \left\{ \begin{matrix} \rho_n^{+*}(x)\ G_{ns}(E,P_y,P_z,\omega)\ \rho_n^{+}(x') \\ + \\ \rho_n^{+}(x)\ G_{ns}(E,P_y,P_z,\omega)\ \rho_n^{+*}(x') \end{matrix} \right\} \phi(x') \quad ,(48)$$

where

$$G_{ns} = 4\pi q_s^2 \frac{\left(\omega_n \frac{\partial F_s}{\partial E} + k_y \frac{\partial F_s}{\partial P_y} + k_z \frac{\partial F_s}{\partial P_z}\right)}{\omega - \omega_n} \quad , \qquad (49)$$

and $\rho_n^{+}(x)$ is given by Eq. (16).

We perform the x' integration by using the following identities

$$\exp\left(a \frac{\partial}{\partial x}\right) f(x) = f(x + a) \quad ,$$

$$\int_{-\infty}^{+\infty} dx\ g(x) \exp\left(a \frac{\partial}{\partial x}\right) f(x) = \int_{-\infty}^{+\infty} dx\ f(x) \exp\left(-a \frac{\partial}{\partial x}\right) g(x) \quad .$$

The first identity follows from a power series expansion of each side. The second identity follows from a power series expansion of $\exp[a\,(\partial/\partial x)]f(x)$, integrating by parts to transfer the derivative operations onto g, and resumming.

Hence we have [deleting the superscripts in $\delta x^{+}(\tau,x_o)$ and $\Omega_n^{+}(\tau)$, and denoting the time average by brackets]

$$\langle \exp i\Omega_n(\tau)\ \delta[x-x_o-\delta x(\tau,x_o)\ ]\rangle = \langle \exp[i\Omega_n(\tau)-\delta x(\tau,x_o)\frac{\partial}{\partial x}]\ \delta(x - x_o)\rangle \quad .$$

Thus,

$$\delta\rho(x) = \frac{1}{2}\int d\underset{\sim}{v}_o dx_o dx' \sum_{n,s} \Big\{ \langle \exp\left[-i\Omega_n(\tau) + \delta x(\tau)\frac{\partial}{\partial x_o}\right] \delta(x-x_o)\rangle\, G_{ns} \langle \exp\left[i\Omega_n(\tau) - \delta x(\tau)\frac{\partial}{\partial x'}\right] \delta(x'-x_o)\rangle$$

$$+ \int \langle \exp\left[i\Omega_n(\tau) + \delta x(\tau)\frac{\partial}{\partial x_o}\right] \delta(x-x_o)\rangle\ G_{ns} \langle \exp\left[-i\Omega_n(\tau) - \delta x(\tau)\frac{\partial}{\partial x'}\right] \delta(x'-x_o)\rangle \Big\}\ \phi(x')$$

$$= \frac{1}{2}\int d\underset{\sim}{v}_o dx_o dx' \sum_{n,s}\Big\{<\delta(x-x_o)\exp\Big[-\frac{\partial}{\partial x_o}\delta x(\tau,x_o)-i\Omega_n(\tau)\Big]> G_{ns} <\delta(x'-x_o)\exp\Big[i\Omega_n(\tau)+\delta x(\tau,x_o)\frac{\partial}{\partial x'}\Big]>$$

$$+ <\delta(x-x_o)\exp\Big[-\frac{\partial}{\partial x_o}\delta x(\tau) + i\Omega_n(\tau)\Big]>G_{ns} <\delta(x'-x_o) \exp\Big[-i\Omega_n(\tau)+\delta x(\tau)\frac{\partial}{\partial x'}\Big]>\Big\}\phi(x'). \qquad (50)$$

Now, the delta functions do not have operators acting on them, so that x_o and x' integrations can be trivially performed to yield,

$$\delta\rho(x) = \frac{1}{2}\int d\underset{\sim}{v}_o \sum_{n,s}\Big\{<\exp\Big[-\frac{\partial}{\partial x}\delta x(\tau, x) - i\Omega_n(\tau)\Big]> G_{ns} <\exp\Big[i\Omega_n(\tau)+\delta x(\tau,x)\frac{\partial}{\partial x}\Big]>$$

$$+ <\exp\Big[-\frac{\partial}{\partial x}\delta x(\tau,x) + i\Omega_n(\tau)\Big]> G_{ns} <\exp\Big[-i\Omega_n(\tau)+\delta x(\tau,x)\frac{\partial}{\partial x}\Big]>\Big\}\phi(x). \qquad (51)$$

One can readily ascertain that this is a self-adjoint differential form. We emphasize this result in view of the fact that one often encounters non-self-adjoint differential equations in the literature. If formally $\Omega_n(\tau) \simeq \delta x(\tau)(\partial/\partial x) \ll 1$, we can expand $\exp[\pm i\Omega_n(\tau) \pm \delta x(\tau)(\partial/\partial x)]$

in a power series and truncate to second order. This would incorporate the conventional finite Larmor radius differential equation but would be more general as it would be valid for arbitrary frequency. A more accurate expansion would be to keep $\exp[i\Omega_n(\tau)]$ exact and expand only in the differential operator.

The more complicated electromagnetic plasma response [Eq. (31)] may be obtained in the same manner as Eq. (51). Deleting the manipulations, we find the result

$$\sum_j \int dx' \, L_{pij}(x,x',\omega)\, A_j(x') = -\frac{1}{2c^2} \sum_j \iint d\underset{\sim}{v}_o \sum_{n,s} \Big[\Big\langle \exp\Big(-\frac{\partial}{\partial x}\,\delta x\Big)\, v_i \exp(-i\Omega_n)\Big\rangle$$

$$\times\; G_{ns} \Big\langle v_j \exp(i\Omega_n) \exp\Big(\delta x \frac{\partial}{\partial x}\Big)\Big\rangle + \sigma_i\sigma_j \Big\langle \exp\Big[-\frac{\partial}{\partial x}\,\delta x\Big]\, v_i \exp(i\Omega_n)\Big\rangle$$

$$\times\; G_{ns} \Big\langle v_j \exp(-i\Omega_n) \exp\Big(\delta x \frac{\partial}{\partial x}\Big)\Big\rangle\Big]\, A_j(x) \quad , \qquad (52)$$

which is manifestly self-adjoint.

To illustrate our formalism we consider the simple case of a plasma situated in a uniform external magnetic field $\underset{\sim}{B} = B\hat{\underset{\sim}{z}}$, with no external forces. The time averages in Eqs. (51) and (52) may be readily computed from the Cartesian velocity components given by

$$v_x(\tau) = v_\perp \cos(\theta_o - \Omega_s\tau), \quad v_y(\tau) = v_\perp \sin(\theta_o - \Omega_s\tau)\ , \quad v_z(\tau) = \text{const.} \qquad (53)$$

where $\Omega_s = q_s B/m_s c$.
Integrating Eq. (53) to obtain the trajectories and substituting the results in Eq. (51), we find the perturbed charge density,

$$\delta\rho(x) = \int dv_\perp v_\perp d\theta_o dv_\parallel \sum_{n,s} \exp\left(-\frac{\partial}{\partial x} a_s \sin\theta_o\right) [J_n \exp(in\lambda)]$$

$$\times G_{ns} [J_n \exp(-in\lambda)] \exp\left(a_s \sin\theta_o \frac{\partial}{\partial x}\right) \phi(x)$$

where J_n is the nth order Bessel function differential operator $J_n = J_n\{[(-d^2/dx^2) + k_y^2]^{1/2} a_s\}$, $a_s = v_\perp/\Omega_s$ is the gyroradius, $\Omega_s = q_s B/m_s c$ is the gyrofrequency, $\theta_o = \theta(t = 0)$, and λ is a differential operator defined by $\tan\lambda = (-i/k_y)(\partial/\partial x)$. Similiarly, for the electromagnetic response we find,

$$\sum_j \int dx' \, L_{pij}(x,x',\omega) \, A_j(x') = -\frac{1}{2c^2} \sum_j \int d\underset{\sim}{v}_o \sum_{n,s} \left\{\exp\left(-\frac{\partial}{\partial x} a_s \sin\theta_o\right) [q_{in} \exp(in\lambda)]\right.$$

$$\times G_{ns} \exp\left(a_s \sin\theta_o \frac{\partial}{\partial x}\right) \tilde{q}_{jn} \exp(-in\lambda)$$

$$+ \sigma_i\sigma_j \exp\left(-\frac{\partial}{\partial x} a_s \sin\theta_o\right) \tilde{q}_{in} \exp(in\lambda)$$

$$\left.\times G_{ns} \exp\left(a_s \sin\theta_o \frac{\partial}{\partial x}\right) q_{jn} \exp(-in\lambda)\right\} A_j(x) \quad , \qquad (55)$$

where

$$q_n = \begin{Bmatrix} i \dfrac{v_\perp}{2} [J_{n-1} \exp(-i\lambda) - J_{n+1} \exp(i\lambda)] \\ \dfrac{v_\perp}{2} [J_{n-1} \exp(-i\lambda) + J_{n+1} \exp(i\lambda)] \\ v_z J_n \end{Bmatrix} \qquad (56)$$

$$\underset{\sim}{p}_n = \begin{Bmatrix} i\,\frac{v_\perp}{2}\,[J_{n-1}\exp(i\lambda) - J_{n+1}\exp(-i\lambda)] \\ \frac{v_\perp}{2}\,[J_{n-1}\exp(i\lambda) + J_{n+1}\exp(-i\lambda)] \\ v_z J_n \end{Bmatrix} \tag{56}$$

with the operators $\tilde{\underset{\sim}{q}}_n$ and $\tilde{\underset{\sim}{p}}_n$ in Eq. (55) being obtained by the replacement $i \rightarrow -i$ in Eq. (56). The plasma response for more complicated cases may be calculated similiarly once the trajectories are specified.

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APPENDIX

Here we explicitly display the kernel for two alternate bases. The first is for the case where the vector potential components are along and perpendicular to the field lines. The vector potential is given by

$$\underset{\sim}{A}(x) = A_1'(x)\hat{\underset{\sim}{x}} + A_2'(x)(\hat{\underset{\sim}{b}} \times \hat{\underset{\sim}{x}}) + A_3'(x)\ \hat{\underset{\sim}{b}}(x) \quad , \tag{A1}$$

where $\hat{b}(x) = \underset{\sim}{B}(x)/B(x)$. For this case $U_{ij}(x)$ is

$$U_{ij}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b_z(x) & -b_y(x) \\ 0 & b_y(x) & b_z(x) \end{pmatrix} \quad , \tag{A2}$$

where $b_{z,y}(x) = B_{z,y}(x)/B(x)$. By directly applying the transformation to Eq. (31), we obtain

$$L'_{vij} = \left\{ \begin{matrix} k_y^2 + k_z^2 - \frac{\omega^2}{c^2} + \sum_s \frac{\omega_{ps}^2}{c^2} & , & -ik_\eta(x')\frac{\partial}{\partial x'} & , & -ik_\xi(x')\frac{\partial}{\partial x'} \\ ik_\eta(x)\frac{\partial}{\partial x} & , & \frac{\partial}{\partial x}\frac{\partial}{\partial x'} + \sum_s \frac{\omega_{ps}^2}{c^2} + k_\xi(x)\ k_\xi(x') - \frac{\omega^2}{c^2} & , & -k_\eta(x)\ k_\xi(x') \\ ik_\xi(x)\frac{\partial}{\partial x} & , & -k_\eta(x')\ k_\xi(x) & , & \frac{\partial}{\partial x}\frac{\partial}{\partial x'} + \sum_s \frac{\omega_{ps}^2}{c^2} + k_\eta(x)\ k_\eta(x') - \frac{\omega^2}{c^2} \end{matrix} \right\} \delta(x - x') \quad , \tag{A3a}$$

$$L'_{pij} = -\sum_s \frac{4\pi q_s^2}{c^2} \int d\underset{\sim}{v}_o$$

$$\times \left[\int dx_o \sum_n \frac{\left(\omega_n \frac{\partial F_s}{\partial E} + k_y \frac{\partial F_s}{\partial E_y} + k_z \frac{\partial F_s}{\partial P_z}\right)}{\omega - \omega_n}\right] G_{ij}(x,x',n) \quad , \qquad \text{(A3b)}$$

and $$G_{ij}(x,x',n) = \frac{1}{2}\left[g_i^*(x|n)\, g_j(x'|n) + \sigma_i g_i(x|n)\, g_j^*(x|n)\, \sigma_j\right]$$

$$g_i(x|n) = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} d\tau'\, v_i'(\tau') \exp i\Omega_n^+(\tau')\, \delta[x - x_o - \delta x(\tau)] \quad , \qquad \text{(A4)}$$

$$k_\eta(x) = k_y b_z(x) - k_z b_y(x) \;,\; k_\xi(x) = k_y b_y(x) + k_z b_z(x) \quad ,$$

$$v_\eta = \underset{\sim}{v} \cdot (\hat{\underset{\sim}{b}} \times \hat{\underset{\sim}{x}}) \;,\; v_\xi = \underset{\sim}{v} \cdot \hat{\underset{\sim}{b}} \;,\; v_1' = v_x \;,\; v_2' = v_\eta \;,\; v_3' = v_\xi \quad ,$$

A second representation for $\underset{\sim}{A}$, related to the one used by Berk and Dominguez,[1] is

$$\underset{\sim}{A} = -\underset{\sim}{\nabla}_\perp \phi_1' + \hat{\underset{\sim}{b}} \times \underset{\sim}{\nabla}\phi_2' + \phi_3' \hat{\underset{\sim}{b}} \quad , \qquad \text{(A5)}$$

where $\underset{\sim}{\nabla}_\perp = \underset{\sim}{\nabla} - \hat{\underset{\sim}{b}}\hat{\underset{\sim}{b}} \cdot \underset{\sim}{\nabla}$. The relation between the ϕ_i' and A_i array is

$$A_i(x) = \sum_j U_{ij}\left(x, -\frac{\partial}{\partial x}\right) \phi_j'(x) \quad ,$$

where $A_1 = \underset{\sim}{A} \cdot \hat{\underset{\sim}{x}}$, $A_2 = \underset{\sim}{A} \cdot (\hat{\underset{\sim}{b}} \times \hat{\underset{\sim}{x}})$, $A_3 = \underset{\sim}{A} \cdot \hat{\underset{\sim}{b}}$ and

$$U_{ij}\left(x, \frac{\partial}{\partial x}\right) = \begin{pmatrix} \frac{\partial}{\partial x} & -ik_\eta & 0 \\ -ik_\eta & -\frac{\partial}{\partial x} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \qquad (A6)$$

The self-adjoint operator for the ϕ' array is then,

$$K_{ij}(x,x',\omega) = \sum_{m,n} \sigma_i \sigma_m U_{mi}\left(x, \frac{\partial}{\partial x}\right) U_{nj}\left(x', \frac{\partial}{\partial x'}\right) L'_{mn}(x,x',\omega) \quad , (A7)$$

Applying these operations we then explicitly find for the local contribution,

$$
\underset{\approx}{K}_V = \left\{ \begin{array}{c|c|c}
\frac{\partial}{\partial x}\frac{\partial}{\partial x'}\left[k^2 - \frac{\omega^2}{c^2} + \sum_s \frac{\omega_{p_s}(x)\omega_{p_s}(x')}{c^2}\right] - [k_\eta^2(x) + k_\eta^2(x')]\frac{\partial}{\partial x}\frac{\partial}{\partial x'} & -ik_\eta(x')\frac{\partial}{\partial x}\left(k^2 - \frac{\omega^2}{c^2} + \sum_s \frac{\omega_p^2(x)}{c^2}\right) + ik_\eta^2(x)k_\eta(x')\frac{\partial}{\partial x} & -ik_\eta(x)\,[k_\eta(x)k_\xi(x')] \\
+k_\eta(x)k_\eta(x')\left[\frac{\partial}{\partial x}\frac{\partial}{\partial x'} + k_\xi(x)k_\xi(x') - \frac{\omega^2}{c^2} + \sum_s \frac{\omega_{ps}^2}{c^2}\right] & +i\frac{\partial}{\partial x'}\Big\{[k_\eta(x') - k_\eta(x)]\frac{\partial}{\partial x}\frac{\partial}{\partial x'} & -ik_\xi(x')\frac{\partial}{\partial x}\frac{\partial}{\partial x'} \\
 & -k_\eta(x)\left[k_\xi(x)k_\xi(x') - \frac{\omega^2}{c^2} + \sum_s \frac{\omega_{p_s}^2(x)}{c^2}\right]\Big\} & \\
\hline
ik_\eta(x)\frac{\partial}{\partial x'}\left[k^2 - \frac{\omega^2}{c^2} - \sum_s \frac{\omega_p^2(x')}{c^2}\right] - ik_\eta^2(x')k_\eta(x)\frac{\partial}{\partial x'} & k_\eta(x)k_\eta(x')\left(k^2 - \frac{\omega^2}{c^2}\right) - \sum_s \omega_{p_s}(x)\frac{\omega_{p_s}(x')}{c^2} - \frac{\partial}{\partial x'}k_\eta(x)k_\eta(x')\frac{\partial}{\partial x'} & k_\eta(x)k_\xi(x')\frac{\partial}{\partial x'} \\
-i\frac{\partial}{\partial x}\Big\{[k_\eta(x) - k_\eta(x')]\frac{\partial}{\partial x}\frac{\partial}{\partial x'} - k_\eta(x')\left[k_\xi(x)k_\xi(x') - \frac{\omega^2}{c^2} - \sum_s \frac{\omega_p^2(x')}{c^2}\right]\Big\} & -\frac{\partial}{\partial x}k_\eta(x')k_\eta(x)\frac{\partial}{\partial x} + \frac{\partial}{\partial x}\frac{\partial}{\partial x'}\Big[\frac{\partial}{\partial x}\frac{\partial}{\partial x'} & +\frac{\partial}{\partial x}\,[k_\eta(x)k_\xi(x')] \\
 & + k_\xi(x)k_\xi(x') - \frac{\omega^2}{c^2} + \sum_s \frac{\omega_{p_s}^2}{c^2}\Big] & \\
\hline
ik_\xi(x)\frac{\partial}{\partial x}\frac{\partial}{\partial x'} + ik_\eta(x)\,[k_\eta(x')k_\xi(x)] & k_\eta(x')k_\xi(x)\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}k_\eta(x')k_\xi(x) & \frac{\partial}{\partial x}\frac{\partial}{\partial x'} + k_\eta(x)k_\eta(x') - \frac{\omega^2}{c^2} + \sum_s \frac{\omega_{p_s}^2}{c^2}
\end{array} \right\} \delta(x - x') \qquad \text{(A8)}
$$

where the derivative inside the array act on all quantities to the right, but terminate after the delta function and $k^2 = k_y^2 + k_z^2$. The contribution from the plasma response is,

$$K_{pij}(x,x',\omega) = -\sum_s \frac{4\pi q_s^2}{c^2} \int d\underset{\sim}{v}_o \, dx_o \sum_{n,s}$$

$$\times \frac{\left(\omega_n \frac{\partial F_s}{\partial E} + k_y \frac{\partial F_s}{\partial P_y} + k_z \frac{\partial F_s}{\partial P_z}\right)}{\omega - \omega_n} Q_{ij}(x,x'|n) \qquad \text{(A9)}$$

where

$$Q_{ij}(x,x',|n) = \frac{1}{2}\left[q_i^{+*}(x|n)\, q_j^{+}(x'|n) + \sigma_i \sigma_j q_j^{+}(x'|n)\, q_j^{+*}(x'|n)\right] ,$$

$$\begin{Bmatrix} q_1^+(x|n) \\ q_2^+(x|n) \\ q_3^+(x|n) \end{Bmatrix} = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} d\tau' \begin{Bmatrix} \frac{\partial}{\partial x} v_x^+(\tau') - ik_\eta(x)\, v_\eta^+(\tau') \\ -ik_\eta(x)\, v_x^+(\tau') - \frac{\partial}{\partial x} v_\eta^+(\tau') \\ v_\xi^+(\tau') \end{Bmatrix} \exp i\Omega_n^+(\tau')\, \delta[x - x_o - \delta x^+(\tau')]$$

(A10)

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