Twisting Space-Time: Relativistic Origin of Seed Magnetic Field and Vorticity

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We demonstrate that a purely ideal mechanism, originating in the space-time distortion caused by the demands of special relativity, can break the topological constraint (leading to helicity conservation) that would forbid the emergence of a magnetic field (a generalized vorticity) in an *ideal* nonrelativistic dynamics. The new mechanism, arising from the interaction between the inhomogeneous flow fields and inhomogeneous entropy, is universal and can provide a finite seed even for mildly relativistic flows.

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The origin of magnetic fields or any kind of generalized vorticity is one of the challenging unsolved problems of theoretical physics [1–3]. The mathematical and dynamical similarity between magnetic fields and fluid vorticity (axial vector fields) imparts both elegance and usefulness to the concept of generalized vorticity. Unless explicitly stated, generalized vorticity, denoted by Ω, will symbolize all physical quantities of this nature.

The origin problem has its genesis in the fact that the circulation associated with Ω must vanish for every ‘‘ideal force’’ including the entropy-conserving thermodynamic force. The reasons lie deep in the Hamiltonian structure governing the dynamics of an ideal fluid; the constrained dynamics implies the conservation of a ‘‘topological charge’’ that measures the generalized vorticity of the fluid [4–6]—the invariance of the generalized helicity, which, for a nonrelativistic charged flow, takes the familiar form $K = \int \boldsymbol{P} \cdot \Omega dx$, where $\boldsymbol{P} = m\boldsymbol{V} + (q/c)\boldsymbol{A}$ is the canonical momentum and $\Omega = \nabla \times \boldsymbol{P}$ is the generalized vorticity or generalized magnetic field (m: mass of a particle; q: charge of a particle; $\boldsymbol{V}$: fluid velocity; $\boldsymbol{A}$: vector potential; $\boldsymbol{B}$: magnetic field) [7,8]. Consequently, in any ideal leading order model, Ω (consisting of both magnetic and kinematic components) cannot emerge from a zero initial value.

The problem of unearthing a primary generation mechanism for the magnetic field, found to be important in every scale hierarchy of the Universe, has defied a satisfactory solution to date [3]. In particular, isolating processes that might create a seed magnetic field, which the so-called dynamo mechanism [9] could greatly ‘‘amplify,’’ constitutes a major quest in astrophysical plasma research. Since the topological constraint on the ideal fluid forbids the vorticity to emerge, one resorts to ‘‘nonideal dynamics’’ to effect a change. A typical example is the baroclinic mechanism [10], or Biermann battery [11], involving nonideal thermodynamics in which the gradients of pressure and temperature have different directions [12,13]. A velocity-space nonequilibrium distribution also provides a source of magnetic field via the so-called Weibel instability [14]. In early cosmology, inflation [15,16], a QCD phase transition [17,18], or a radiation effect [19] could create a source. While these mechanisms may, and likely will, play important roles in magnetic-field generation at some scales, none of these could be considered a universal mechanism operating at all scales [3,9,20].

The search for such a universal mechanism provided the stimulus for this Letter in which we make a clean break with the standard practice: Instead of relying on nonideal mechanisms (such as the baroclinic effect), which are too weak to account for the observed cosmic magnetic fields [3], we will show that Ω can be generated in strictly ideal dynamics, as long as the dynamics is explicitly embedded in the space-time dictated by the demands of special relativity. The generalized vorticity is, then, generated through a source term born out of the special-relativistic ‘‘modifications’’ to the interaction of an inhomogeneous flow with inhomogeneous entropy. The use of ideal dynamics is justified because the new ideal term turns out to be much larger than the standard nonideal damping terms (entropy production due to resistivity, for example). To set the stage for a proper relativistic calculation, we begin with some nonrelativistic preliminaries and see how an ideal mechanics restricts the topology of fields.

The circulation $\oint_L \delta Q$, associated with a physical quantity δQ, calculated along the loop L, may be zero or finite depending on whether δQ equals an exact differential $d\varphi$ (φ being a state variable) or not. For example, if $\delta Q = T d\sigma$ (T: temperature; σ: entropy), the circulation is generally finite and measures the heat gained in a quasistatic thermodynamic cycle.

A circulation theorem pertains to a ‘‘movement’’ of loops. Along the time-dependent loop $L(t)$, convected by the fluid motion, the rate of change of circulation associated with the canonical momentum $\oint_{L(t)} \boldsymbol{P} \cdot d\boldsymbol{x}$ is identically zero in an ideal fluid. In fact, if two loops $L(t)$ and $L(t')$ are connected by the ‘‘flow’’ $d\boldsymbol{x}/dt = \boldsymbol{V}$, the rate of change of circulation is calculated as

$$\frac{d}{dt} \oint_{L(t)} \boldsymbol{P} \cdot d\boldsymbol{x} = \oint_{L(t)} [\partial_t \boldsymbol{P} + (\nabla \times \boldsymbol{P}) \times \boldsymbol{V}] \cdot d\boldsymbol{x}. \quad (1)$$

Coupling it with the ideal equation of motion

$$\partial_t \boldsymbol{P} + (\nabla \times \boldsymbol{P}) \times \boldsymbol{V} = -\nabla \mathcal{E},$$

with the effective energy $\mathcal{E} = mV^2/2 + \phi + h$ (ϕ: potential energy; h: molar enthalpy), shows that the rate of change of circulation equals the circulation of an exact fluid-dynamic force derived from the energy density, i.e., $\oint_{L(t)} \nabla\mathcal{E} \cdot d\boldsymbol{x} = \oint_{L(t)} d\mathcal{E} = 0$. In the standard nonrelativistic description of an ideal fluid, therefore, if the initial state has no circulation (vorticity), the later state will also be vorticity-free (Kelvin's circulation theorem). For the vorticity to be created, the "force" on the fluid must not be an exact differential.

How will special relativity affect the ideal dynamics? Interestingly, the very basic kinematic modification—the Jacobian weight $\gamma^{-1} = \sqrt{1-(V/c)^2}$ reflecting the space-time unity imposed by special relativity—destroys the exactness of the ideal thermodynamic force; the loop integral $\oint_{L(t)} dH$ (for some exact differential dH) transforms to $\oint_{L(t)} \gamma^{-1} dH$, which is no longer zero. Thus Ω could be created within purely ideal dynamics.

For a geometric visualization of the new creation mechanism, we now examine the fundamental reconstruction of the notion of circulation in relativistic dynamics. In the relativistic space-time, the loop $L(t)$ pertaining to a "synchronic space" (t = constant cross section of space-time in a reference frame) ceases to be the appropriate geometric object along which the circulation must be evaluated. The loop moves in space-time with a 4-velocity $U^\mu = (\gamma, \gamma V^j/c)$ (V^j: the reference-frame velocity), and the relativistic circulation must be described as a function of the proper time s. In Fig. 1, the respective evolutions of the

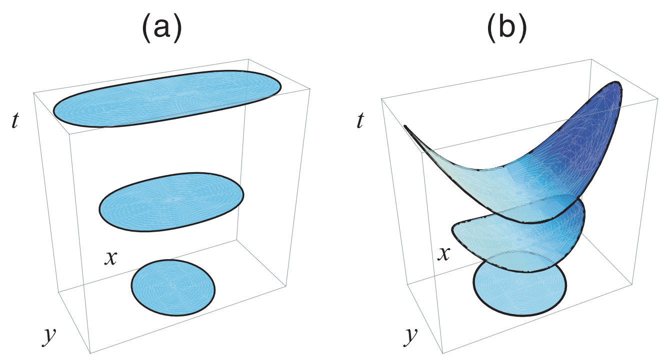

FIG. 1 (color online). Two figures compare the evolution of a surface and its boundary (loop) moved in space-time, respectively, by (a) the nonrelativistic velocity ($dx_j/dt = V_j$: 3-vector) and (b) the relativistic 4-velocity ($dx_\mu/ds = U_\mu$). The figures are drawn in the x-y-t coordinate with $\boldsymbol{V}/c = (\tanh x, 0, 0)$ (thus $\gamma = \mathrm{sech}^{-1} x$). In the Lorentz-covariant theory, the circulation theorem applies to a loop $L(s)$ that is moved by the 4-velocity U_μ. However, the vorticity (or magnetic field) is a reference-dependent quantity defined on the synchronic cycle $L(t)$, requiring a mapping from the relativistically distorted $L(s)$ to $L(t)$; this map multiplies the thermodynamic force by a Jacobian weight γ^{-1}, breaking the exactness of the differential form.

"synchronic loop" $L(t)$ and the "relativistic loop" $L(s)$ are compared. The synchronicity of the loop $L(s)$ is broken by the nonuniformity of the proper time. The circulation of a 4-vector $\wp^\mu$ along the relativistic loop $L(s)$ obeys

$$\frac{d}{ds}\left(\oint_{L(s)} \wp^\mu dx_\mu\right) = \oint_{L(s)} (\partial^\mu \wp^\nu - \partial^\nu \wp^\mu) U_\nu dx_\mu. \quad (2)$$

If $\wp^\mu$ is an appropriate momentum, the relativistic equation of motion relates the integrand $(\partial^\mu \wp^\nu - \partial^\nu \wp^\mu)U_\nu$ with an effective force. If the force is exact, the relativistic circulation will be conserved; the ideal fluid does, indeed, obey an appropriate relativistic Kelvin circulation theorem. However, vorticity (or magnetic field) is defined on synchronic space (hence, it is reference-dependent); its circulation still pertains to the synchronic loop $L(t)$. The field must be mapped from the naturally distorted $L(s)$ back to $L(t)$—this reciprocal distortion, represented by a Jacobian γ^{-1}, imparts a shear to the thermodynamic force (i.e., changes dH to $\gamma^{-1}dH$), destroying its exactness.

These formal considerations will, now, be translated into an explicit calculation showing how relativity helps us to circumvent the "no-circulation" theorem. A covariant theory of vorticity generation follows from the recently formulated unified theory of relativistic, hot magnetofluids [21]. The central construction of this theory is the relativistic generalized 4-momentum $\wp^\mu = mcfU^\mu + (q/c)A^\mu$ (A^μ: 4-vector potential) and the antisymmetric tensor

$$M^{\mu\nu} = \partial^\mu \wp^\nu - \partial^\nu \wp^\mu = mcS^{\mu\nu} + (q/c)F^{\mu\nu}, \quad (3)$$

where $S^{\mu\nu} = \partial^\mu(fU^\nu) - \partial^\nu(fU^\mu)$ is the flow-field tensor representing both the inertial and thermal forces and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic tensor. The factor f represents the thermally induced increase in effective mass: an increasing function of temperature T, $f \approx 1$ in the nonrelativistic limit, rising to $f \approx 6.66$ for $T = 1$ MeV [4,22]. The generalized vorticity $\hat{\Omega}$ (or the generalized magnetic field $\hat{\boldsymbol{B}}$) is defined by $\nabla \times \wp$ [or $(c/q)\nabla \times \wp$], where $\wp$ is the vector part of $\wp^\mu$. The equation of motion is written succinctly as [21,23]

$$cM^{\mu\nu}U_\nu = T\partial^\mu \sigma. \quad (4)$$

Substituting (4) into (2) shows that the rate of change of circulation of $\wp^\mu$ is balanced by the integral along $L(s)$ of $(T/c)\partial^\mu \sigma$. It is the vector part of (4)

$$q\left[\hat{\boldsymbol{E}} + \left(\frac{\boldsymbol{V}}{c}\right) \times \hat{\boldsymbol{B}}\right] = \frac{cT}{\gamma}\nabla\sigma \quad (5)$$

that explicitly shows the relativistic modification of the force $T\nabla\sigma$ by the factor γ^{-1}. Here, the generalized electric field $\hat{E}^j = E^j + (mc/q)S^{0j}$ satisfies Faraday's law $\partial_t \hat{\boldsymbol{B}} = -\nabla \times \hat{\boldsymbol{E}}$. The appearance of γ^{-1} on the right-hand side is due to the mapping back of the relativistic space-time onto the synchronic space in which the conventional circulation and the vorticity are to be calculated. To evaluate the rate of change of $\hat{\boldsymbol{B}}$ (with respect to the reference time t), we must go back to (5), whose curl reveals the source for magnetic-field generation:

$$\mathfrak{S} = -\nabla \times \left(\frac{cT}{q\gamma}\nabla\sigma\right) = -\nabla\left(\frac{cT}{q\gamma}\right) \times \nabla\sigma, \quad (6)$$

which may be broken into the familiar baroclinic term $\mathfrak{S}_B = -(c/q\gamma)\nabla T \times \nabla\sigma$ and the relativistically induced new term

$$\mathfrak{S}_R = -\left(\frac{cT}{q}\right)\nabla\gamma^{-1} \times \nabla\sigma = -\left(\frac{c\gamma T}{2q}\right)\nabla\left(\frac{V}{c}\right)^2 \times \nabla\sigma. \quad (7)$$

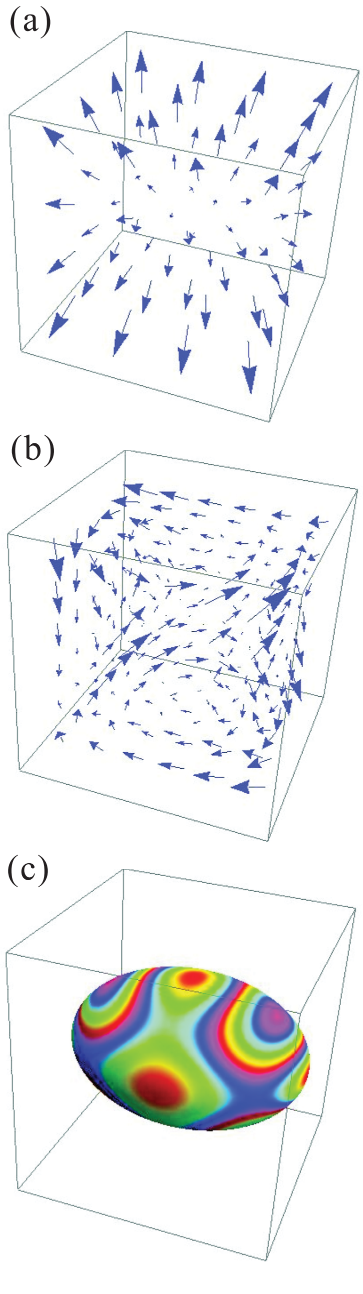

FIG. 2 (color). Relativistic generation of the vorticity for a barotropic flow accelerated (for an infinitesimal time) by an internal energy $\mathcal{E} = x^2 + 2y^2 + 2.5z^2$. (a) The vector image of $\boldsymbol{V} \propto \nabla\mathcal{E}$. (b) The generated vorticity $\mathfrak{S}_R$ after an infinitesimal time. (c) The magnitude of the enstrophy density (square of the vorticity) plotted on an isobaric surface of $\mathcal{E}$.

The discovery of $\mathfrak{S}_R$ is the principal result of this Letter. The following conclusions are readily deducible. (1) For homogeneous entropy, there is no vorticity drive—either baroclinic or relativistic. (2) As long as the kinetic energy is inhomogeneous, its interaction with inhomogeneous entropy keeps $\mathfrak{S}_R$ nonzero, even in a barotropic fluid. In Fig. 2, we show an example of the vorticity generation from an initial (incident) vorticity-free flow. Here we assume the simplest scenario. (i) We start with an inhomogeneous scalar, the energy density (enthalpy) $\mathcal{E}$. (ii) This drives a potential (vorticity-free) flow $\boldsymbol{V} \propto \nabla\mathcal{E}$. In Fig. 2(a), the vector image of $\boldsymbol{V}$ is drawn for $\mathcal{E} = ax^2 + by^2 + cz^2$. (iii) If the fluid is barotropic, only the relativistic source $\mathfrak{S}_R \propto \nabla\gamma^{-1} \times \nabla\mathcal{E}$ survives (here we assume $f \approx 1$). In Fig. 2(b), we show $\mathfrak{S}_R$, the generated vorticity after an infinitesimal time. The enstrophy density $|\mathfrak{S}_R|^2$ is shown in Fig. 2(c). We observe that the inhomogeneity of the flow velocity (a direct consequence of the inhomogeneous scalar field) yields a vorticity (an axial vector field) by the relativity-induced twist of space-time. (3) When baroclinic drive is nonzero and, in addition, the kinematic and thermal gradients are comparable, we can estimate

$$\frac{|\mathfrak{S}_R|}{|\mathfrak{S}_B|} \approx \frac{(V/c)^2}{1-(V/c)^2}. \quad (8)$$

For highly relativistic flows (cosmic particle-antiparticle plasmas, electron-positron plasmas in the magnetosphere of neutron stars, relativistic jets, etc.), $\mathfrak{S}_R$ will be evidently dominant and can be far larger than the conventional estimates for the baroclinic mechanism. One must also remember that most long-lived plasmas will tend to have $\nabla T \times \nabla\sigma = 0$ because of the thermodynamic coupling of temperature and entropy. In this large majority of physical situations, $\mathfrak{S}_R$ may be the only vorticity generation mechanism; no physical constraints will force the alignment of the gradients of kinematic γ and statistical σ. Thus, the relativistic drive is truly universal. (4) Finally, we compare the strength of the new relativistic drive with damping from the nonideal dissipative processes to determine general conditions when such a drive will be able to overcome the inherent dissipative tendencies. A systematic and rigorous inclusion of dissipative processes in a relativistic system is nontrivial, and we will not attempt to do it here. We will, instead, attempt a simple heuristic approach. The resistive dissipative term $\mathfrak{D} = \nabla \times (\eta\boldsymbol{J}) = (c/4\pi)\nabla \times (\eta\nabla \times \boldsymbol{B})$, which is pertinent for the nonrelativistic evolution equation, will be assumed to be valid for the relativistic equation. Notice that, in the initial stages when one is looking to create the "seed" field, the resistive dissipation term will be necessarily negligible because the vorticity is zero in the beginning. Being proportional to the generated vorticity, the resistive dissipation term will become progressively large as the fluid builds up vorticity. At some arbitrary stage in the development, a ratio of the strengths of the source $\mathfrak{S}_R$ and the sink $\mathfrak{D}$ may be written as

$$\frac{|\mathfrak{S}_R|}{|\mathfrak{D}|} \approx \frac{(\gamma V/c)^2 (T/mc^2)}{(V_A/c)(\nu/\omega_p)}, \tag{9}$$

where we have introduced the Alfvén speed V_A, the collision frequency ν, and the plasma frequency ω_p. We may deduce the following. (a) The denominator of (9) is always very small. For all plasmas where the ideal fluid models are valid, $\nu/\omega_p \ll 1$ (for an electron plasma, for instance, $\nu/\omega_p = 10^{-9} n^{1/2}/T^{3/2}$, where the density is in Gaussian units and the temperature in electron volts). Similarly, in all problems of interest to this Letter, $V_A/c \ll 1$. (b) The numerator, of course, has a large range of variation. For plasmas that are very relativistic [in both directed and random (thermal) motion], the numerator is much greater than unity. (c) Even for nonrelativistic plasmas, the drive $\mathfrak{S}_R$ can easily overcome damping. For a hypothetical electron fluid with $n = 10^{10}/\mathrm{cm}^3$, $T = 20$ eV ($T/mc^2 = 4 \times 10^{-5}$), $V/c = 10^{-2}$, one can calculate $|\mathfrak{S}_R|/|\mathfrak{D}| \simeq B^{-1}$ (in gauss). Thus the drive remains dominant until one reaches the magnetic fields of 1 G or so. Notice that the ratio is really independent of the density and is immensely boosted up by higher temperatures. The "relativistic drive" has turned out to be strong even for plasmas that are quite mildly relativistic.

We have thus found that a recourse to special relativity uncovers an ideal, ubiquitous, fundamental vorticity generation mechanism. The exploration of this mechanism is likely to help us understand, *inter alia*, the origin of the magnetic fields in astrophysical and cosmic settings.

We end this Letter by making a few comments about the finer points concerning vorticity, the generalized vorticity, and the relativistic generalized vorticity. As the physical system becomes more and more complicated (from an uncharged fluid to a charged fluid to a relativistic charged fluid), one must invent more and more sophisticated physical variables so that the fundamental dynamical structure (vortical form), epitomized in (4) is maintained. We do this because the very beautiful vortical structure is so thoroughly studied that reducing a more complicated system to this form immediately advances our understanding of new larger physical systems or, possibly, of more advanced space-time geometries.

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[22] In the relativistic regime, the definition of vorticity must be appropriately generalized: Based on the Lorentz-covariant magnetofluid unified theory [21], the canonical momentum $\boldsymbol{P} = mf\gamma\boldsymbol{V} + (q/c)\boldsymbol{A}$ modifies through the relativistic factors γ and f. The latter represents the increase of the effective mass by large random motions; we may write $f = f(T) = K_3(mc^2/T)/K_2(mc^2/T)$ for a Maxwellian fluid, where K_j is the modified Bessel function. The factor f is related to molar enthalpy through $h = mcf$ (volume enthalpy is $n_R h$, where n_R is the fluid density in the rest frame); see D. I. Dzhavakhrishvili and N. L. Tsintsadze, Sov. Phys. JETP **37**, 666 (1973); V. I. Berezhiani and S. M. Mahajan, Phys. Rev. E **52**, 1968 (1995).
[23] The curl of $\wp$ is the relativistic generalized vorticity. Normalizing differently, we may define the relativistic generalized magnetic field $\hat{\boldsymbol{B}} = \boldsymbol{B} + (mc/q)\nabla \times (f\gamma\boldsymbol{V})$. The corresponding (Faraday-law-consistent) relativistic generalized electric field is $\hat{\boldsymbol{E}} = \boldsymbol{E} - (mc/q)[\partial_t(f\gamma\boldsymbol{V}) + c\nabla(f\gamma)]$. In terms of these electromagnetic fields, the equation of motion (the vector part of the unified equation of motion) reads as the balance between the generalized Lorentz force (including the inertial force) and the relativistically modified thermodynamic force; see (5).