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INTERACTION OF THE PRECESSIONAL WAVE WITH FREE-BOUNDARY ALFVÉN SURFACE WAVES IN TANDEM MIRRORS

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Abstract

We consider a symmetric tandem mirror plugging a long central cell, with plugs stabilized by a hot component plasma. The system is taken to have a flat pressure profile, with a steep edge gradient. We then consider the interaction of the precessional mode with Alfvén waves generated in the central cell. This analysis is non–eikonal and is valid when $m\Delta/r < 1$ (m is the azimuthal mode number, r the plasma radius and Δ the radial gradient scale length) for long–wavelength radial modes. We find that without FLR effects the precessional mode is always destabilized by the excitation of the Alfvén waves for $m \geqslant 2$. For $m=1$, it is possible to achieve stabilization with conducting walls. A discussion is given of how FLR affects stabilization of the $m \geqslant 2$ long–wavelength modes and of finite–Larmor–radius stabilization of modes described in the eikonal approximation.

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I. INTRODUCTION

In tandem mirrors it would be desirable to operate in an azimuthally symmetric mode. In order to obtain stability of such a system it has been suggested that one use a hot component plasma that is dynamically decoupled from the response of the background plasma. Unfortunately, even if gross MHD stability is achieved the system can still produce negative energy excitations[1–3] that will cause instability when interacting with positive dissipation or positive energy waves. This case was analyzed in the eikonal limit[4] for the excitation of shear–Alfvén waves in the central cell.

A principal purpose of this paper is to extend the eikonal analysis to long wavelength modes. If we model the plasma as being flat with a sharp density gradient, we can perform a long–wavelength layer analysis if $m\Delta/r \ll 1$, where m is the azimuthal mode number, r the plasma radius and Δ the plasma boundary layer thickness. In this case we excite surface Alfvén waves that damp due to a resonance at the local Alfvén speed near the edge of the plasma. We find that both the positive wave excitation and the damping lead to mechanisms which destabilize the precessional mode of the hot plasma. The growth rates in various regimes are calculated. A somewhat similar calculation has recently been discussed by Timofeev.[5]

Recent work[6] has shown how conducting walls can, in principal, stabilize the m=1 mode by converting the precessional mode to a positive energy wave (the MHD energy is then also favorable). This mechanism is included in our analysis and can stabilize the m=1 mode under suitable conditions. Conducting walls do not appreciably alter the dispersion relation for the higher–m modes. However, if the hot component plasma consists of only a few Larmor radii, finite Larmor radius and internal compressional effects can stabilize in a similar robust manner. This mechanism is illustrated in the eikonal limit. In the layer mode limit, $m\Delta/r \ll 1$, an appropriate analysis has been performed in the z-pinch model[7], where there is no dependence along a field line. Incorporating this result enables us to conjecture with some assurance the final combined result. We conclude that there can be a window in parameter space where a hot plasma component can operate stably, free of negative energy waves. This conjecture could be tested in present day experiments with hot-electron plasmas, but would require hot ion plasmas of several MeV in later fusion-sized machines.

II. GOVERNING EQUATION FOR THE mth AZIMUTHAL NORMAL MODE

We consider a long-thin, low-β, axisymmetric plasma equilibrium[8,9] and use flux coordinates to express the magnetic field as

$$\underset{\sim}{B}(z) = B(z)\hat{\underset{\sim}{b}} = \nabla\psi\times\nabla\vartheta$$

where z is distance along the magnetic axis, $\psi = \frac{1}{2}B(0)r^2$ is the axial magnetic flux and ϑ is the ignorable azimuthal coordinate. The geometry, indicated schematically in Fig. 1, is that of a solenoidal central cell of length $2L_c$ bounded by simple-mirror end cells of length $L_e \ll L_c$ in which the hot species is trapped. The plasma density and pressure are taken to be constant out to the plasma edge at $\psi=\psi_p$, and thereafter to fall smoothly to zero over a boundary layer of thickness $\Delta\psi$. There is a flux-conserving wall at $\psi=\psi_w$. We will investigate modes that are primarily flute-like in the end-plug regions. As a result the quantities in the end cell will ultimately depend on averages over the cell.

To simplify the analysis we take as the governing equation for the mth azimuthal mode of the perturbed potential

$$\frac{\partial}{\partial\psi}\left[\frac{\partial}{\partial z}\left(\psi\frac{\partial}{\partial\psi}\right)\right]\frac{\partial\varphi}{\partial z} - \frac{m^2}{4\psi}\frac{\partial^2\varphi}{\partial z^2} + \frac{\partial}{\partial\psi}\left(\frac{\rho}{B^2}\omega^2\psi\frac{\partial\varphi}{\partial\psi}\right)$$

$$-\frac{m^2}{4\psi}\frac{\rho}{B^2}\omega^2\varphi + G\varphi = 0 \qquad (1)$$

where

$$G = -m^2\frac{\kappa_\psi}{B}\frac{\partial}{\partial\psi}\left(P_w + HP_h\right) \quad . \qquad (2)$$

Here $\kappa_\psi \equiv \kappa/Br$ and κ is the field line curvature and r the radius. The quantity G agrees with Ref. 10 after we average G in the plugs over z with φ constant. This is exactly what we do to solve this problem and to this extent Eq. (1) leads to a rigorously correct form. We have expressed the curvature vector in the form $\hat{b}\cdot\nabla\hat{b} = \kappa_\psi\nabla\psi$, and the pressures of the warm and hot species as

$$P_{w,h} = \frac{1}{2}\left(P_\perp + P_\parallel\right)_{w,h} \ ,$$

with $\perp$ and $\parallel$ taken with respect to $\hat{b}$, the unit vector along the magnetic field. The function $H(\omega)$ appearing in Eq. (2) is[4,10]

$$H(\omega) = \frac{(\frac{\omega}{\omega_\kappa} + \tilde{\beta}_w)\{1 - \frac{\kappa_s}{|m|\kappa}\,[1+(\psi_p/\psi_w)^{|m|}]\}}{\frac{\omega}{\omega_\kappa} + \tilde{\beta}_w + \frac{\kappa_s}{|m|\kappa}\,[1+(\psi_p/\psi_w)^{|m|}] - 1} \tag{3}$$

where ω_κ is the average curvature-drift frequency and is given by

$$\frac{1}{\omega_\kappa} = \frac{q \int dz \, \frac{\partial(n_h/B)}{\partial\psi}}{m \int \frac{dz}{B}\, \kappa_\psi \, \frac{\partial(P_{\perp h}+P_{\parallel h})}{\partial\psi}} ,$$

$\tilde{\beta}_w$ is a parameter characterizing the β value of the warm plasma[11,12] defined as

$$\tilde{\beta}_w = \frac{\int dz \, \frac{\partial P_{\perp h}}{B\partial\psi} \, \frac{\partial(P_{\perp w}/B^2)}{\partial\psi}}{\int dz \, \frac{\kappa_\psi}{B} \, \frac{\partial(P_{\perp h}+P_{\parallel h})}{\partial\psi}} ,$$

and κ_s is the self-curvature caused by the bowing out of the equilibrium with the specific definition given by[6]

$$\frac{\kappa_s}{\kappa} = \frac{\frac{1}{2}\int \frac{dz}{B^2} \, \frac{\partial P_{\perp h}}{\partial\psi} \, \frac{\partial^2}{\partial z^2}\left(\frac{P_{\perp h}}{B^2}\right)}{\int \frac{dz}{B}\, \kappa_\psi \, \frac{\partial(P_{\perp h}+P_{\parallel h})}{\partial\psi}} .$$

Since the drive coefficient G is proportional to both κ_ψ and $\partial P/\partial\psi$, it is non-vanishing only in the boundary layer in the end cell. We also define a parameter g,

$$g = 1 - \frac{\kappa_s}{|m|\kappa}\left[1+\left(\frac{\psi_p}{\psi_w}\right)^{|m|}\right] .$$

III. DISPERSION RELATION

The axial structure of the eigenmode has a different character in the central and end cells. In the long solenoidal central cell where flux-tube bending and inertia are competitive effects and the local drive is small, the modes tend to be standing waves with axial wavelengths less than or comparable to L_c. In the short end cells, on the other hand, where flux-tube bending dominates the energetics ($\omega \simeq \omega_\kappa << v_e/L_e$, v_e = end cell Alfvén velocity), the modes are flute-like. We exploit this structure to obtain a dispersion relation by solving Eq. (1) in the central cell and match the result to a near-flute mode in the end cell, where the presence of the driving term makes the equation more complicated.

Toward this end, we introduce a Fourier representation in the central cell. Since the symmetry of Eq. (1) allows modes that are either even or odd in z, we set

$$\varphi(\psi,z) = \sum_{n=0}^{\infty} \varphi_n(\psi)\ CS(k_n z), \tag{4}$$

where

$$CS(k_n z) = \begin{cases} \cos\left(\frac{n\pi}{L_c} z\right), & n \geqslant 0, \text{ even modes} \\ \sin\left[\frac{(n-\frac{1}{2})\pi}{L_c} z\right], & n \geqslant 1, \text{ odd modes} \end{cases} \tag{5}$$

and

$$\varphi_n(\psi) = \frac{1}{(1+\delta_{n,o})} \frac{1}{L_c} \int_{-L_c}^{L_c} dz \ \varphi(\psi,z) \ CS(k_n z) \ . \tag{6}$$

The Fourier transformation of Eq. (1) then leads to an equation for $\varphi_n(\psi)$:

$$\frac{\partial}{\partial\psi}[\psi \frac{\rho_c}{B_c^2} (\omega^2-k_n^2 v_c^2) \frac{\partial\varphi_n}{\partial\psi}] - \frac{m^2}{4\psi} \frac{\rho_c}{B_c^2} (\omega^2-k_n^2 v_c^2)\varphi_n$$

$$= \frac{-1}{(1+\delta_{n,o})} \frac{2}{L_c} CS(k_n L_c) \ \{\frac{\partial}{\partial\psi} [\psi \frac{\partial}{\partial\psi} \frac{\partial\varphi(L_c)}{\partial z}] - \frac{m^2}{4\psi} \frac{\partial\varphi(L_c)}{\partial z} \} \tag{7}$$

where c refers to quantities in the central cell, v_c is the Alfvén speed in the central cell. Note that $CS(k_n L_c)=\pm 1$ and the right side of Eq. (7) involves the eigenfunction only at $z=L_c$. To eliminate the z-derivatives in the boundary term we integrate Eq. (1) over the end cell to obtain

$$\frac{\partial}{\partial\psi} [\psi \frac{\partial}{\partial\psi} \frac{\partial\varphi(L_c)}{\partial z}] - \frac{m^2}{4\psi} \frac{\partial\varphi(L_c)}{\partial z}$$

$$= \int_{L_c}^{L_c+L_e} dz [\frac{\partial}{\partial\psi} (\psi \frac{\rho\omega^2}{B^2} \frac{\partial\varphi}{\partial\psi}) - \frac{m^2}{4\psi} \frac{\rho\omega^2}{B^2} \varphi + G\varphi]$$

The flute-like nature of φ in the end cell allows us to replace the z-averaged eigenfunction on the right side with $\varphi(L_c)$, giving

$$\frac{\partial}{\partial\psi}\left[\psi\frac{\partial}{\partial\psi}\frac{\partial\varphi(L_c)}{\partial z}\right]-\frac{m^2}{4\psi}\frac{\partial\varphi(L_c)}{\partial z}=L_e\Big\{\frac{\partial}{\partial\psi}\left[\psi\frac{\rho_e\omega^2}{B_e^2}\frac{\partial\varphi(L_c)}{\partial\psi}\right]$$

$$-\frac{m^2}{4\psi}\frac{\rho_e\omega^2}{B_e^2}\varphi(L_c)+G_e\,\varphi(L_c)\Big\}\ , \tag{8}$$

where $L_eG_e=\int_{L_c}^{L_c+L_e}dz\,G(z)$ and $L_e\rho_e/B_e^2=\int_{L_c}^{L_c+L_e}dz\rho/B^2$. Substitution of Eq. (8) in Eq. (7) then gives an inhomogeneous equation for the ψ-dependence of the nth Fourier mode:

$$\frac{\partial}{\partial\psi}\left[\psi\frac{\rho_c}{B_c^2}(\omega^2-k_n^2v_c^2)\frac{\partial\varphi_n}{\partial\psi}\right]-\frac{m^2}{4\psi}\frac{\rho_c}{B_c^2}(\omega^2-k_n^2v_c^2)\varphi_n$$

$$=-\frac{2}{1+\delta_{n,0}}\frac{L_e}{L_c}\,CS(k_nL_c)\,\Big\{\frac{\partial}{\partial\psi}\left[\psi\frac{\rho_e\omega^2}{B_e^2}\frac{\partial\varphi(L_c)}{\partial\psi}\right]$$

$$-\frac{m^2}{4\psi}\frac{\rho_e\omega^2}{B_e^2}\varphi(L_c)+G_e\,\varphi(L_c)\Big\} \tag{9}$$

Outside the boundary layer, where ρ is independent of ψ, the solution of Eq. (9) is of power law form. The solution that is regular on axis and vanishes at the wall is

$$\varphi_n(\psi) = \begin{cases} \varphi_n(\psi_p)(\psi/\psi_p)^{|m|/2}, & 0 \leqslant \psi \leqslant \psi_p \\ \\ \varphi_n(\psi_p+\Delta\psi)\, \dfrac{(\psi/\psi_w)^{-m/2} - (\psi/\psi_w)^{m/2}}{[(\psi_p+\Delta\psi)/\psi_w]^{-m/2} - [(\psi_p+\Delta\psi)/\psi_w]^{m/2}}, & \psi_p+\Delta\psi \leqslant \psi \leqslant \psi_w \end{cases}$$

Thus, the logarithmic derivatives of φ_n at the inside and outside edge of the layer are

$$\frac{1}{\varphi_n}\frac{d\varphi_n}{d\psi} = \begin{cases} \dfrac{|m|}{2}\dfrac{1}{\psi_p}, & \psi = \psi_p \\ \\ -\dfrac{m}{2}\dfrac{\Lambda_m}{\psi_p}, & \psi = \psi_p + \Delta\psi \end{cases} \tag{10}$$

where the constant Λ_m accounts for the effect of the wall:

$$\Lambda_m = \frac{(\psi_w/\psi_p)^m + 1}{(\psi_w/\psi_p)^m - 1} + \mathcal{O}\left(\frac{\Delta\psi}{\psi_p}\right) \tag{11}$$

To complete the determination of φ_n we integrate between $\psi_p \leqslant \psi < \psi_p + \Delta\psi$ and find

$$\rho_e(\omega^2 - k_n^2 v_c^2)\,\psi\,\frac{\partial\varphi(\psi)}{\partial\psi} = \rho_c(\omega^2 - k_n^2 v_{co}^2)\,\frac{|m|}{2}\,\varphi_n(\psi_p)$$

$$+ \frac{2B_c^2}{1+\delta_{n,o}}\,\frac{L_e}{L_c}\,CS(k_n L_c)\Big[-\rho_e\,\frac{\omega^2}{B_e^2}\,\psi\,\frac{\partial\varphi(\psi,L_c)}{\partial\psi} + \frac{|m|}{2}\,\frac{\rho_{eo}\omega^2\varphi(\psi,L_c)}{B_e^2}$$

$$-2\int_{\psi_p}^{\psi} d\psi'\,G_e(\psi')\varphi(\psi,L_c)\Big] + \mathcal{O}\Big[\Big(\frac{m\Delta\psi}{\psi}\Big)^2\Big] \tag{12}$$

where the subscript "o" denotes the value of quantities for $\psi<\psi_p$. At $\psi=\psi_p$ Eqs. (10) and (12) give,

$$\Lambda_{|m|}\ \rho_{co}k_n^2\ v_{co}^2\ \varphi_n(\psi_p+\Delta\psi) = \rho_{co}(\omega^2-k_n^2v_{co}^2)\varphi_n(\psi_p)$$

$$+\frac{2}{1+\delta_{n,o}}\ B_c^2\ \frac{L_e}{L_c}\ CS(k_nL_c)\ \left[\frac{\rho_{eo}\omega^2}{B_e^2}-\frac{2}{m}\int_{\psi_p}^{\psi_p+\Delta\psi} d\psi\ G_e\right]\ \varphi(\psi_p,L_c) \qquad (13)$$

It remains to determine $\varphi_n(\psi+\Delta\psi)$. To do this we integrate Eq. (12) with the assumption that the inertia terms on the right hand side are small and can be neglected and this approximation is ultimately justified when $L_e/L_c<<1$. One then finds

$$\varphi_n(\psi_p+\Delta\psi) = \Phi_n(\psi_p) + \int_{\psi_p}^{\psi_p+\Delta\psi} \frac{d\psi}{\rho_c(\omega^2-\ k_n^2v_c^2)}\ \{\rho_{co}(\omega^2-k_n^2v_{co}^2)\ \frac{|m|}{2\psi_p}\ \varphi_n(\psi_p)$$

$$-\frac{2}{1+\delta_{n,o}}\ B_c^2\ \frac{L_e}{L_c}\ CS(k_nL_c)\int_{\psi_p}^{\psi}\frac{d\psi'}{\psi'}\ G_e(\psi')\varphi(L_c,\psi_p)$$

As long as $\omega^2-k_n^2v_{co}^2 > 0$ the denominator of the integrand vanishes somewhere inside the boundary layer. To treat the singularity we let ω have a small positive imaginary part and use

$$\mathrm{Im}\ \frac{1}{\rho_c(\omega^2-k_n^2v_c^2)} = -\ \frac{\pi}{\omega^2}\ \frac{\delta(\psi-\psi_n)}{|\partial\rho_c/\partial\psi|}\ \mathrm{sgn}(\omega)\ ,$$

where $\psi=\psi_n$ is the resonant surface for the nth Fourier harmonic, $k_n^2v_c^2(\psi_n) = (\mathrm{Re}\omega)^2$, to obtain

$$\varphi_n(\psi_p+\Delta\psi) = \varphi_n(\psi_p) - i\,\frac{\pi}{\omega^2}\,\frac{\vartheta_n \mathrm{sgn}(\omega)}{|d\rho_c/d\psi|_{\psi=\psi_n}}\,\{\frac{|m|}{2}\,\frac{1}{\psi_p}\,\rho_{co}(\omega^2-k_n^2v_{co}^2)\varphi_n(\psi_p)$$

$$-\frac{2}{1+\delta_{n,o}}\,B_c^2\,\frac{L_e}{L_c}\,CS(k_nL_c)\,\int_{\psi_p}^{\psi_n}\frac{d\psi'}{\psi'}\,G_e(\psi')\,\varphi(\psi_p,L_c)\} \tag{14}$$

where $\vartheta_n \equiv \vartheta[(\mathrm{Re}\omega)^2-k_n^2v_{co}^2)]$ is the Heaviside step function. [We are primarily interested in the imaginary part of $\varphi_n(\psi_p+\Delta\psi)-\varphi_n(\psi_p)$. The small real part leads to a small shift in the real frequency of $\mathcal{O}(m\Delta\psi/\psi_p)$.] Elimination of $\varphi_n(\psi_p+\Delta\psi)$ between Eqs. (13) and (14) then gives,

$$\varphi_n(\psi_p) = A_n(\omega)\,CS(k_nL_c)\,\varphi(\psi_p,L_c) \tag{15}$$

where

$$A_n(\omega) = \frac{2}{1+\delta_{n,o}}\,[\{\gamma_{MHD}^2(\psi_p) + i\vartheta_n\sigma_n[\gamma_{MHD}^2(\psi_p)-\gamma_{MHD}^2(\psi_n)]\}H(\omega) + \alpha\omega^2]$$

$$\times\,[(\Lambda_{|m|}+1)k_n^2v_{co}^2-\omega^2-i\vartheta_n\sigma_n(\omega^2-k_n^2v_{co}^2)]^{-1}\,, \tag{16}$$

$$\gamma_{MHD}^2(\psi) = -2|m|\,\frac{L_e}{L_cB_e}\,v_{co}^2\,\frac{\bar{\kappa}_\psi}{B_e}\,P_h(\psi) > 0\,, \tag{17}$$

$$\sigma_n = \frac{\pi}{2}\,m\,\frac{\rho_{co}}{\psi_p}\,\frac{k_n^2v_{co}^2}{|d\rho_c/d\psi|_{\psi=\psi_n}}\,\Lambda_m\,\frac{\mathrm{sgn}(\omega)}{\omega^2} << 1,\quad \alpha = \frac{L_e}{L_c}\,\frac{v_{co}^2}{v_{eo}^2} << 1 \tag{18}$$

In (16) the quantity $\bar{\kappa}_\psi$ is the average curvature in the boundary layer of the end cell, and we have neglected P_w relative to HP_h. The dimensionless parameter $\sigma_n \sim \mathcal{O}(m\Delta\psi/\psi_p)$ measures the rate at which central-cell surface waves resonant at $\psi=\psi_n$ are damped.[13,14] It is shown in Appendix A that the damping rate for such waves is $\gamma_n = \sigma_n \mathrm{Re}(\omega)\Lambda_{|m|}/2(1+\Lambda_{|m|})$. The end-cell ion inertia parameter α is small in the ratio L_e/L_c, and therefore is negligible in the long-central-cell limit.

The dispersion relation follows from using Eqs. (14)-(18) in Eq. (4) and requiring continuity of φ at $\psi=\psi_p$, $z=L_c$. We thereby obtain

$$1 = \sum_{n=0}^{\infty} \frac{2}{(1+\delta_{n,o})} \frac{[\gamma_{MHD}^2(\psi_p)+i\vartheta_n\sigma_n\Delta\gamma_{MHD}^2]H(\omega)+\alpha\omega^2}{(\Lambda_{|m|}+1)k_n^2v_{co}^2-\omega^2-i\vartheta_n\sigma_n(\omega^2-k_n^2v_{co}^2)} \tag{19}$$

where $\Delta\gamma_{MHD}^2 = \gamma_{MHD}^2(\psi_p)-\gamma_{MHD}^2(\psi_n) > 0$. In the following section we will use (19) to determine growth rates in the limit $|\mathrm{Im}\ \omega|<<\mathrm{Re}\ \omega$ for cases in which $|\mathrm{Im}\ \omega|$ is both large and small relative to the separation between resonant frequencies, $(\Lambda_{|m|}+1)^{1/2}\ \pi v_{co}/L_c$.

IV. DESTABILIZATION OF PRECESSIONAL MODES

In our analysis of the dispersion relation we assume that:

1) The curvature-drift frequency of the hot species is sufficiently high that MHD-like modes are stable, i.e.,

$$\gamma_{MHD}/\omega_\kappa << 1 \text{ and } \tilde{\beta}_w << 1.$$

2) The real frequency of the modes under investigaton is near the precessional frequency, and the imaginary part much smaller

$$\omega = \omega_o + \delta\omega,$$

$$\omega_o = \omega_\kappa\{1- \kappa_s[1+(\psi_p/\psi_w)^{|m|}]/|m|\kappa - \tilde{\beta}_w\} \equiv \omega_\kappa(g-\tilde{\beta}_w)$$

$$\delta\omega = \delta\omega_o + i\gamma,$$

$$|\delta\omega| << \omega_o . \tag{20}$$

Note that it follows from (20) that

$$H(\omega) \approx g^2 \frac{\omega_\kappa}{\delta\omega} \tag{21}$$

and

$$\sigma_n(\omega) = \sigma_n(\omega_o) .$$

We also define the parameter

$$\Delta\omega_n \equiv \omega_o - (\wedge_{|m|}+1)^{1/2}k_n v_{co}, \tag{22}$$

which measures the extent to which the mode is out of resonance with the nearest axial harmonic. Then

$$\omega_o^2 - (\wedge_{|m|}+1)k_n^2 v_{co}^2 = 2\omega_o\Delta\omega_n(1- \frac{\Delta\omega_n}{\omega_o}) .$$

3) The condition the growth rate is much smaller than the real frequency requires

$$\frac{g^2\gamma_{MHD}^2}{(g-\tilde{\beta}_w)\tilde{\omega}_\kappa^2} \approx g^2(g-\tilde{\beta}_w)\,\frac{\gamma_{MHD}^2 L_c^2}{v_c^2}, \quad \text{when} \quad \frac{v_c}{L_c\tilde{\omega}_\kappa(g-\tilde{\beta}_w)} \approx 1 .$$

In this limit we can consider the perturbed response as a separate precessional mode in the end cell and cold-plasma shear Alfven wave in the central cell, with a weak interaction between them. As $g-\tilde{\beta}_w$ becomes small, the precessional frequency becomes less than the minimum Alfven frequency and we lose the resonance of the two modes. When $g-\tilde{\beta}_w \to 0$, we assume $\omega \ll v_c/L_c$, and then we need only keep the n=0 term in Eq. (19). The dispersion relation is then that of the layer mode discussed in Ref. 15, with a frequency and growth rate $\omega \sim \gamma \sim (g\gamma_{MHD})^{2/3}\omega_\kappa^{1/3}$. The inequality, $\omega \ll v_c/L_c$, then restricts the axial length L_c under consideration to $L_c \ll \dfrac{v_c}{\tilde{\omega}_\kappa^{1/3}\gamma_{MHD}^{2/3}g^{2/3}}$. For the remainder of the paper we will only analyze modes with $\omega \gtrsim v_c/L_c$, as the low frequency limit has been discussed previously.[15]

(4) The end-cell ion-inertia term α is negligible which follows if $L_e/L_c \ll 1$.

A. $|\mathrm{Im}\,\omega| \ll (\Lambda_{|m|}+1)^{1/2}\,\pi\, v_{co}/L_c$

We first evaluate the growth rate, γ, when it is small compared to the separation between resonant frequencies.

When $|\delta\omega|$ is small enough to be neglected in the denominator in (19), viz.

$$\left|\frac{\delta\omega}{\Delta\omega_n}\right| << 1, \tag{23}$$

we find,

$$\frac{\delta\omega}{\omega_o} = 2g^2 \sum_{n=0}^{\infty}$$

$$\frac{[(\wedge_{|m|}+1)k_n^2 v_{co}^2-\omega_o^2]\gamma_{MHD}^2(\psi_p)-\vartheta_n\sigma_n^2(\omega_o^2-k_n^2v_{co}^2)\Delta\gamma_{MHD}^2+i\sigma_n[\wedge_{|m|}k_n^2v_{co}^2\Delta\gamma_{MHD}+(\omega_o^2-k_n^2v_{co}^2)\vartheta_n\gamma_{MHD}(\psi_n)]}{(1+\delta_{n,o})(g-\tilde{\beta}_w)\{[(\wedge_{|m|}+1)k_n^2v_{co}^2-\omega_o^2]^2+\vartheta_n\sigma_n^2(\omega_o^2-k_n^2v_{co}^2)^2\}} \tag{24}$$

First we assume $g>\tilde{\beta}_w$ and therefore $\omega_o>0$. Then γ is manifestly positive definite, and the mode is always unstable in this limit. Note that the growth rate vanishes with σ_n, which identifies the mechanism responsible for destabilizing the negative-energy precessional mode in this limit as damping of central-cell surface waves resonant at the Alfvén frequency at $\psi=\psi_n$. If $g-\tilde{\beta}_w<0$, Im $\delta\omega$ is manifestly negative, and we have stability. The reader will observe in the forthcoming calculations that stability always arises when $g-\tilde{\beta}_w<0$.

To check that (24) is consistent with (23a,b) we approximate the sum by its largest term and get ($n\neq0$),

$$\frac{\delta\omega}{\Delta\omega_n} = \frac{-g^2\gamma_{MHD}^2(\psi_p)}{(g-\tilde{\beta}_w)\Delta\omega^2}\ \frac{\left[-\left(1+\dfrac{\sigma_n\omega_o\gamma_n\Delta\gamma_{MHD}^2}{(\Delta\omega_n)^2\gamma_{MHD}(\psi_p)}\right)+i\,\dfrac{\gamma_n}{\Delta\omega}\right]}{[1+(\gamma_n/\Delta\omega)^2]} \tag{25}$$

Comparison with (23) then gives the validity condition

$$g^2\gamma_{MHD}^2 << (g-\tilde{\beta}_w)\Delta\omega_n^2 \left(1+\left|\frac{\gamma_n}{\Delta\omega_n}\right|\right). \tag{26}$$

If $\Delta\omega$, $\gamma_n << v_c/L_c$, the mode is sufficiently resonant so that only one term contributes significantly to the sum. Then take $\delta\omega$ and γ_n as the same order as $\Delta\omega_n$. Keeping the leading order part of the resonant term in (19) gives

$$\frac{\delta\omega}{\omega_o} = \frac{-g^2}{(g-\tilde{\beta}_w)} \frac{\gamma_{MHD}^2(\psi_p)+i\sigma_n\Delta\gamma^2{}_{MHD}}{\omega_o(\Delta\omega_n+\delta\omega) + i\gamma_n\omega_o} .$$

Multiplying through numerator and denominator by $\omega_o(\Delta\omega_n+\delta\omega)-i\gamma_n\omega_o$ then gives ($n\neq 0$)

$$\frac{\delta\omega}{\omega_o} = -\frac{g^2\gamma_{MHD}^2(\psi_p)}{(g-\tilde{\beta}_w)} \frac{\left(\frac{\Delta\omega_n}{\omega_o}+\frac{\delta\omega}{\omega_o}-\frac{i\gamma_n}{\omega_o}\right)}{(\Delta\omega_n+\delta\omega)^2+\gamma_n^2} . \tag{27}$$

This result overlaps with Eqs. (24) and (25). If $g^2\gamma_{MHD}^2/[\Delta\omega_n^2(g-\tilde{\beta}_w)]<<1$, then $(\delta\omega+\Delta\omega)$ can be neglected on the right hand side of Eq. 27 and we find,

$$\delta\omega \rightarrow i\,\frac{g^2\ \gamma_{MHD}^2(\psi_p)\gamma_n}{(g-\tilde{\beta}_w)(\gamma_n^2+\Delta\omega_n^2)} . \tag{28}$$

Note that here the growth rate can be <u>inversely</u> proportional to γ_n. In the opposite limit, $\gamma_n^2 << |\Delta\omega_n+\delta\omega|^2$, where γ_n^2 can be neglected in the denominator of (27), we have

$$\delta\omega^2+\Delta\omega_n\delta\omega+\frac{g^2\gamma_{MHD}^2(\psi_p)}{(g-\tilde{\beta}_w)} = i\,\frac{g^2\gamma_{MHD}^2(\psi_p)}{(g-\tilde{\beta}_w)}\,\frac{\gamma_n}{\Delta\omega_n+\delta\omega} \quad . \tag{29}$$

If we neglect the right side we find the solutions

$$\delta\omega_\pm = -\frac{\Delta\omega_n}{2} \pm \left[\left(\frac{\Delta\omega_n}{2}\right)^2 - \frac{g^2\gamma_{MHD}^2(\psi_p)}{(g-\tilde{\beta}_w)}\right]^{1/2} \quad . \tag{30}$$

Thus, if

$$\frac{g^2\gamma_{MHD}^2(\psi_p)}{(g-\tilde{\beta}_w)} > \left(\frac{\Delta\omega_n}{2}\right)^2 \quad , \tag{31}$$

there is instability, with

$$\gamma = \left[\frac{g^2\gamma_{MHD}^2(\psi_p)}{(g-\tilde{\beta}_w)} - \left(\frac{\Delta\omega_n}{2}\right)^2\right]^{1/2} , \tag{32}$$

the maximum growth rate occurring exactly at resonance. When the opposite of equation (31) applies, we need to keep the right hand side of Eq. (29) to find instability. Then setting,

$$\delta\omega = \delta\omega_\pm + \delta\omega_{1\pm},$$

with $|\delta\omega_1| << |\delta\omega_\pm|$, and $\delta\omega_\pm$ given by Eq. (30). We find that the solution for $\delta\omega_1$ is,

$$\delta\omega_{1\pm} = \frac{\pm i 2\gamma_n g^2 \gamma^2_{MHD}(\psi_p)}{(g-\tilde{\beta}_w)\Delta\omega^2_n \lambda(1\pm\lambda)} \tag{33}$$

and

$$\lambda^2 = 1 - \frac{4g^2\gamma^2_{MHD}}{\Delta\omega^2_n(1+\delta_{n,o})(g-\tilde{\beta}_w)} \quad . \tag{34}$$

Thus, the root near $\omega_+(\omega_-)$ is always unstable (stable) if $g-\tilde{\beta}_w>0$. If $g-\tilde{\beta}_w<0$ observe that both roots are stable as $\lambda>1$. Note that in the limit $\gamma_{MHD}(\psi_p)/\Delta\omega_n \to 0$, the unstable root agrees with Eq. (28).

B. $|\gamma| > (\Lambda_{|m|}+1)^{1/2}\ \pi\ v_{co}/L_c$

If the growth rate is larger than the separation between resonant frequencies no single term dominates the sum over axial harmonics. Then in the limit where the damping of central cell surface modes is weak enough that σ_n can be neglected in the dispersion relation, the summation can be expressed in closed form. With the aide of identities found in Ref. 16, we find that Eq. (19) becomes

$$1 = \frac{2\gamma_{MHD}^2(\psi_p)g\left(\frac{\omega}{\omega_\kappa}+\tilde{\beta}_w\right)}{\left(g-\frac{\omega}{\omega_\kappa}-\tilde{\beta}_w\right)}\sum_{n=0}^{\infty}\frac{1}{(1+\delta_{n,o})}\frac{1}{[\omega^2-(\Lambda_{|m|}+1)k_n^2v_{co}^2]}$$

$$= \frac{\gamma_{MHD}^2(\psi_p)g\left(\frac{\omega}{\omega_\kappa}+\tilde{\beta}_w\right)L_c\,CT\left(\frac{\omega L_c}{(1+\Lambda_{|m|})^{1/2}v_{co}}\right)}{\left(g-\frac{\omega}{\omega_\kappa}-\tilde{\beta}_w\right)(\Lambda_{|m|}+1)^{1/2}v_{co}\omega}$$

$$\approx \frac{g^2\gamma_{MHD}^2(\psi_p)\,L_c\,CT\left(\frac{\omega L_c}{(1+\Lambda_{|m|})^{1/2}v_{co}}\right)}{(g-\tilde{\beta}_w)\,\delta\omega\,(\Lambda_{|m|}+1)^{1/2}\,v_{co}} \qquad (35)$$

where CT = tan(−cot) for odd (even) modes and $\omega = \omega_\kappa(g-\tilde{\beta}_w)+\delta\omega$. For $|\mathrm{Im}\omega| > (\Lambda_{|m|}+1)^{1/2}\,v_{co}/L_c$ the trigonometric functions are well approximated by

$$CT\left[\frac{\omega L_c}{(\Lambda_{|m|}+1)^{1/2}\,v_{co}}\right] = i\ ,$$

in which case (35) gives

$$\delta\omega = i\gamma = \frac{i}{g-\tilde{\beta}_w}\,\frac{g^2\gamma_{MHD}^2 L_c}{(\Lambda_{|m|}+1)^{1/2}\,v_{co}}\ . \qquad (36)$$

Thus, we find instability when $g-\tilde{\beta}_w>0$, even in the absence of damping due to Alfven resonance excitations. Here the negative-energy precessional mode is destabilized by excitation of many Alfven surface modes.

The validity condition on (36) follows by demanding $|\gamma| > (\Lambda_{|m|}+1)^{1/2}\, v_{co}/L_c$,

$$\frac{\gamma_{MHD}^2 g^2 L_c^2}{(\Lambda_{|m|}+1) v_{co}^2} > g - \tilde{\beta}_w \ ,$$

while the condition for neglecting σ_n in the dispersion relation is $\gamma_n \ll \gamma$. Of course it has been assumed throughout our analysis that $\gamma \ll \omega_o$.

V. CONCLUSION

We have analyzed the interaction of long-wavelength precessional modes of a hot component plasma, with Alfvén waves in the central cell of the tandem. By assuming that the wave interaction of the precessional mode in the plugs with the cold plasma waves of the central cell are weak, we have calculated the growth rate, γ, that arises due to this interaction. Roughly, the growth rate can be expressed by the interpolation formula,

$$\gamma = \min \left\{ \frac{g^2 \gamma_{MHD}^2 \gamma_n}{(g-\beta_w)(\gamma_n^2 + \Delta\omega_n)} \ , \ \frac{2\gamma_n \left(\frac{g\gamma_{MHD}^2}{\Delta\omega_n} \right)^2}{\lambda(1-\lambda)(g-\beta_w)} \ , \right.$$

$$\left. \max \left[\frac{g\gamma_{MHD}}{(g-\beta_w)^{1/2}} \ , \ \frac{g^2 \gamma_{MHD}^2 L_c}{(g-\beta_w)(\Lambda_{|m|}+1)^{1/2} v_{co}} \right] \right\} \tag{37}$$

These growth rates are depicted schematically in Fig. 2. These results are qualitatively similar to that obtained from the eikonal theory, in that the negative energy precessional mode is destabilized either by exciting a surface Alfvén wave, or through the dissipation associated with the Alfvén resonance. However, with a conducting wall in the environment of the hot plasma it is possible to convert the m=1 precessional wave to a positive energy wave, and in that case the Alfvén wave excitation in the central cell is stable.[6] It should be noted that Eq. (37) is not valid if $g-\tilde{\beta}_w$ is too small, as the assumption $\gamma/\omega_o \ll 1$ fails. This regime is the threshold for the onset of MHD-like instability.

This stabilization mechanism is not effective for higher m numbers, but it is still possible to convert to positive-energy waves using finite Larmor radius. For example, the previous eikonal analysis[4] leads to a form similar to Eq. (35),

$$1 = \frac{\left(\frac{\omega}{\omega_\kappa} + \tilde{\beta}_w\right)L_c}{\left[1 - \frac{\omega}{\omega_\kappa} - \tilde{\beta}_w\right]} \frac{\gamma_{MHD}(eik)}{v_c \omega} CT\left(\frac{\omega L_c}{v_c}\right) \tag{38}$$

where

$$\gamma^2_{MHD}(eik) = \frac{\int \frac{ds}{B} \kappa_\psi \frac{\partial}{\partial \psi} (P_{\perp h} + P_{\parallel h})}{\int \frac{ds k_\perp^2 \rho}{B^2}}$$

where $k_\perp$ is the perpendicular wave number. In obtaining Eq. (38), the hot component FLR terms derived previously,[4] were neglected. However, when including the FLR terms, the dispersion relation becomes

$$1 = \frac{\left(\frac{\omega}{\omega_\kappa} + \bar{\beta}_w\right) L_c \left(1 - k_\perp^2 a_h^2 \frac{\beta_h{}'}{2\kappa\Delta}\right)\gamma_{MHD}^2(eik)}{\left[1 - \frac{\omega}{\omega_\kappa} - \bar{\beta}_w - k_\perp^2 a_h^2 \frac{\beta_h}{2\kappa\Delta}\right] v_c\omega} CT\left(\frac{\omega L_c}{v_c}\right) \tag{39}$$

where a_h is the hot-electron Larmor radius. The structure of this dispersion relation is the same as Eq. (35), and complete stabilization results if $k_\perp^2 a_h^2\, \beta_h/(2\kappa\Delta)>1$.

Eq. (39) is still an eikonal result. For the layer mode analysis the results are not complete. Recent analysis[7] applies to the layer mode FLR problem when $m\Delta/r << 1$, $m^2 >> 1$ and an FLR term of the form $m^2\beta_h a_h^2/(2\kappa\Delta r_p^2)$ arises in the hot component response with $r_p=(2\psi_p/B)^{1/2}$. However, as the m=1 mode is nearly rigid so that there should be no FLR term, we expect the FLR term to be proportional to m^2-1. Thus, the conjectured dispersion relation is,

$$1 = \frac{\left(\frac{\omega}{\omega_\kappa} + \bar{\beta}_w\right)\left(1 - \frac{\kappa_s}{|m|\kappa}\left[1+(\psi_p/\psi_w)^{|m|}\right] - \frac{(m^2-1)a_h^2\beta_h}{2\kappa\Delta r_p^2}\right)\gamma_{MHD}^2(\psi_p)}{1 - \frac{\omega}{\omega_\kappa} - \bar{\beta}_w - \frac{(m^2-1)a_h^2\beta_h}{2\kappa\Delta r_p^2} - \frac{\kappa_s}{|m|\kappa}\left[1+(\psi_p/\psi_w)^{|m|}\right]} \cdot \frac{L_c\, CT\left(\frac{\omega L_c}{(\Lambda_{|m|}+1)^{1/2} v_{co}}\right)}{\omega v_{co}(\Lambda_{|m|}+1)^{1/2}} \tag{40}$$

Thus, if $\frac{(m^2-1)a_h^2\beta_h}{2\kappa\Delta r_p^2} + \frac{\kappa_s}{|m|\kappa}\left[1+ (\psi_p/\psi_w)^{|m|}\right] > 1$, one achieves complete stabilization with a close fitting conducting wall and with a hot component whose Larmor radius is a reasonably large fraction of the radial scale length. In reactor conditions one requires ions in the MeV regime

Appendix A: Damped Central-Cell Surface Modes

Here we derive the damping rate for waves resonant at the Alfvén freqency in the boundary layer of a cylindrical plasma column from our general dispersion relation Eq. (19). The appropriate limit is $\gamma^2_{MHD}(\psi_p)$, $\Delta\gamma^2_{MHD}$, $\alpha \to 0$, in which case the denominator in (19) must vanish for some axial mode number. Thus, the dispersion relation is simply

$$(\Lambda_{|m|}+1)k_n^2v_{co}^2 - \omega^2 + i\sigma_n(\omega^2-k_n^2v_{co}^2) = 0$$

Writing $\omega=\omega_n-i\gamma_n$ and discarding terms of $\sigma(\gamma_n^2)$, we obtain

$$\omega_n^2 = (\Lambda_{|m|}+1)k_n^2v_{co}^2, \tag{A1}$$

$$\gamma_n = \frac{\sigma_n}{2}\frac{\Lambda_{|m|}}{\Lambda_{|m|}+1}\omega_n > 0 . \tag{A2}$$

(Note that for convenience we have defined γ_n to be positive.) The waves have a real frequency equal to the Alfvén frequency on that surface in the boundary layer where the density is $\rho_o/(1+\Lambda_{|m|})$, i.e., the resonant surface $\omega_n^2-k_n^2B_o^2/\rho(\psi_n) = 0$. They are damped at a rate proportional to the thickness of the boundary layer [recall Eq. (17)]. These waves have been studied by Chen and Hasegawa[15] and Hasegawa[16]; Eqs. (A1,A2) are equilivalent to their results, corrected for the presence of a flux-conserving wall.

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Figure Captions

1. Geometry of the axisymmetric tandem mirror. Axial profiles of B and the number density of warm and hot particles is shown in (a). In (b) the ψ-dependence of B, the magnetic field, ρ, the density, and p, the pressure, is indicated.
2. Schematic diagram of the parameter regimes in which various limits of the dispersion relation (indicated by the circled equation numbers) are valid.

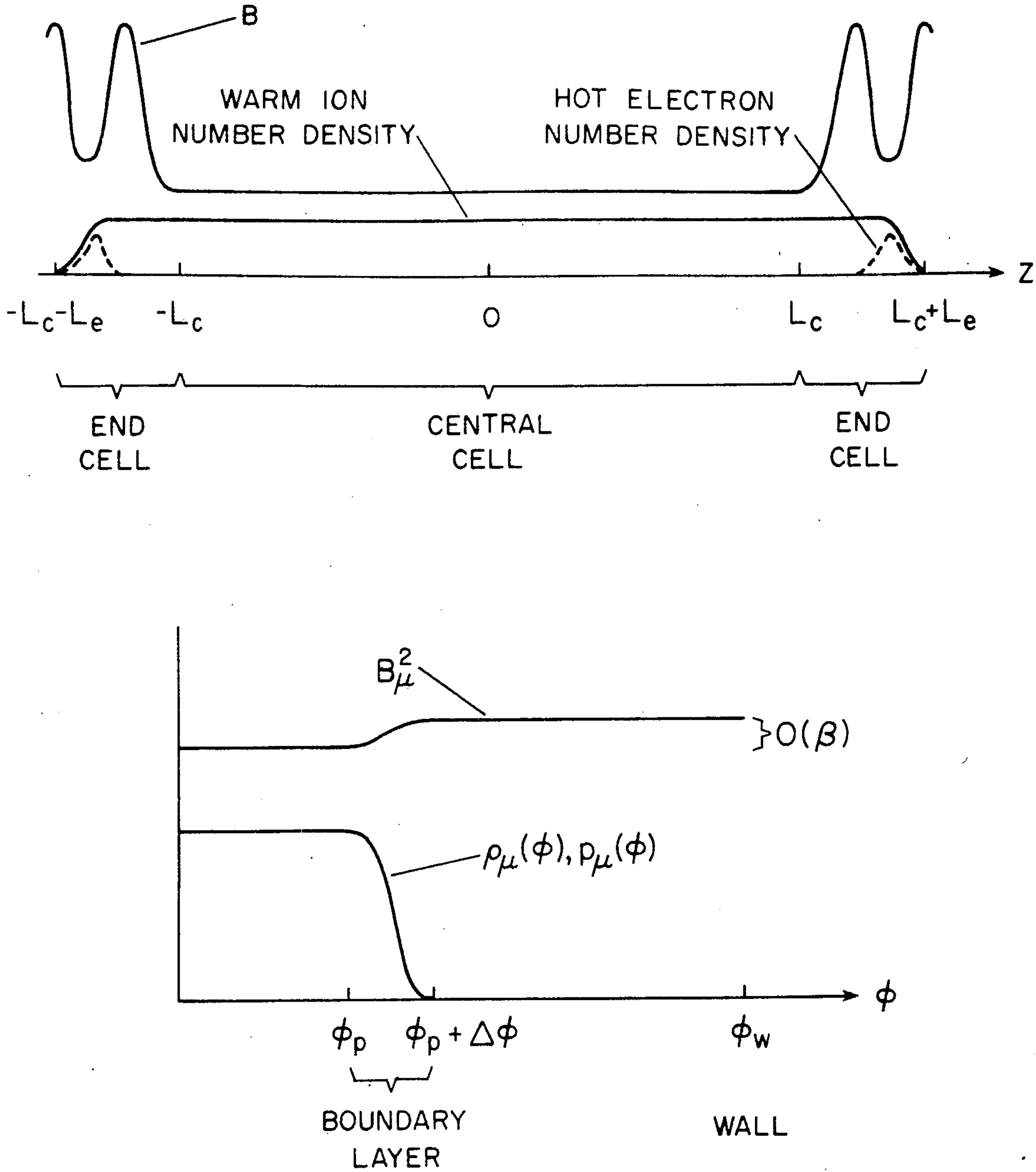
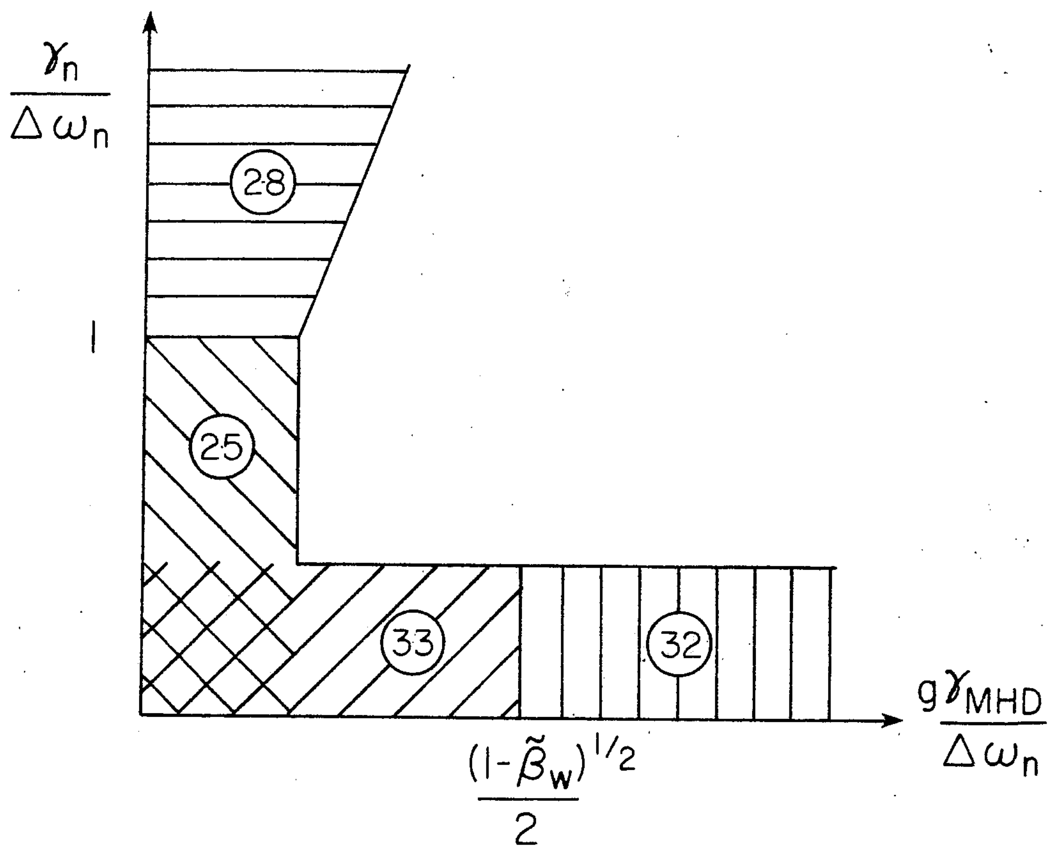

B
WARM ION
NUMBER DENSITY
HOT ELECTRON
NUMBER DENSITY
Z
$-L_c-L_e$
$-L_c$
0
L_c
L_c+L_e
END
CELL
CENTRAL
CELL
END
CELL
B_μ^2
$\}O(\beta)$
$\rho_\mu(\phi), p_\mu(\phi)$
ϕ
ϕ_p
$\phi_p+\Delta\phi$
ϕ_w
BOUNDARY
LAYER
WALL

FIG. 1



$\frac{\gamma_n}{\Delta\omega_n}$
1
28
25
33
32
$\frac{g\gamma_{MHD}}{\Delta\omega_n}$
$\frac{(1-\tilde{\beta}_w)^{1/2}}{2}$

FIG. 2