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PRESSURE INDUCED ISLANDS IN THREE-DIMENSIONAL TOROIDAL PLASMA

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Dr. Fred L. Ribe
Editor, The Physics of Fluids
AERL Building, FL–10
University of Washington
Seattle, WA 98195

Dear Dr. Ribe:

I have read the comments of the referee for the paper FP–14242, "Pressure Induced Islands in Three-Dimensional Toroidal Plasmas" by John R. Cary and M. Kotschenreuther. The referee's comments indicate that he did not carefully read the original manuscript. Every comment was either inconsequential or addressed in the original manuscript. Nevertheless, I have taken this to be evidence that the manuscript could be improved with more emphasis of certain key issues. Therefore, I have rewritten the manuscript slightly.

A point by point dicussion of the referee's comments follows. For this discussion I have labeled the substantive comments on the second page of the referee's report by continuing the numbering.

1. The small island approximation relies on the island being narrow relative to the smaller of the plasma radius and $k_\perp^{-1}$. This is now explicitly stated in Eq. (30).

2. Given the additional discussion of App. A, the existence of an intermediate regime is now manifest as discussed in the vicinity of Eqs. (65).

3. One need not assume ℓ to be large for the intermediate regime to exist. One needs only that $\varepsilon_{\ell m}$ be small as is shown to be the case in App. A.

3. and 4. The finite β corrections to $\varepsilon_{\ell m}$ are, of course, neglected in the low–β theory. Only at sufficiently high–β, a regime beyond the scope of this paper, do these corrections become important. In fact, no definitive calculation has as yet given the corrections to the high-order $\varepsilon_{\ell m}$'s to high order in β. Therefore, the statement made by the referee cannot be substantiated.

5. This statement contradicts itself. One cannot call a renormalized theory a linear theory. In any case, the island calculation is definitely nonlinear, as discussed in Ref. 22, Sec. 3.2.

6. This claim was previously substantiated in Sec. IVB, which has been relabeled Sec. V.

7. The large aspect ratio and associated approximations were used only for purposes of estimation, e.g., Secs. IID and IIE.

8. The referee apparently missed the point that the threshold was due to considering only low-order islands.

9. The present analysis developed entirely independently of the recent work of Reiman and Boozer. Still, we do not object to adding a reference. A brief description of their work is included in Sec. VI.

I ask that you reconsider this manuscript for publication in <u>The Physics of Fluids</u>. I also request that you send it to a new referee.

Finally, I ask that you send future correspondence to me at my new address: Department of Astrophysical, Planetary, and Atmospheric Sciences, Campus Box 391, University of Colorado, Boulder, Colorado 80309.

Thank you for your consideration.

Sincerely yours,

John R. Cary

John R. Cary

/sc
Enc.

Pressure Induced Islands in Three-Dimensional Toroidal Plasma

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ABSTRACT

The production of magnetic islands by plasma pressure in three-dimensional toroidal systems is analyzed. Far from the rational surfaces a procedure based on linearization in the plasma pressure applies. This yields the solution in terms of δ-function currents at the surface. These currents are found by a nonlinear analysis valid in the vicinity of the island. The result is a set of coupled nonlinear equations determining the island widths. Scaling is found by using the approximation of nearly circular flux surfaces. The results indicate that for typical stellarators, which have a small ratio $\iota\ell_0/m_0$ of field line rotational transform to coil rotational transform, the island size depends dramatically on whether a magnetic well is present. In this case, if a magnetic well is present, islands are insignificant; in contrast, if a magnetic hill is present, island overlap occurs for arbitrarily low pressure.

I. Introduction

The stellarator is a toroidal plasma confinement system[1] with rotational transform produced by external windings. It has the advantage that it may be operated steady state, since it needs no internal toroidal current to provide rotational transform. However, with this advantage comes the necessity of being fully three-dimensional, which greatly complicates the analysis to the point where no rigorous theory of three-dimensional plasma equilibria exists.

Indeed, as emphasized by Grad,[2] three-dimensional magnetic fields may not have good flux surfaces even in the absence of plasma. This has not been thought to be a practical problem because surface of section analyses of various systems[3-5] have shown the existence of fairly good surfaces. However, as discussed by Cary[6,7] these good surfaces all have rotational transform values much less than the theoretical limit of $\iota_s = m_0/\ell_0$ for an (ℓ_0, m_0) stellarator.

Recently, a practical solution to the problem of finding vacuum fields with dense flux surfaces has been presented.[8] This solution is based on a technique for identifying and measuring the stochasticity inducing islands produced by toroidal coupling. Numerical techniques are used to minimize the islands by varying the coil winding law. As a result, one obtains coil winding laws that produce magnetic fields with no visible stochasticity over a wide range of rotational transform and inverse aspect ratio ε.

With the vacuum field problem solved, it is necessary to ask what the effect of plasma will be. One would especially like to know whether the plasma will modify the magnetic field in such a way that islands and stochasticity are found. As a first cut it is reasonable to analyze this problem with the assumption of scalar pressure, in which case the equation of plasma equilibrium is $\underset{\sim}{J}\times\underset{\sim}{B} = \underset{\sim}{\nabla}P$. Such a theory is strictly applicable only for sufficiently large collision frequency, such that a particle scatters before it can complete a superbanana orbit.[9] However, such an analysis may point to features general to all regimes.

It should be noted that while previous analyses of three-dimensional equilibria exist, they have not addressed the question of islands. Either good surfaces were assumed from the outset[1,10] and consequences were derived, or the stellarator expansion[11] was used. The stellarator expansion cannot address the island question because the averaging procedure employed a priori ignores the existence of island structures.

The present analysis of the scalar pressure equilibrium equation $\underset{\sim}{J}\times\underset{\sim}{B} = \underset{\sim}{\nabla}P$ is based on a low pressure expansion about an integrable vacuum field, i.e., a magnetic field $\underset{\sim}{B}_0$ such that $\underset{\sim}{\nabla}\times\underset{\sim}{B}_0 = 0$, and for which there exists a flux function ψ such that field lines lie on the contour surfaces of ψ: $\underset{\sim}{B}_0\cdot\underset{\sim}{\nabla}\psi = 0$. In a way common to the analysis of nearly integrable Hamiltonian systems, one keeps in first order effects of only a finite number of the low-order resonant perturbations. Then, in regions far from the corresponding resonant surfaces one can find the plasma current by linearization.[12] From this "exterior" point of view, the narrow currents associated with the islands can be represented by δ-functions. To find these island currents one must solve $\underset{\sim}{J}\times\underset{\sim}{B} = \underset{\sim}{\nabla}P$ by a different technique, which relies on being close to a particular resonant surface, where the effects of the associated nonresonant fields may be neglected. Finally, by integrating Ampere's law to obtain the magnetic field perturbation $\nabla \times \mathbf{B}_1 = \mathbf{J}_1$, one finds a nonlinear equation for the island width.

In general, the resulting equation for the island width depends on details of the original vacuum field and the pressure profile. Moreover, complications arise because the perturbations of various helicities are coupled. To make further progress, decoupling is achieved by invoking the approximation of nearly circular flux surfaces. As a result one obtains a nonlinear algebraic equation governing the widths of the islands of the various helicities.

The resulting equation shows that at low values of β, the island width scales as $\beta^{1/2}$. Once β exceeds a transition value β_t, the island size depends markedly on whether the island is in a region where there is a magnetic well,[13,14] i.e., $d^2V/d\psi^2 < 0$, where V is the volume inside a flux surface ψ. The widths of islands in good regions, where a well is present, saturate at $\beta \sim \beta_t$; that is, the island width for $\beta > \beta_t$ is approximately the same as it is for $\beta = \beta_t$. Applied to the proposed stellarator AFT,[3] these results indicate that islands in the inner region will be insignificant.

Quite a different result is obtained for islands in regions where there is a magnetic hill. Instead of saturating, the island width scales linearly with β for $\beta > \beta_t$. More important is the scaling with mode number ℓ. For large ℓ, β_t approaches zero and the island width scales as β/ℓ. Thus, as one considers more rational surfaces $\iota = m/\ell$, $\ell=1,\ldots,L$, one always finds island overlap because the mean spacing between rationals scales as $1/L^2$. Therefore, the present results indicate that three-dimensional scalar pressure equilibria do not exist in regions where there is a magnetic hill.

These results may be interpreted in a more familiar way by first neglecting the 3D-terms, i.e., the terms due to the lack of symmetry. In this case a scalar pressure equilibrium with nested magnetic surfaces (i.e.,without resonant perturbations) exists. This equilibrium is stable (unstable) to resistive interchange modes at low β if there is a magnetic well (hill). The 3D-terms drive resonant perturbations which produce islands. In the stable case the islands are generally small not only because the 3D-terms are small, but also because of the effects of good curvature. In the unstable case the 3D-terms initiate a linear instability which grows until it saturates due to nonlinear effects. For sufficiently unstable systems, i.e., $\beta > \beta_t$, the island width is independent of the 3D-terms and is identical to that found for the width of saturated resistive instabilities of axisymmetric systems.[15]

The present calculation does not address what actually occurs in this instance, except to say that magnetic stochasticity arises, because the present calculation relies on the islands not overlapping. We expect that these effects lead to a soft β-limit, where as the plasma pressure is raised, stochasticity and plasma transport increase.

The outline of this paper is as follows. In Sec. II the exterior current is found by linearization. Finding the magnetic fields due to these currents is in general a difficult numerical problem. To make further progress and obtain scaling, we find these magnetic fields with the approximation of nearly circular flux surfaces. To complete the analysis, the island currents are found in Sec. III. This involves determining the pressure profile near the island, which is accomplished by considering the Pfirsch-Schluter transport in this region. In

Sec. IV the analyses of the interior and exterior regions are combined. This yields a nonlinear equation for the island width.

II. Exterior solution

When only a finite number of helicities are considered, one can define the exterior region as the region far from the corresponding resonant surfaces. In this region, the flux surfaces in the presence of the perturbation have the same topology as the vacuum flux surfaces. This allows one to find the plasma currents by linearization. In fully three-dimensional systems obtaining this linear solution is a difficult numerical problem. Therefore, Sec. IID shows how analytic solutions may be obtained when the flux surfaces are nearly circular.

A. Vacuum magnetic field coordinates

Being curl-free, a toroidal vacuum field can be written as the gradient of a multivalued scalar potential,

$$\underset{\sim}{B}_0 = \underset{\sim}{\nabla}\Phi \ .$$

According to Ampere's law, this scalar potential must increase by an amount $\Delta\Phi = I$ equal to the total current through the center of the torus. Thus we can use a multiple of the scalar potential,

$$\varphi \equiv \frac{2\pi}{I}\,\Phi \ ,$$

as a toroidal angle. With the definition $\gamma \equiv I/2\pi$, the vacuum field can be written

$$\underset{\sim}{B}_0 = \gamma\underset{\sim}{\nabla}\varphi \ . \tag{1}$$

To select the two remaining coordinates, we require our system to be a flux coordinate system[16] so that $\underset{\sim}{B}_0$ has the form

$$\underset{\sim}{B} = \underset{\sim}{\nabla}\psi \times \underset{\sim}{\nabla}\vartheta + ᵼ(\psi)\underset{\sim}{\nabla}\varphi \times \underset{\sim}{\nabla}\psi \ , \tag{2}$$

where $ᵼ(\psi)$ is the rotational transform. Thus, magnetic field lines stay on surfaces of constant ψ, where $2\pi\psi$ is the toroidal flux enclosed by the surface.

Similar coordinates were introduced by Boozer[10] for analysis of three-dimensional scalar-pressure equilibria. (We note, however, that in the present vacuum field case the toroidal angle is periodic with a ψ-independent period of 2π.) These coordinates differ from the more familiar Hamada coordinates,[17] in that the Jacobian is not a function of the surface alone. In fact, Hamada coordinates need not exist[18] for integrable, three-dimensional, vacuum magnetic fields.

We introduce these coordinates because of their simple metric properties. For example, the dot product of Eqs. (1) and (2) yields the relation,

$$B_0^2 = \gamma/\mathscr{J} \ , \tag{3}$$

between the vacuum field $\underset{\sim}{B}_0$ and the Jacobian $\mathscr{J} = (\underset{\sim}{\nabla}\psi \times \underset{\sim}{\nabla}\vartheta \cdot \underset{\sim}{\nabla}\varphi)^{-1}$. Furthermore, from Eq. (1) we find the covariant components of $\underset{\sim}{B}_0$,

$$B_{0\psi} = 0 \quad , \quad B_{0\vartheta} = 0 \quad , \quad B_{0\varphi} = \gamma \ , \tag{4}$$

while from Eq. (2) we find the contravariant components of $\underset{\sim}{B}_0$:

$$B_0^\psi = 0 \quad , \quad B_0^\vartheta = \iota/\mathcal{J} \quad , \quad B_0^\varphi = 1/\mathcal{J} \quad . \tag{5}$$

These results, together with the lowering operator, allow us to conclude that all the relevant metric information is known once one is given $\iota(\psi), \gamma$, and any three components of the metric tensor. For example, suppose one knows the metric elements $g_{\psi\psi}$, $g_{\psi\vartheta}$, and $g_{\vartheta\vartheta}$ as functions of ψ, ϑ and φ. Then the Jacobian is given by

$$\mathcal{J} = \gamma(g_{\psi\psi}g_{\vartheta\vartheta} - g_{\psi\vartheta}^2) \quad , \tag{6}$$

the magnetic field strength is given by Eq. (3), and the remaining elements of the metric tensor are given by

$$g_{\psi\varphi} = -\iota g_{\psi\vartheta} \quad , \tag{7a}$$

$$g_{\vartheta\varphi} = -\iota g_{\vartheta\vartheta} \quad , \tag{7b}$$

and

$$g_{\varphi\varphi} = \gamma\mathcal{J} + \iota^2 g_{\vartheta\vartheta} \quad . \tag{7c}$$

The Jacobian contains much of the relevant information about the vacuum magnetic field properties. For example, the specific volume $V' \equiv dV/d\psi$ is simply the surface integral of the Jacobian:

$$V'(\psi) = \int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi \; \mathcal{J}(\psi,\vartheta,\varphi) \quad . \tag{8}$$

Furthermore, the specific volume of a closed flux tube of rotational transform $\iota(\psi_{\ell m}) = m/\ell$ is given by

$$\oint \frac{d\ell}{B} = \frac{1}{\ell} \int_0^{2\pi\ell} d\varphi \; \mathcal{J}(\psi_{\ell m}, \vartheta + m\varphi/\ell, \varphi) \quad . \tag{9}$$

Thus, the "$\oint d\ell/B$-criterion" for magnetohydrodynamic equilibrium,[19] that the quantity of Eq. (9) be a surface function, is equivalent to requiring that the resonant amplitudes of the Fourier expansion of the Jacobian,

$$\mathscr{J}(\psi,\vartheta,\varphi) = \sum_{\ell,m} \mathscr{J}_{\ell m}(\psi) e^{i\ell\vartheta - im\varphi} , \tag{10}$$

vanish at resonance:

$$\mathscr{J}_{\ell m}(\psi_{\ell m}) = 0 . \tag{11}$$

This condition is <u>not</u> generally satisfied for vacuum magnetic fields.

B. Calculation of the current by linearization

To solve the magnetohydrodynamic equilibrium equation $\underset{\sim}{J}\times\underset{\sim}{B} = \underset{\sim}{\nabla}P$, we assume that P is small such that linearization is possible

$$\underset{\sim}{J}_1\times\underset{\sim}{B}_0 = \underset{\sim}{\nabla}P_1 .$$

Upon taking the dot product of this equation with $\underset{\sim}{B}_0$, we conclude, in the usual way,[1] that P_1 is a function of ψ alone. From the cross-product of this equation with $\underset{\sim}{B}_0$ we determine the perpendicular part of the current:

$$\underset{\sim}{J}_{1\perp} = P_1'(\psi) \ \mathscr{J}(\psi,\vartheta,\varphi)\underset{\sim}{\nabla}\varphi\times\underset{\sim}{\nabla}\psi . \tag{12}$$

It remains to find the parallel part of the current. To do so we write

$$\underset{\sim}{J}_{1\parallel} = Q(\psi,\vartheta,\varphi)\left[\underset{\sim}{\nabla}\psi\times\underset{\sim}{\nabla}\vartheta + \iota(\psi)\underset{\sim}{\nabla}\varphi\times\underset{\sim}{\nabla}\psi\right] . \tag{13}$$

With Fourier expansion,

$$Q = \sum_{\ell m} Q_{\ell m}(\psi)e^{i\ell\vartheta - im\varphi} , \tag{14}$$

the equation of charge conservation $\underset{\sim}{\nabla}\cdot\underset{\sim}{J} = 0$ becomes

$$[\ell\iota(\psi)-m]Q_{\ell m} = -\ell P_1'(\psi) \quad \mathscr{J}_{\ell m} . \tag{15}$$

Several aspects of this equation should be noted. First, the (ℓ=0, m=0) component is undetermined. This is usually required to vanish by assuming that the net toroidal current has died away in a steady-state stellarator. Second, the general solution of Eq. (15),

$$Q_{\ell m}(\psi) = \frac{-\ell P_1'(\psi) \ \mathscr{J}_{\ell m}(\psi)}{\ell\iota(\psi)-m} + \hat{Q}_{\ell m} \quad \delta(\psi-\psi_{\ell m}) , \tag{16}$$

is singular at the appropriate rational surface $\psi_{\ell m}$. While both terms are singular, both singularities are integrable. Third, the multiplier of the δ-function singularity, $\hat{Q}_{\ell m}$, is undetermined. In Sec. III the singularity will be resolved and the value of $\hat{Q}_{\ell m}$ will be determined.

C. Magnetic field perturbation

To complete the description of the exterior region, one must solve for the magnetic field perturbation,

$$\underset{\sim}{\nabla}\times\underset{\sim}{B}_1 = \underset{\sim}{J}_1 \ ,$$

where, according to Eqs. (12), (13), and (16),

$$\underset{\sim}{J}_1 = P_1'(\psi) \quad \mathscr{I}_{00}(\psi)\underset{\sim}{\nabla}\varphi\times\underset{\sim}{\nabla}\psi + \sum_{\substack{\ell,m \\ \neq(0,0)}} \{\hat{Q}_{\ell m}\delta(\psi-\psi_{\ell m}) - P'(\psi) \quad \mathscr{I}_{\ell m}(\psi)/[\text{ϵ}(\psi)-m/\ell]\}$$

$$\times \exp(i\ell\vartheta-im\varphi) \ \underset{\sim}{\nabla}\psi\times\underset{\sim}{\nabla}(\vartheta-m\varphi/\ell) \ . \tag{17}$$

The three-dimensionality of the system gives rise to currents with every helicity (ℓ,m). These give rise to a vector potential

$$\underset{\sim}{A}_1 = \sum_{\ell,m} [\ \mathscr{A}_{\vartheta}(\psi,\ell,m)\underset{\sim}{\nabla}\vartheta + \mathscr{A}_{\varphi}(\psi,\ell,m)\underset{\sim}{\nabla}\varphi]\exp(i\ell\vartheta-im\varphi) \ , \tag{18}$$

containing every helicity. (Note that the gauge $A_\psi = 0$ has been chosen.)

The solution for $\mathscr{A}_\vartheta$ and $\mathscr{A}_\varphi$ in terms of the as yet undetermined $\hat{Q}_{\ell m}$'s can be obtained by a Green's function integration. As the singularities are integrable, the solution for $\mathscr{A}$ is continuous. Because the equations are linear, the solution has the form

$$\mathscr{A}_{\vartheta}(\psi,\ell,m) = C_{\vartheta}(\psi,\ell,m) + \sum_{\ell',m'} D_{\vartheta}(\psi,\ell,m,\ell',m')\hat{Q}_{\ell',m'} \ , \tag{19}$$

where the coefficients C and D depend on the details of the vacuum field and the pressure profile. As Eq. (19) indicates, in general, all helicities of the vector potential are coupled by the D coefficients, since $\hat{Q}_{\ell' m'}$ will be found to depend on $\underset{\sim}{A}_{1\ell' m'}$.

D. Solution for nearly circular flux surfaces

In order to make further analytical progress, we introduce the approximation of nearly circular flux surfaces. First, we introduce a coordinate r satisfying

$$\psi = \frac{1}{2}\,(\gamma/R_0)r^2 \ , \tag{20}$$

with R_0 a constant corresponding to the mean radius of the magnetic axis. (This choice is dictated by Eq. (1) which indicates that $B_0^2 \cong \gamma^2/R_0^2$.) Second, we introduce a coordinate $z = R_0\varphi$. With these coordinates, one can always write the operator $\underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\times$ in Ampere's law, $\underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\times\underset{\sim}{A}_1) = \underset{\sim}{J}_1$, using the metric tensor. However, if the flux surfaces are nearly circular and the inverse aspect ratio is small, then to lowest order in the three-dimensionality one can use the metric for circular coordinates in the operator $\underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\times$. This approximation decouples the various helicities in Eq. (19).

To implement this procedure we introduce the resonant coordinate,

$$\alpha = \vartheta - m\varphi/\ell \tag{21a}$$

$$\zeta = \varphi \ , \tag{21b}$$

for finding the (ℓ,m) harmonic,

$$\underset{\sim}{A}_{1\ell m} = \mathscr{A}_\alpha(\psi,\ell,m)\nabla\alpha + \mathscr{A}_\zeta(\psi,\ell,m)\nabla\zeta \ . \tag{22}$$

With the aforementioned approximations, Ampere's law yields

$$\frac{\partial \mathscr{A}_\alpha}{\partial r} = \frac{mr^2/\ell R_0}{1 + (mr/\ell R_0)^2} \frac{\partial \mathscr{A}_\zeta}{\partial r} ,$$

and

$$\frac{\partial}{\partial r} \frac{r/R_0}{1 + (mr/\ell R_0)^2} \frac{\partial \mathscr{A}_\zeta}{\partial r} - \frac{\ell^2}{rR_0} \mathscr{A}_\zeta = -\hat{Q}_{\ell m}\delta(r - r_{\ell m}) + \frac{dP}{dr} \frac{\mathscr{I}_{\ell m}(r)}{\iota(r) - m/\ell} .$$

This equation can be solved with the Green's function

$$G(r,r') = -\frac{m^2 rr'}{\ell^2 R_0} I'_\ell(mr_</R_0) K'_\ell(mr_>/R_0) ,$$

where I and K are modified Bessel functions. The Green's function allows us to find the coefficients for A_ζ,

$$\mathscr{A}_\zeta(r,\ell,m) = C_\zeta(r,\ell,m) + \sum_{\ell',m'} D_\zeta(r,\ell,m,\ell',m')\hat{Q}_{\ell',m'} , \qquad (23)$$

analogous to Eq. (19). The results are

$$C_\zeta(r,\ell,m) = -\frac{m^2 r}{\ell^2 R_0} \int dr'r' \, I'_\ell(mr_</R_0) K'_\ell(mr_>/R_0)$$

$$\times \frac{dP}{dr}(r') \, \mathscr{I}_{\ell m}(r')/.[\iota(r') - m/\ell] \qquad (24)$$

and

$$D_\zeta(r,\ell,m,\ell',m') = -\delta_{\ell\ell'}\,\delta_{mm'}\,\frac{m^2 r}{\ell^2 R_0}\int dr'r'I_\ell'(mr_</R_0)K_\ell'(mr_>/R_0)$$

$$\times\,\delta(r-r_{\ell m})\ . \tag{25}$$

where $\delta_{\ell\ell'}$ is the Kronecker δ.

For large $\ell(\ell>>1)$, the asymptotic expansions of $I_\ell'(\ell x)$ and $K_\ell'(\ell x)$ may be used. This allows us to determine

$$C_\zeta(r_{\ell m},\ell,m) \cong -\frac{R_0}{2\ell^2}\left(\frac{dP}{dr}\mathscr{I}_{\ell m}\right)^{-1}\frac{d}{dr}\left[r\left(\frac{dP}{dr}\mathscr{I}_{\ell m}\right)^2\Big/\frac{d\iota\!\!-}{dr}\right] \tag{26}$$

and

$$D_\zeta(r_{\ell m},\ell,m) \cong (R_0/2\ell)\left[1+(mr/\ell R_0)^2\right]^{1/2}\ . \tag{27}$$

E. Scaling of the coefficients

The precise evaluation of C_ζ and D_ζ depends on the particular profiles of $\mathscr{I}_{\ell m}$ and dP/dr. To estimate C_ζ and D_ζ we assume the stellarator to have rotational transform of order unity and small inverse aspect ratio $\varepsilon \sim a/R_0 << 1$. We take the mean magnetic field to be B_0 and we use β to denote the peak ratio of plasma pressure to magnetic pressure, $\beta \equiv 2\Delta P/B_0^2$. We then find

$$C_\zeta(\ell,m) \sim R_0^2 B_0 \beta\varepsilon_{\ell m}/(\ell^2\Delta\iota\!\!-)\ , \tag{28}$$

where $\varepsilon_{\ell m}$ is a typical value of $\mathscr{J}_{\ell m}/\mathscr{J}_{00}$, and $\Delta\iota\!\!\!- \equiv \iota\!\!\!-(a) - \iota\!\!\!-(0)$ is the total shear. Similarly, we find

$$D_\zeta \sim \frac{R_0}{2\ell} . \tag{29}$$

It is important that D_ζ is always positive.

III. Interior solution

The analysis of the last section provided the solution for the magnetic field in terms of the δ-function island current $\hat{Q}_{\ell m}$. In this section we determine $\hat{Q}_{\ell m}$ by considering the details of the island region. For notational convenience we introduce the subscript "r" to denote the resonant $(\ell,m) = (\ell_r,m_r)$ and resonant surface ψ_r which satisfies $\iota(\psi_r) = m_r/\ell_r$. Similarly we define $\iota_r' \equiv d\iota/d\psi$ evaluated at $\psi = \psi_r$ and $\varepsilon_r \equiv |\varepsilon_{\ell_r,m_r}|$.

This analysis of the island region relies on the smallness of the island size. This assumption, essentially that the island width is small compared with the machine size and $k_\perp^{-1}$, allows one to neglect the ψ-variation of equilibrium quantities and the perturbed fields within the island region. In terms of the island width $\delta\iota$ in rotational transform and the total shear $\Delta\iota$ of the vacuum field, this assumption is

$$\delta\iota << \Delta\iota/\left(\ell^2+m^2\varepsilon_t^2\right)^{1/2} . \tag{30}$$

A. Interior current

From Eq. (2) we deduce that the unperturbed magnetic field comes from the vector potential

$$\underset{\sim}{A}_0 = \psi\underset{\sim}{\nabla}\vartheta - F(\psi)\underset{\sim}{\nabla}\varphi = \psi\underset{\sim}{\nabla}\alpha - \left[F(\psi) - \iota_r\psi\right]\underset{\sim}{\nabla}\zeta$$

where $\iota(\psi) = dF/d\psi$, and the variables α and ζ refer to the transformation Eqs. (21-22) for the particular resonance under consideration. As discussed in the last section, plasma pressure modifies the vector potential by adding the new terms of Eq. (18). As a result, the field lines no longer stay on surfaces of constant ψ. Instead there is a new invariant function $\overline{\chi}$. To find this new invariant function we use the noncanonical perturbation theory of Ref. 20. For ψ near the resonant surface ψ_r, this invariant function is

$$\chi = \frac{1}{2}\,\iota_r'(\psi-\psi_r)^2 - \sum_{k=-\infty}^{\infty} \mathscr{A}_\zeta(\psi,k\ell_r,km_r)e^{ik\ell_r\alpha}$$
$$- \iota_r'(\psi-\psi_r) \sum_{k=-\infty}^{\infty} \mathscr{A}_\alpha(\psi,k\ell_r,km_r)e^{ik\ell_r\alpha}$$
$$- \iota_r'(\psi-\psi_r) \sum_{(\ell,m)}{}' \frac{\mathscr{A}_\zeta+(\iota_r - \frac{m}{\ell})\mathscr{A}_\alpha}{\iota_r - \frac{m}{\ell}}\; e^{i\ell\vartheta-im\varphi} \;. \tag{31}$$

to lowest order in $\underset{\sim}{A}_1$. The notation $\sum'$ indicates that resonant terms, $(\ell,m) = k(\ell_r,m_r)$, are to be excluded from the sum.

As one can see from Eq. (31), the resonant component of $A_{1\zeta}$ gives the largest part of χ. The effect of the $A_{1\alpha}$ component and the nonresonant components is small provided one is close to ψ_r, cf. (30). This leads us to introduce an averaging operator that selects only the resonant terms:

$$\overline{f}(\psi,\alpha) \equiv \frac{1}{2\pi\ell_r}\int_0^{2\pi\ell_r} d\zeta\; f(\psi,\vartheta{=}\alpha{+}m_r\zeta/\ell_r,\zeta) \;. \tag{32a}$$

We further define its complement,

$$\tilde{f} \equiv f-\overline{f} \;. \tag{32b}$$

In addition we introduce the new variable,

$$x \equiv \psi - \psi_r \quad . \tag{33}$$

As a result, the new invariant is given approximately by

$$\overline{\chi} \cong \overline{\chi} \cong \frac{1}{2}\,\iota_r' x^2 - \overline{A}_{1\zeta}(x,\alpha) \quad . \tag{34}$$

The new magnetic surfaces, which are the contours of $\overline{\chi}$, are topologically different from the vacuum surfaces by the presence of an island chain. The magnetic surfaces for a typical perturbation, $\overline{A}_{1\zeta}(x,\alpha) = \chi_c \cos(\ell_r \alpha)$, are shown in Fig. 1. At the boundary or separatrix of the island, $\overline{\chi} = \chi_{sx} = \chi_c\ \mathrm{sign}(\iota)$. Inside, $\iota_r'\overline{\chi} < |\iota_r'\chi_c|$, while outside $\iota_r'\overline{\chi} > |\iota_r'\chi_c|$. This indicates the double-valuedness of the function $\psi(\overline{\chi})$. The extent of this island in the variable ψ is $\delta\psi = 2|\chi_c/\iota_r'|^{1/2}$. This indicates an extent in rotational transform of

$$\delta\iota = 2|\iota_r'\chi_c|^{1/2} \quad . \tag{35}$$

As in any toroidal system with shear,[1] the pressure must be a function of the flux invariant alone, $P(\overline{\chi})$. This allows us to find the perpendicular current via force balance:

$$\underset{\sim}{J}_\perp = \frac{P'(\overline{\chi})}{B^2}\left[-B_\zeta \frac{\partial\overline{\chi}}{\partial\alpha}\,\underset{\sim}{\nabla}\alpha\times\underset{\sim}{\nabla}\zeta + B_\zeta\frac{\partial\overline{\chi}}{\partial\psi}\,\underset{\sim}{\nabla}\zeta\times\underset{\sim}{\nabla}\psi + \left(B_\psi\frac{\partial\overline{\chi}}{\partial\alpha} - B_\alpha\frac{\partial\overline{\chi}}{\partial\psi}\right)\underset{\sim}{\nabla}\psi\times\underset{\sim}{\nabla}\alpha\right] \quad ,$$

upon neglecting the ζ-dependence of $\overline{\chi}$. To find the parallel current, we write

$$\underset{\sim}{J}_{\|} = Q\underset{\sim}{B} \ , \tag{36}$$

and set $\underset{\sim}{\nabla}\cdot\underset{\sim}{J}_{\|} = -\underset{\sim}{\nabla}\cdot\underset{\sim}{J}_{\perp}$. The resulting equation for Q is

$$\left(1 + \frac{\partial A_{1\alpha}}{\partial\psi}\right)\frac{\partial Q}{\partial\zeta} + \frac{\partial\bar{\chi}}{\partial\psi}\frac{\partial Q}{\partial\alpha} - \frac{\partial\bar{\chi}}{\partial\alpha}\frac{\partial Q}{\partial\psi} - \frac{\partial\tilde{A}_{1\zeta}}{\partial\psi}\frac{\partial Q}{\partial\alpha} + \left(\frac{\partial\tilde{A}_{1\zeta}}{\partial\alpha} - \frac{\partial\tilde{A}_{1\alpha}}{\partial\zeta}\right)\frac{\partial Q}{\partial\psi}$$

$$= - P'(\bar{\chi})\left[\frac{\partial\bar{\chi}}{\partial\alpha}\frac{\partial}{\partial\psi}\frac{B_\zeta}{B^2} - \frac{\partial\bar{\chi}}{\partial\psi}\frac{\partial}{\partial\alpha}\frac{B_\zeta}{B^2} + \frac{\partial}{\partial\zeta}\left(\frac{B_\psi}{B^2}\frac{\partial\bar{\chi}}{\partial\alpha} - \frac{B_\alpha}{B^2}\frac{\partial\bar{\chi}}{\partial\psi}\right)\right] \ . \tag{37}$$

The preceding expressions for the current are fully nonlinear. To make further progress we invoke the low-β approximation. This we do by neglecting $\mathcal{O}(\beta^2)$ terms in Eq. (37). That is we use B_0^2 for B^2, γ for B_ζ, we neglect the last term on the right of Eq. (37). The result is

$$\left(1 + \frac{\partial A_{1\alpha}}{\partial\psi}\right)\frac{\partial Q}{\partial\zeta} + \frac{\partial\bar{\chi}}{\partial\psi}\frac{\partial Q}{\partial\alpha} - \frac{\partial\bar{\chi}}{\partial\alpha}\frac{\partial Q}{\partial\psi} - \frac{\partial\tilde{A}_{1\zeta}}{\partial\psi}\frac{\partial Q}{\partial\alpha} + \left(\frac{\partial\tilde{A}_{1\zeta}}{\partial\alpha} - \frac{\partial\tilde{A}_{1\alpha}}{\partial\zeta}\right)\frac{\partial Q}{\partial\psi}$$

$$= - P'(\bar{\chi})\left(\frac{\partial\bar{\chi}}{\partial\alpha}\frac{\partial\mathcal{J}}{\partial\psi} - \frac{\partial\bar{\chi}}{\partial\psi}\frac{\partial\mathcal{J}}{\partial\alpha}\right) \ . \tag{38}$$

Secondly, we write $Q = \bar{Q}+\tilde{Q}$ and take the ζ-dependent and ζ-independent parts of Eq. (38). Neglecting $\mathcal{O}(\beta^2)$ terms we find

$$\frac{\partial\tilde{Q}}{\partial\zeta} = -P'(\bar{\chi})\left(\frac{\partial\bar{\chi}}{\partial\alpha}\frac{\partial\tilde{\mathcal{J}}}{\partial\psi} - \frac{\partial\bar{\chi}}{\partial\psi}\frac{\partial\tilde{\mathcal{J}}}{\partial\alpha}\right) \tag{39a}$$

and

$$\frac{\partial\bar{\chi}}{\partial\psi}\frac{\partial\bar{Q}}{\partial\alpha} - \frac{\partial\bar{\chi}}{\partial\alpha}\frac{\partial\bar{Q}}{\partial\psi} = -P'(\bar{\chi})\left(\frac{\partial\bar{\chi}}{\partial\alpha}\frac{\partial\bar{\mathcal{J}}}{\partial\psi} - \frac{\partial\bar{\chi}}{\partial\psi}\frac{\partial\bar{\mathcal{J}}}{\partial\alpha}\right) \ . \tag{39b}$$

For consistency we should apply the same approximations to our result for $\underset{\sim}{J}_\perp$. This yields

$$\underset{\sim}{J}_\perp = -\mathscr{J}P'(\bar{\chi})\left(\frac{\partial\bar{\chi}}{\partial\alpha}\,\underset{\sim}{\nabla}\alpha\times\underset{\sim}{\nabla}\zeta - \frac{\partial\bar{\chi}}{\partial\psi}\,\underset{\sim}{\nabla}\zeta\times\underset{\sim}{\nabla}\psi\right) . \qquad (40)$$

It is important to recognize that we have not linearized even though we have used the low-β approximation. The linearization of Sec. II does not allow topology changes. It is equivalent to setting $\bar{\chi} = \frac{1}{2}\,\text{ϵ}_r'x^2$, thereby neglecting the term $\partial\bar{\chi}/\partial\alpha$, which, as we will now see, resolves the singularity.

The solution to Eq. (39a) is straightforwardly obtained by Fourier expansion.

$$\tilde{Q} = iP'(\bar{\chi}) \sum_{(\ell,m)}{}' \left(\frac{\partial\bar{\chi}}{\partial\alpha}\,\frac{\partial\mathscr{J}_{\ell m}}{\partial\psi} - i\ell\,\frac{\partial\bar{\chi}}{\partial\psi}\,\mathscr{J}_{\ell m}\right) \frac{\exp(i\ell\vartheta - im\varphi)}{(\ell\text{ϵ}_r - m)} \qquad (41)$$

To find $\bar{Q}$ we note that Eq. (39b) can be put into the form

$$\frac{\partial\bar{\chi}}{\partial\psi}\,\frac{\partial}{\partial\alpha}\,(\bar{Q}-P'\bar{\mathscr{J}}) - \frac{\partial\bar{\chi}}{\partial\alpha}\,\frac{\partial}{\partial\psi}\,(\bar{Q}-P'\bar{\mathscr{J}}) = 0 ,$$

which has the general solution

$$\bar{Q} = P'(\bar{\chi})\bar{\mathscr{J}} + K(\bar{\chi}) \qquad (42)$$

with the function $K(\bar{\chi})$ arbitrary. Thus, the parallel and perpendicular currents are known in terms of the functions $P'(\bar{\chi})$ and $K(\bar{\chi})$.

B. Nonlinear Island Equation

One can derive a nonlinear Grad-Shafranov-like equation for the island region by invoking the narrow island approximation. This allows us to assume $\partial^2 A_{1\zeta}/\partial\psi^2 >> |\nabla\alpha||\nabla\psi|\partial^2 A_{1\zeta}/\partial\psi\partial\alpha$ and to neglect the first term in the sum of Eq. (41). As a result, Ampere's law yields

$$\frac{\partial^2 \bar{A}_{1\zeta}}{\partial\psi^2} = -\bar{h}[P'(\bar{\chi})\mathscr{J}(\psi,\alpha) + K(\bar{\chi})]$$

$$-P'(\bar{\chi})\frac{\partial\bar{\chi}}{\partial\psi}\sum_{\substack{(\ell,m)\\(\ell',m')}}{}' \frac{\ell m h^*_{\ell'm'}\exp[i(\ell-\ell')\alpha]\delta_{(\ell-\ell')m_r,(m-m')\ell_r}}{(m_r/\ell_r-m/\ell)}, \tag{43a}$$

where

$$h(\alpha,\zeta) \equiv \gamma(\mathscr{J}|\nabla\psi|^2)^{-1}_{\psi=\psi_r} = \bar{h} + \sum_{\ell,m}{}' h_{\ell m}e^{i\ell\vartheta-im\varphi}, \tag{43b}$$

This nonlinear equation resolves the singularity of Sec. II and determines the structure of the magnetic field near the island. It remains to determine the functions $P'(\bar{\chi})$ and $K(\bar{\chi})$.

C. Determination of the current function K

In a steady-state stellarator the inductive electric field has decayed away. This implies that the integral of $\underset{\sim}{J}\cdot\underset{\sim}{B}$ over the volume between two flux surfaces vanishes

$$\int_{\bar{\chi}_0<\bar{\chi}<\bar{\chi}_0+d\bar{\chi}} d\alpha d\zeta d\psi\, Q(\psi,\eta,\zeta) = 0 .$$

Given our expressions for Q we find

$$K(\bar{\chi}) = -P'(\bar{\chi}) \; <\bar{\mathcal{J}}> \; ,$$

where the averaging process is defined by

$$<f(\psi,\alpha)> \;\equiv\; \frac{\int d\alpha \; f(\psi(\bar{\chi},\alpha),\alpha)/\frac{\partial\bar{\chi}}{\partial\psi}(\psi(\bar{\chi},\alpha),\alpha)}{\int \; d\alpha/\frac{\partial\bar{\chi}}{\partial\psi} \; (\psi(\bar{\chi},\alpha),\alpha)} \; . \tag{44}$$

Therefore, we have the expression,

$$\bar{Q} = P'(\bar{\chi})(\;\bar{\mathcal{J}} - <\bar{\mathcal{J}}>) \; , \tag{45}$$

for the parallel current function.

D. Determination of the pressure profile

As discussed in Sec. IIIA the pressure is constant on a surface of constant $\bar{\chi}$. This is due to the large parallel transport, which implies, for example, that pressure is constant on the island separatrix. In determining the pressure profile, particle sources are neglected, because their effect is small when the island region is small. This implies that the pressure profile is constant within the island separatrix because it is constant everywhere on a bounding surface. Outside the separatrix, the particle flux induced by resistivity is constant. In addition, far from the island the pressure gradient must reduce to the value of the exterior region. In combination these facts yield the pressure profile.

The consequences of resistivity are found from Ohm's law,

$$-\nabla\Phi + V\times B = \eta_{\parallel} J_{\parallel} + \eta_{\perp} J_{\perp} . \tag{46}$$

As resistivity is a higher order effect, the modifications of $J_{\parallel}$ and $J_{\perp}$ due to resistivity may be neglected. From the perpendicular part of Eq. (46) one finds the particle flow velocity,

$$V_{\perp} = \eta_{\perp} B\times J_{\perp}/B^2 + B\times\nabla\Phi/B^2 . \tag{47}$$

Thus, if $\Phi = \bar{\Phi}(\psi,\alpha) + \tilde{\Phi}(\psi,\alpha,\zeta)$ is known, one can find the particle flux,

$$\Gamma = \rho \int_{\partial\bar{\chi}} d\vartheta d\varphi (\nabla\bar{\chi}\times\nabla\alpha\cdot\nabla\zeta)^{-1} V_{\perp}\cdot\nabla\bar{\chi} = \Gamma_c + \Gamma_r + \Gamma_n , \tag{48}$$

as the sum of the classical flux,

$$\Gamma_c \equiv \eta_{\perp}\rho \int_{\partial\bar{\chi}} d\vartheta d\varphi (\nabla\bar{\chi}\times\nabla\alpha\cdot\nabla\zeta)^{-1} B\times J_{\perp}\cdot\nabla\bar{\chi}/B^2 , \tag{49}$$

the resonant Pfirsch–Schluter flux,

$$\Gamma_r \equiv \rho \int_{\partial\bar{\chi}} d\vartheta d\varphi (\nabla\bar{\chi}\times\nabla\alpha\cdot\nabla\zeta)^{-1} B\times\nabla\bar{\Phi}\cdot\nabla\bar{\chi}/B^2 , \tag{50}$$

and the nonresonant Pfirsch–Schluter flux,

$$\Gamma_n \equiv \rho \int_{\partial\bar{\chi}} d\vartheta d\varphi (\nabla\bar{\chi}\times\nabla\alpha\cdot\nabla\zeta)^{-1} B\times\nabla\tilde{\Phi}\cdot\nabla\bar{\chi}/B^2 . \tag{51}$$

The notation indicates that these integrals are over constant-$\bar{\chi}$ surfaces.

The calculation of the classical flux involves using the expression (40) for $\underset{\sim}{J}_\perp$, and the narrow-island approximation (30), which allows one to use $|\nabla\overline{\chi}|^2 \cong |\nabla\psi|^2(\partial\overline{\chi}/\partial\psi)^2$ and $\partial\overline{\chi}/\partial\psi \cong \iota_r' x$ in the integral of Eq. (49). The result for Γ_c is

$$\Gamma_c = -[\eta_\perp \rho P'(\overline{\chi})\iota_r'/\gamma] \int_{\partial\overline{\chi}} d\vartheta d\varphi \; x(\overline{\chi},\alpha)(\mathcal{J}^2|\nabla\psi|^2)_{\psi_r} \; . \tag{52}$$

The evaluation of the Pfirsch-Schluter transport is more difficult because we must first use Eq. (46) to evaluate Φ. From Eq. (46) we find

$$\nabla \cdot (\Phi\underset{\sim}{B}) = \eta_\parallel \underset{\sim}{J}\cdot\underset{\sim}{B} = \eta_\parallel Q B^2 \; .$$

Splitting Φ in the usual way, $\Phi = \overline{\Phi}+\tilde{\Phi}$, and invoking the narrow island assumption we find

$$\overline{\Phi}(\psi,\alpha) = f(\overline{\chi}(\psi,\alpha),\alpha) \; ,$$

where $$\frac{\partial f}{\partial\alpha} = -\gamma\eta_\parallel \; \overline{Q}/(\partial\overline{\chi}/\partial\psi) \; ,$$

and $$\tilde{\Phi} = -i\gamma\eta_\parallel P'(\overline{\chi}) \frac{\partial\overline{\chi}}{\partial\psi} \sum_{(\ell,m)}{}' \; \ell \, \mathcal{J}_{\ell m} \exp(i\ell\vartheta - im\varphi)/(\ell\iota_r - m)^2 \; .$$

These results allow us to find the Pfirsch-Schluter transport. With the definition,

$$x(\overline{\chi},\alpha) \equiv \sum_k x_k(\overline{\chi})\exp(ik\ell_r\alpha) \; , \tag{53}$$

we find

$$\Gamma_r = -2\pi\eta_\parallel \gamma\rho P'(\overline{\chi}) \int_{\partial\overline{\chi}} d\alpha\ (\overline{\mathscr{J}} - \langle\overline{\mathscr{J}}\rangle)^2/(\partial\overline{\chi}/\partial\psi) \ , \tag{54}$$

and

$$\Gamma_n = -(2\pi)^2\eta_\parallel\gamma\rho P'(\overline{\chi}) \imath_r' \sum_k \sum_{(\ell,m)}{}' \frac{x_k \mathscr{J}_{\ell m} \mathscr{J}^*_{\ell+k\ell_r, m+km_r}}{(m_r/\ell_r - m/\ell)^2} \ . \tag{55}$$

As mentioned in the beginning of this section, the total particle flux must be constant, $\Gamma(x) = \Gamma_0$. To determine this constant, the pressure gradient is required to reduce to its global value P_r' defined by $P_r' = dP/d\psi$ evaluated at an intermediate value of ψ close to the particular surface but several island widths away. This procedure yields

$$\frac{dP}{d\overline{\chi}} = P_r'/[g_1(\overline{\chi}) + g_2(\overline{\chi})] \ , \tag{56}$$

where

$$g_1(\overline{\chi}) \equiv (\imath_r'/G)\Big[\int \frac{d\vartheta d\varphi}{(2\pi)^2}\, x(\overline{\chi},\alpha)(\mathscr{J}^2|\nabla\psi|^2)_{\psi=\psi_r}$$

$$+ \frac{\gamma^2\eta_\parallel}{\eta_\perp} \sum_{k,(\ell,m)}{}' \frac{x_k \mathscr{J}_{\ell m} \mathscr{J}^*_{\ell+k\ell_r, m+km_r}}{(m_r/\ell_r - m/\ell)^2}\Big] \ , \tag{57a}$$

$$g_2(\overline{\chi}) \equiv (\gamma^2\eta_\parallel/\eta_\perp G) \int \frac{d\alpha}{2\pi} (\overline{\mathscr{J}} - \langle\overline{\mathscr{J}}\rangle)^2 \left(\frac{\partial\overline{\chi}}{\partial\psi}\right)^{-1} \ , \tag{57b}$$

and

$$G \equiv \int \frac{d\vartheta d\varphi}{(2\pi)^2} \left(\mathscr{g}^2 |\underset{\sim}{\nabla}\psi|^2\right)_{\psi=\psi_r} + \frac{\gamma^2\eta_\parallel}{\eta_\perp} \sum_{\ell,m}{}' \frac{|\mathscr{I}_{\ell m}|^2}{(m_r/\ell_r - m/\ell)^2} . \qquad (58)$$

E. Constant-A_ζ approximation

While the general calculation proceeds from the nonlinear Eq. (43), for the present case of narrow islands in low-β systems, one can use the "constant-ψ" approximation[21] of linear tearing mode theory. In this approximation we write

$$\bar{A}_{1\zeta} = \sum_k \chi_k(\psi)\exp(ik\ell_r\alpha) ,$$

and neglect the variation of $\chi_k(\psi)$ throughout the island layer. Thus, $\hat{Q}_{\ell_r,m_r}$ is found by simply integrating the parallel current (45) through the layer.

$$\hat{Q}_{k\ell_r,km_r} = \int_{-\infty}^{\infty} d\psi \int_0^{2\pi} \frac{d\alpha}{2\pi} \exp(-ik\ell_r\alpha)P'(\bar{\chi})(\bar{\mathscr{J}} - \langle\bar{\mathscr{J}}\rangle) . \qquad (59)$$

The contribution due to $\hat{Q}$ of (41) is small because of the approximation (30). Since the ψ-variation of $\bar{A}_\zeta$ has been neglected, $\bar{\chi}$ is an even function of ψ, and $P'(\bar{\chi})$ is an odd function of ψ. Thus we may use $\bar{\mathscr{J}} - \langle\bar{\mathscr{J}}\rangle \cong \bar{\mathscr{J}}'(\psi_r,\alpha)(x-\langle x\rangle)$ in the integrand of Eq. (59). Furthermore, as $\varepsilon_{\ell m} << 1$ is generally true, we find

$$\hat{Q}_{k\ell_r,km_r} = -2\, \mathscr{J}_{00}'(\psi_r) \int_{\bar{\chi}_{sx}}^{\pm\infty} d\bar{\chi} \frac{dP}{d\bar{\chi}} \frac{\langle x\rangle}{\mp_r'} \int_0^{2\pi} \frac{d\alpha}{2\pi} \frac{\exp(-ik\ell_r\alpha)}{x(\bar{\chi},\alpha)} , \qquad (60)$$

where the +(-) sign in the limit of the integal is chosen if $\overline{\chi}_{sx}$ is positive (negative).

F. Single-harmonic approximation

To make further progress we assume that only a single harmonic in $\overline{A}_{1\zeta}$ is significant. Without loss of generality, we choose

$$\overline{A}_{1\zeta} = \chi_c \cos(\ell_r \alpha) \quad . \tag{61}$$

The width of the island in rotational transform is given by Eq. (35). Similarly, we define

$$\hat{Q} = \hat{Q}_c \cos(\ell_r \alpha) + \hat{Q}_s \sin(\ell_r \alpha) \quad . \tag{62}$$

The symmetry of $x(\overline{\chi},\alpha)$ in Eq. (60) implies that $\hat{Q}_s = 0$. This means that the external currents, through the coefficient C of Section II, determine the phase of the perturbation. The internal currents given by

$$\hat{Q}_c = -4[\mathcal{J}_{00}'(\psi_r)/\text{\char"2B}_r'] \int_{\chi_{sx}}^{\pm\infty} d\overline{\chi}\ P'(\overline{\chi})\langle\cos(\ell_r \alpha)\rangle \ , \tag{63}$$

add in phase to the exterior perturbation.

G. Estimation of the profile functions

To evaluate the δ-function multiplier $\hat{Q}_c$ in Eq. (63) we must estimate the profile functions g_1 and g_2 of Eqs. (56-58) that determine the pressure profile. The function g_1 is readily obtained for typical stellarators with $\iota \ell_0/m_0 \ll 1$ (cf. App. A), since the resonant coefficients in Eq. (57a) are very small. Thus one obtains

$$g_1(\bar{\chi}) \cong \iota_r x_0(\bar{\chi}) \ .$$

With this term alone, Eq. (56) has a simple physical explanation. The pressure gradient averaged over the modified surfaces,

$$\int_{\partial\bar{\chi}} \frac{d\alpha}{2\pi} \frac{dP}{d\psi} = P'(\bar{\chi}) \int_{\partial\bar{\chi}} \frac{d\alpha}{2\pi} \frac{\partial\bar{\chi}}{\partial\psi} = P'(\bar{\chi})\iota'_r x_0(\bar{\chi}) \ ,$$

is constant.

With the single harmonic approximation,

$$\bar{\mathcal{J}} = \mathcal{J}_{00}(\psi) + \mathcal{J}_1(\psi_r)\cos(\ell_r(\alpha+\alpha_0)) \ ,$$

the second function $g_2(\bar{\chi})$ contains four terms,

$$g_2(\bar{\chi}) = (\gamma^2\eta_{\parallel}/\eta_{\perp}G\iota'_r)\Big[\mathcal{J}'^2_{00} \int \frac{d\alpha}{2\pi} \frac{x^2 - \langle x\rangle^2}{x} - 2\mathcal{J}'_{00}\ \mathcal{J}_1 \cos\ell_r\alpha_0\langle\cos\ell_r\alpha\rangle$$

$$+ \ \mathcal{J}_1^2 \cos^2\ell_r\alpha_0 \int \frac{d\alpha}{2\pi} \frac{\cos^2(\ell_r\alpha)-\langle\cos(\ell_r\alpha)\rangle^2}{x(\bar{\chi},\alpha)}$$

$$+ \ \mathcal{J}_1^2 \sin^2\ell_r\alpha_0 \int \frac{d\alpha}{2\pi} \frac{\sin^2\ell_r\alpha}{x(\bar{\chi},\alpha)}\Big] \ . \tag{64}$$

Near the island, the first term scales linearly with the island size and is independent of ε_r. Thus, it essentially produces a modification of the function $g_1(\bar{\chi})$. The last two terms are the previously noted[10] resonant Pfirsh-Schluter transport terms modified by the presence of the island. In the absence of the island, these terms become infinite as one approaches the resonant surface, thereby flattening the pressure profile in that region. This singularity is eliminated once the island structure is accounted for. Nevertheless, these terms remain but scale inversely with the island size. The remaining term behaves like a cross term with an intermediate scaling.

The last three terms in Eq. (64) are important only when the islands are extremely small, that is, at very low values of β. These terms can be neglected if they are small relative to the first term in Eq. (64) or to the function g_1. Together, these conditions allow us to neglect the last three terms in Eq. (64) provided either

$$\delta ғ >> \frac{d\ell nV'}{dғ} \varepsilon_r \tag{65a}$$

or

$$\delta ғ >> \frac{d\ell nV'}{dғ} \varepsilon_r \left| \eta_\parallel [\eta_\perp \varepsilon_t^2 + 2\eta_\parallel \varepsilon_t^2 ғ_r^{-2} + 2\eta_\parallel \varepsilon_h^2 (ғ_r - m_0/\ell_0)^{-2}]^{-1} \right|^{1/2} \tag{65b}$$

applies. [In these estimates we have kept only the $\mathcal{I}_{1,0}$ and $\mathcal{I}_{\ell_0,m_0}$ terms in the sum of Eq. (58).] Usually, the restriction (65a) applies. The regime where neither of Eqs. (65) applies is uninteresting since then the islands are extremely small, $\delta ғ \lesssim \varepsilon_r d\ell nV'/dғ$, as seen from App. A. Therefore, for the remainder of the section we work in the intermediate regime where Eq. (30) and one of Eqs. (65) applies. In this regime we find the pressure profile

$$P'(\chi) = (P_r'/\tau_r')/[x_0(\bar{\chi}) + \hat{g}(x_0 - \langle x\rangle)] \tag{66a}$$

where

$$\hat{g} = \left(\frac{1}{\varepsilon_t}\frac{d\ell nV'}{d\tau}\right)^2 \frac{\eta_\parallel}{\eta_\perp + 2\eta_\parallel/\tau_r^2 + 2\eta_\parallel(\varepsilon_h/\varepsilon_t)^2/(\tau_r - m_0/\ell_0)^2} . \tag{66b}$$

In typical systems, $\hat{g} \lesssim 1$.

H. Calculation of $\hat{Q}_c$

To complete the calculation of $\hat{Q}_c$ we insert the profile (66) into the integral (63). The result is

$$\hat{Q}_c = -\frac{1}{2}\left(\mathscr{I}_{00}' P_r'/\tau_r'^2\right)\delta\tau u(\hat{g})\,\mathrm{sign}(\chi_c) \tag{67}$$

where

$$u(\hat{g}) \equiv \frac{4}{|\tau_r'\chi_c|^{1/2}} \int_{\chi_{sx}}^{\pm\infty} d\bar{\chi}\, \frac{\langle\cos(\ell_r\alpha)\rangle}{x_0 + \hat{g}(x_0 - \langle x\rangle)} .$$

A graph of $u(\hat{g})$ is shown in Fig. 2. As is illustrated in the figure, $u(0) \cong 1$, and $u(\hat{g})$ remains of order unity for very large values of $\hat{g}$.

IV. Combined interior and exterior solutions

Together, the results of Secs. II and III give the widths of the magnetic islands. A closed set of equations is obtained by inserting the interior result (67) for $\hat{Q}_{\ell m}$ into Eq. (23). In general, this set of equations is nonlinear and couples all of the considered helicities. In this section we obtain scaling laws by using the approximation of nearly circular flux surfaces.

By combining the result (67) for $\hat{Q}_{\ell m}$ with the exterior Eqs. (23–25), we obtain

$$4\iota_r' \chi_{\ell m} = \beta[\hat{c}_{\ell m} + \hat{d}_{\ell m} 2|\iota_r' \chi_{\ell m}|^{1/2} \mathrm{sign}(\iota' \chi_{\ell m})] \tag{68}$$

where $\chi_{\ell m}$ is $\chi_c(\psi_{\ell m})$ with χ_c as defined by Eq. (61), and the coefficients $\hat{c}$ and $\hat{d}$ are given by

$$\hat{c}_{\ell m} = (4\iota_r'/\beta) C_\zeta(\psi_{\ell m}, \ell, m)$$

and

$$\hat{d}_{\ell m} = -2P_r' \mathcal{J}_{00}' u(\hat{g}) D_\zeta(\psi_{\ell m} \ell, m)/|\beta \iota_r'| \sim -(1/\ell \varepsilon_t^2) |d\ell n V'/d\iota| \mathrm{sign}(P_r' \mathcal{J}_{00}') .$$

The solution to Eq. (68) for the island width $\delta\iota_{\ell m}$ is

$$\delta\iota_{\ell m} = \frac{1}{2} [\beta \hat{d}_{\ell m} + (\beta^2 \hat{d}_{\ell m}^2 + 4\beta|\hat{c}_{\ell m}|)^{1/2}] . \tag{69}$$

From this equation we see that at low values of β, the island width is given by

$$\delta \psi_{\ell m} = |\beta \hat{c}_{\ell m}|^{1/2} \sim |\beta \varepsilon_{\ell m}|^{1/2}/(\varepsilon_t \ell) \ .$$

At values of β greater than the transition value

$$\beta_t \equiv 4|\hat{c}_{\ell m}/\hat{d}^2_{\ell m}| \cong 4\left(\varepsilon_t \frac{d\psi}{d\ell nV'}\right)^2 \varepsilon_{\ell m} \ , \tag{70}$$

the dependence of the island width depends markedly on whether there is a magnetic well $(\hat{d}_{\ell m} < 0)$ or a magnetic hill $(\hat{d}_{\ell m} > 0)$. When a magnetic well is present, the width saturates, so that in the limit of large β (but within the low-β approximation) one finds

$$\delta \psi_{\ell m} = |\hat{c}_{\ell m}/\hat{d}_{\ell m}| \cong \frac{\varepsilon_{\ell m}}{\ell} \frac{d\psi}{d\ell nV'} \ . \tag{71}$$

As discussed in Sec. III this result is applicable only once the island is larger than the size (65) required for flattening to be insignificant. Thus, Eq. (71) applies only when $(d\psi/d\ell nV')^2 > \ell$. If this is not true, the island grows to the size given by (65). That is, the island growth turns off as soon as the flattening effect is diminished and the curvature effect comes into play.

In contrast, if a magnetic hill is present, the island width grows linearly with β for $\beta > \beta_t$:

$$\delta \psi_{\ell m} = \beta \hat{d}_{\ell m} \ . \tag{72}$$

It is interesting that $\hat{c}_{\ell m}$, the driving term from the three dimensionality, has dropped out of this equation. This is consistent with the fact that at low β resistive interchanges are unstable when

$V'' > 0$, and that Eq. (71) provides the nonlinear saturation level for a single mode. The main difference is that $\hat{c}_{\ell m} \neq 0$ eliminates the unstable symmetric equilibrium solution $\delta\tau_{\ell m} = 0$.

Perhaps more important is the fact that Eq. (72) predicts island overlap for arbitrarily low β. This is due to the fact that the island width predicted by Eq. (72) scales as ℓ^{-1},

$$\delta\tau_{\ell m} \sim \beta\ \varepsilon_t^{-2}\ \left|\frac{d\ell nV'}{d\tau}\right| \ell^{-1} \ , \tag{73}$$

while the mean density of islands with $\ell<L$ is given by $dN/d\tau = \frac{1}{2}L^2$. Stochasticity occurs[22] when the overlap parameter $\delta\tau dN/d\tau$ exceeds approximately $2/\pi$. Thus, one expects to find stochasticity once islands with $\ell=1,\ldots,6$ are considered, where

$$L(\beta) \sim \frac{\varepsilon_t^2}{\beta}\left|\frac{d\tau}{d\ell nV'}\right| \ . \tag{74}$$

We note that the flattening effect of Eqs. (66) cannot come into play since the width predicted by Eq. (73) greatly exceeds the necessary width (66).

V. Applications and Discussion

The proposed[3] (ℓ_0=2, m_0=12) stellarator ATF provides a good illustration of these ideas as it has a magnetic well region near the axis and a magnetic hill region towards the edge. For this machine typical values are $\varepsilon_t \sim 0.15$, $\varepsilon_h \sim 0.5$, $\iota \sim 0.5$ and $d\ell nV'/d\iota \sim .15$. With the scalings of App. A it is easy to see that island effects will be insignificant in the (inner) region of magnetic well. In the outer region, the smallness of the coefficients indicates $\beta >> \beta_t$. For ATF, a typical value is $d\iota/d\ell nV' \sim 7$. Thus, the ℓ=1,...,14 islands will be overlapping at β of order 1%.

VI. Discussion

The existence of three-dimensional scalar equilibria has long been an issue. In part this is due to the $\oint d\ell/B$ criterion.[19] (The fact that scalar pressure toroidal systems must satisfy this criterion, while even highly integrable vacuum fields do not,[18] indicate that the introduction of a small amount of pressure is a singular perturbation.) In part the issue arises because one expects islands and stochasticity to be present at some level in a system without special symmetry. Once one accepts the presence of islands, the $\oint d\ell/B$ issue is resolved. The important problem is the calculation of the island size and, hence, the level of stochasticity.

The present low-β calculation is a first step. Two important effects have been found. The plasma produces slowly varying exterior currents far from the island. These currents produce resonant fields in much the same way as an external coil. The plasma also produces sharply peaked currents near the island which can either enhance or limit island size depending on whether the average curvature is bad or good. In typical stellarators with $\iota\ell_0/m_0 << 1$, these currents are sufficiently strong, and the island driving terms $\varepsilon_{\ell m}$ are sufficiently weak (see App. A), that the importance of islands depends dramatically on the presence of a magnetic well.

If the well is present, islands are small. In contrast, when a magnetic hill is present, island overlap occurs for arbitrarily low β.

One should note that this overlap does not mean complete loss of equilibrium unless the low-order islands overlap. At low β, where only the high-order islands overlap, the result is an enhanced diffusion due to stochasticity. The calculation of the diffusion constant is outside the scope of this paper, which <u>a priori</u> assumes that overlap is not present.

The physical picture is that the three-dimensionality gives a driving force for the resistive interchange mode. If the mode is stable, it cannot be driven to significant amplitudes by the small effects represented by the resonant $\varepsilon_{\ell m}$'s. In contrast, if the mode is unstable, it grows to very large amplitude before it saturates.

In stellarators where one cannot assume $\iota \ell_0/m_0 << 1$, one cannot use the well as the only criterion, since the presence of low order rationals no longer allows one to show that the resonant coefficients $\varepsilon_{\ell,m}$ are extremely small. While no such stellarator is currently operating, new optimization techniques[8,18] indicate that one can be built. In this case one must evaluate the island sizes with Eq. (69) and check for overlap.

A slight generalization of this problem is to allow for a small island producing vacuum field perturbation. This essentially modifies Eq. (68) by the replacement $\beta\hat{c}_{\ell m} \rightarrow \beta\hat{c}_{\ell m} + 4\iota_r'\chi_{\ell m}^{vac}$. This indicates that one could tune away islands provided there is a magnetic well so that the island size depends on $\hat{c}_{\ell m}$. However, in the case of a magnetic hill one can not reduce the island below the size of the saturated resistive interchange (72). It also follows from this analysis that the interchange forces will begin to reduce the vacuum island once β begins to exceed $\delta\iota_{vac}\ell\varepsilon_t^2|d\iota/d\ell nV'|$ if the average curvature is good. $(\delta\iota_{vac} \equiv |4\iota_r'\chi_{\ell m}^{vac}|^{1/2}.)$

A future direction for such island calculations is the inclusion of effects of higher order in β. As β is increased the plasma shape distorts, which causes both the $\varepsilon_{\ell m}$'s and V″ to change. (In the course of the work it has come to our attention that Reiman and Boozer[23] have calculated the $\mathcal{O}(\beta)$ corrections to the $\varepsilon_{\ell m}$'s.) Provided the $\varepsilon_{\ell m}$'s remain small, a change in their magnitude should not affect the basic conclusion for typical stellarators with $\iota\ell_0/m_0 \ll 1$: islands are small if there is a well but overlap if there is not. More important is the fact that as β increases the resistive interchange stability parameter is modified[24] to D_R rather than simple V″. In this case our physical picture leads us to believe that at higher values of β, D_R rather than V″ determines the importance of islands.

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Appendix A

Estimation of the Fourier Amplitude $\varepsilon_{\ell m}$

A stellarator magnetic field consists of a constant part, $\langle B\rangle$; a helical part, $B_h \sim \langle B\rangle\varepsilon_h\cos(\ell_0\Theta-m_0\varphi)$; a part that varies due to toroidal effects, $B_t \sim \langle B\rangle\varepsilon_t\cos\Theta$, where ε_t is given roughly by the inverse aspect ratio; and all other terms that can be obtained by nonlinear coupling. The various coefficients, $\langle B\rangle$, ε_h, and ε_t, vary from surface to surface. Since B nowhere vanishes, the coefficients ε_h and ε_t must be less than unity. Typically, ε_h never exceeds 0.2 while ε_t never exceeds 0.15. Thus, ε_h and ε_t can be assumed small. Since $\mathcal{J}= \gamma/B_0^2$, the corresponding terms in the Jacobian are $\mathcal{J}=\mathcal{J}_{00}[1-2\varepsilon_h\cos(\ell_0\Theta-m_0\varphi)-2\varepsilon_t\cos(\Theta)]$ + other terms generated by the nonlinear coupling.

Nonlinear coupling can produce only those variations that are consistent with the m_0-fold periodicity in φ. This implies that $\varepsilon_{\ell m}$ must vanish unless $(\ell,m) = n(\ell_0,m_0) + (k,0)$. Various models of the nonlinear coupling can be used to estimate the size of these amplitudes. From the exponential model, $\mathcal{J}\sim \mathcal{J}_{00}\exp[-2\varepsilon_h\cos(\ell_0\Theta-m_0\varphi) - 2\varepsilon_t\cos\Theta]$, one obtains

$$\varepsilon_{n\ell_0+k,nm_0} \sim \frac{\varepsilon_h^n\varepsilon_t^k}{n!k!} \tag{A1}$$

for the coefficients of Eq. (10). From the inverse model, $\mathcal{J}\sim\mathcal{J}_{00}[1-2\varepsilon_h\cos(\ell_0\Theta-m_0\varphi)]^{-1}[1-2\varepsilon_t\cos(\Theta)]^{-1}$, one finds

$$\varepsilon_{n\ell_0+k,nm_0} \sim \varepsilon_h^n \varepsilon_t^k \ . \tag{A2}$$

In either case the coefficients decrease exponentially with mode number.

The exponential decrease implies that the resonant coefficients are very small in typical stellarators. For example, the largest such coefficient in ATF,[3] a (2,12) stellarator with maximum rotational transform near unity is $\varepsilon_{12,12}$. For ATF, typical values are $\varepsilon_h \sim \varepsilon_t \sim 0.15$. Thus, we estimate $\varepsilon_{12,12} \sim 8.6\times10^{-10}$ on the basis of the more pessimistic estimate (A2).

This small size for the resonant terms is due to the fact that typical stellarators[3-5] rely on $\iota\!\!\!-\ell_0/m_0 << 1$ in order to have good magnetic surfaces in vacuum.[6,7] (This need not be true of optimized fields[8,9] or fields with rotational transform nested between low order rational.) For resonances with n=1, this implies $k/\ell_0 >> 1$. Thus, according to (A2) the associated resonant terms $\varepsilon_{\ell_0+k,m_0} \sim \varepsilon_h \varepsilon_t^k$ must be very small.

In addition there are variations caused by coil winding errors that lack the m_0-field periodicity. Without more knowledge of these errors, the estimation of the corresponding $\varepsilon_{\ell,m}$'s is difficult. One can expect them to be present but exponentially small, as in (A2), for large values of ℓ and m.

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Figure Captions

1) Contours of $\overline{\chi}$ (from Eq. (33)) at an equilibrium resonant surface. ψ and α are the equilibrium radial coordinate and resonant angle, respectively. $\delta\psi$ is the island width, and $\chi = \chi_{sx}$ on the separatrix.

2) Reduction in the island current due to island induced transport.

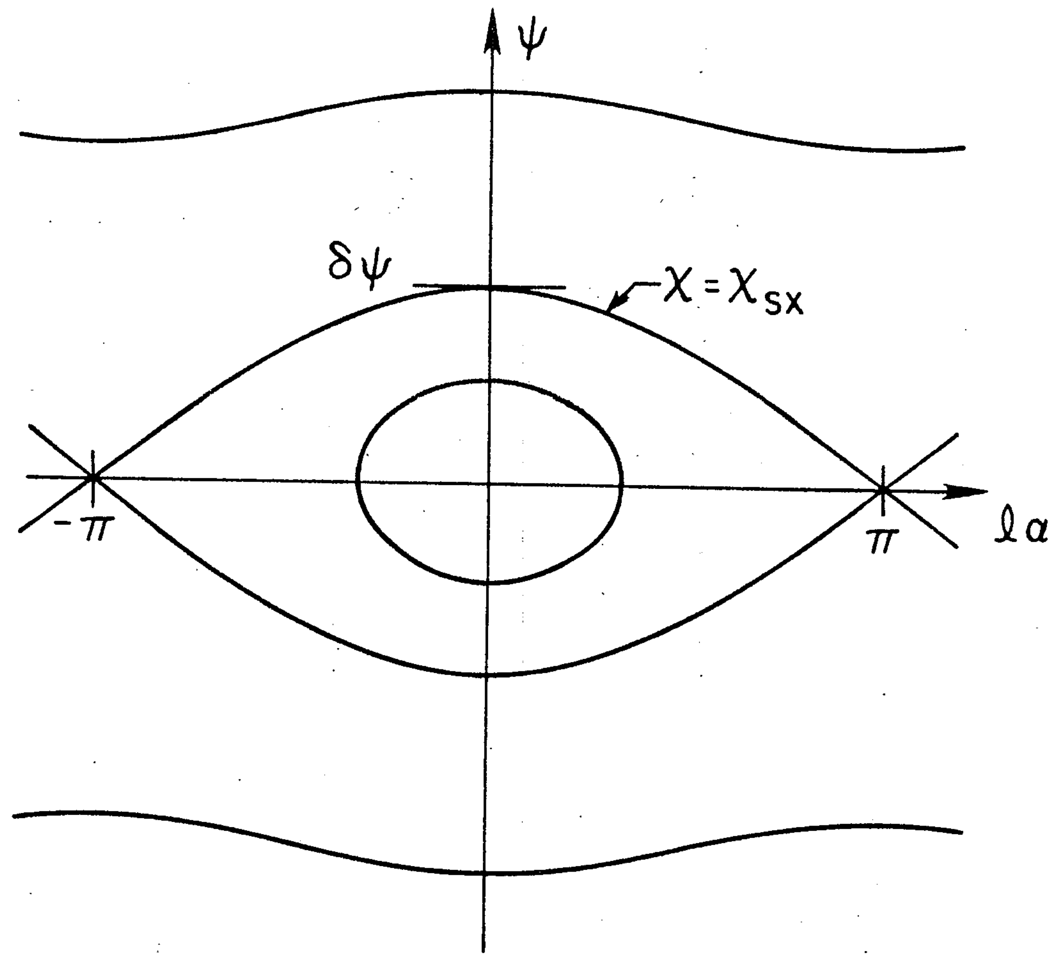

FIG. 1

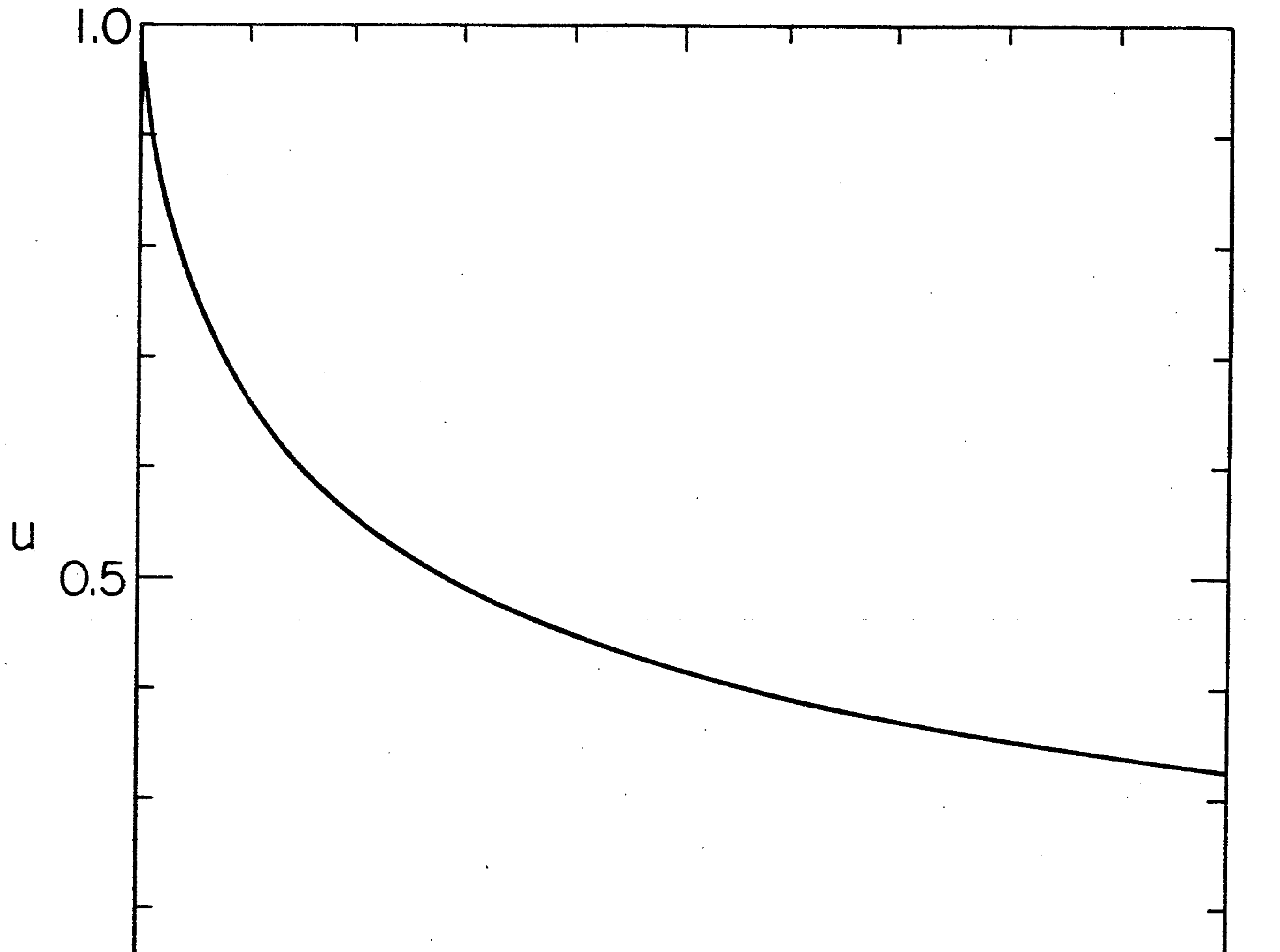

FIG. 2