Stability of High Energy Particle Plasmas to MHD-like Modes

H.L. Berk and H.V. Wong

Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

Abstract

The stability of symmetric mirror systems, such as tandem mirrors and multiple mirrors (or equivalently bumpy tori of large aspect ratio) is investigated when stabilization is attempted with high energy particles. The analysis is derived from a zero Larmor radius variational form, and the stability criteria for eikonal and long wavelength layer modes are obtained. For eikonal modes it is shown that line-bending can stabilize the low ℓ-number modes and together with finite Larmor radius effects discussed elsewhere, complete stabilization is possible. For disc-shaped plasma pressure profiles it is shown that currents induced by conducting walls can stabilize the $\ell=1$ layer mode, while the higher-ℓ layer modes require finite Larmor radius effects for stabilization. For thin ring-like pressure profiles, wall stabilization of the $\ell=1$ mode cannot be achieved, although the line bending term reduces the core beta limit and the growth rate of low ℓ-number layer modes. The coupling of the precessional mode of a plasma ring to the surface Alfvén wave in a multiple mirror plasma is also discussed.

I. INTRODUCTION

In several magnetic configurations such as a symmetric tandem mirror[1] or EBT,[2] one attempts to confine plasma stably in an otherwise unstable MHD trap, by forming a diamagnetic well with a hot component plasma. It is hoped that the hot component can anchor the magnetic field lines in much the same way as the quadruple fields of a minimum-B configuration, but with the advantage of axisymmetry in a tandem mirror and relative simplicity of the magnetic coil geometry in EBT. The conditions which have been satisfied for this to occur are: 1) the dynamical response of the hot component decouples from the background plasma - this requires the "precessional" drift frequency of the hot component be greater than twice the typical MHD growth rate; 2) the diamagnetic well is deep enough to cause magnetic particle drifts in the opposite direction to the "unstable" curvature drifts so that the background plasma is stabilized by the minimum-B principle; 3) the background plasma pressure is below a critical threshold.

These stability criteria were first obtained using simple gravity models[3,4,5,6] where the field lines were straight and curvature was modeled by the inclusion of a fictitious gravity. Later, a z-pinch model[7] was analyzed in which field line curvature appeared naturally in the equilibrium. More recently, the gyrokinetic equations[8,9,10] have been used to consider realistic equilibrium geometries, but are applicable only to perturbations with short perpendicular wavelengths (eikonal approximation).

In this paper, we discuss systematically the low frequency stability of axisymmetric mirror cell equilibria and bumpy cylinder models for a bumpy torus configuration containing a hot plasma component, and we extend previous investigations to include perturbation with small azimuthal and axial mode numbers and the effect of nearby walls. Calculations based on the z-pinch model[7] and a non-systematic extension[11] of the Antonsen-Lee equations[12] for a bumpy cylinder model have indicated that the longest wavelength modes set the most stringent conditions for the decoupling of the hot component. Some of the important results of this paper have already been reported.[13] However, the derivation of the basic equations is different and somewhat more general than Ref. 13. In addition, this paper contains the rigorous and nontrivial justification of the boundary conditions at the ends of a tandem mirror central cell, used in Ref. 14 to describe the coupling of the surface Alfvén wave excited in a central cell to the hot particle precessional mode excited in the plugs. New calculations will be presented for layer modes of plasma rings in bumpy cylinder configurations.

In Sec. II we formulate the low frequency eigenmode equation which describes the coupled compressional and interchange mode. In Sec. III we use the eikonal approximation to determine the dispersion relation, local on a flux surface, for the compressional mode and the interchange mode, with inclusion of hitherto neglected line bending terms arising from the axial variation of the equilibrium. This effect of line bending gives rise to a stabilizing term that becomes more effective the lower the azimuthal mode number of the perturbations. In Sec. IV we derive the dispersion relation for the "layer" mode, a global mode in which the wavelengths of the perturbations are larger than the hot pressure scale lengths of the equilibrium and the eikonal approximation is inapplicable. The dispersion relation is similar to that previously derived except that the growth rates and the stability limit on the background plasma pressure are modified by the effect of line bending for low ℓ-numbers. Especially for $\ell=1$, a significant alteration arises.

For $\ell=1$, with a disc-shaped radial pressure profile, the instability above is proportional to the external vacuum curvature rather than the self-consistent local curvature (additional curvature arises at finite beta due to self-consistent bowing out of field lines from the equilibrium plasma currents). With walls one can completely stabilize the $\ell=1$ mode of a disc-shaped plasma. At higher ℓ-numbers the new term is less important. However, finite Larmor radius terms, if large enough, can stabilize the high ℓ-numbers. The theory developed here uses a small beta expansion. In the MHD regime, the $\ell=1$ stability criterion has recently been obtained at arbitrary beta.[15]

For a ring-shaped plasma we show that line bending does not prevent instability although the critical core beta and growth rates are significantly altered from theories that neglect the line bending terms. We do find that, for sufficiently low ℓ, line bending prevents a new instability that was recently proposed.[16] This instability would arise from the coupling of precessional modes associated with the inner and outer edges of the pressure profile, if the line bending terms are ignored. However, when interaction with shear Alfvén are considered, a new three wave interaction at low core beta arises that produces instability. At moderate core beta the destabilizing interaction is between the Alfvén wave and the precessional mode associated with the outer part of the pressure profile.

In Sec. V we present a self-contained summary of the most important stability criteria obtained in this paper.

II. EIGENMODE EQUATION

We introduce as independent spatial variables the curvilinear field line coordinates (α,θ,s) of the equilibrium magnetic field $\underline{B} = \nabla\alpha \times \nabla\theta$. α is the flux variable, θ the angular variable is the azimuthal angle, and s is the distance along the field line $\left(\frac{\underline{B}_0}{B_0} \cdot \nabla s \equiv \underline{b}\cdot\nabla s = 1\right)$. The mirror cell equilibrium is cylindrically symmetric and hence independent of θ. The magnetic field line structure is such that $\nabla\alpha\cdot\nabla\theta = 0$, $\nabla\theta\cdot\nabla s = 0$. The parallel component of the equilibrium current is zero $\underline{J}_0\cdot\underline{b} = 0$. The metric tensors of the magnetic field geometry are

$$g_{\alpha\alpha} \equiv \frac{\partial \underline{R}}{\partial\alpha}\cdot\frac{\partial \underline{R}}{\partial\alpha} = \frac{1}{r^2B_0^2} + g_{s\alpha}^2$$

$$g_{\theta\theta} \equiv \frac{\partial \underline{R}}{\partial\theta}\cdot\frac{\partial \underline{R}}{\partial\theta} = r^2$$

$$g_{ss} \equiv \frac{\partial \underline{R}}{\partial s}\cdot\frac{\partial \underline{R}}{\partial s} = \underline{b}\cdot\underline{b} = 1$$

$$g_{\alpha\theta} = 0 \qquad g_{s\theta} = 0 \ ,$$

where $\underline{R}(\alpha,\theta,s)$ is the position vector of the field line, and (r,θ,z) are the usual circular cylindrical coordinates. The only non-zero off-diagonal component is $g_{s\alpha}$, which is related to the magnetic field line curvature by

$$\underline{\kappa} \equiv \underline{b}\cdot\nabla\underline{b} = \nabla\alpha \frac{\partial}{\partial s} g_{s\alpha} \;.$$

It may be noted that

$$\nabla\alpha\cdot\nabla\alpha = r^2 B_0^2 \;, \quad \nabla\theta\cdot\nabla\theta = \frac{1}{r^2} \;,$$

$$\nabla s\cdot\nabla s = 1 + g_{s\alpha}^2 r^2 B_0^2 \;,$$

$$\nabla\alpha\cdot\nabla s = -g_{s\alpha} r^2 B_0^2 \;.$$

The equilibrium distribution function $F_0(E,\mu,\alpha)$ is a function of the particle energy E and magnetic moment μ, and the flux variable α. The perpendicular pressure balance equation for the equilibrium is

$$\nabla_\perp P_\perp + \nabla_\perp \frac{B_0^2}{8\pi} = \frac{B_0^2\sigma}{4\pi} \underline{b}\cdot\nabla b \;, \tag{1}$$

where $\nabla_\perp = \nabla - \underline{b}\,\underline{b}\cdot\nabla$, $\sigma = 1 + \frac{4\pi(P_\perp - P_\parallel)}{B_0^2}$, and the perpendicular $P_\perp$ and parallel pressure $P_\parallel$ are defined by

$$P_\perp = \sum \int d^3v \mu B_0 F_0$$

$$= \sum \frac{2\pi B_0}{m^2} \int \frac{d\mu dE}{|v_\parallel|} \mu B_0 F_0$$

$$P_\parallel = \sum \frac{2\pi B_0}{m^2} \int \frac{d\mu dE}{|v_\parallel|} m v_\parallel^2 F_0$$

The summation is over all species as well as positive and negative parallel velocities.

The perturbed field variables are taken to be of the form $\underline{A} = \sum_{\ell} \underline{A}_{\ell}(\alpha,s)\exp(-i\omega t + i\ell\theta)$ where ω is the frequency of the perturbation and ℓ the azimuthal mode number. Hereafter the subscript ℓ on the field variables is suppressed.

In the limit of zero Larmor radius and negligible perturbed $E_{\parallel} = \underline{E}\cdot\underline{b}$ (the parallel component of the electric field), the eigenmode equations for low frequency perturbations of the plasma equilibrium may be expressed in terms of the perpendicular components of the vector potential $\underline{A} = \underline{A}_{\perp}$ or equivalently the magnetic field line displacement vector $\underline{\xi}$ defined by $\underline{A}_{\perp} = \underline{\xi}\times\underline{B}_0$.

The eigenmode equations are derivable from the following quadratic variational form,[17] with ion inertia included:

$$
\begin{aligned}
W(\underline{\xi}^{+},\underline{\xi}) = \int \frac{d\alpha\, d\theta\, ds}{B_0} \Big[& -N_0 m_i \omega^2 \underline{\xi}^{+}\cdot\underline{\xi} + \frac{\sigma}{4\pi} Q_{\perp}^{+}\cdot\underline{Q}_{\perp} + \frac{\tau}{4\pi} Q_L^{+} Q_L \\
& - \underline{\xi}^{+}\cdot\underline{\kappa}\Big(\frac{\sigma}{\tau}\,\underline{\xi}\cdot\hat{\nabla}P_{\perp} + \underline{\xi}\cdot\hat{\nabla}P_{\parallel}\Big)\Big] \\
-\int \frac{d\alpha\, d\theta\, ds}{B_0} \Big[& \sum \int_{\mathrm{hot}} d^3v\, \frac{\partial F_0}{\partial E} \frac{(\omega-\omega_*)}{(\omega-\bar{\omega}_D)} \overline{K}^{+}\overline{K} \\
& + \sum \int_{\mathrm{warm}} d^3v\, \frac{\partial F_0}{\partial E}\Big(1 - \frac{\omega_*}{\omega}\Big) K^{+}K\Big] = 0 \,,
\end{aligned}
\tag{2}
$$

where

$$\underline{Q} = \nabla\times\underline{A} = \nabla\times(\underline{\xi}\times\underline{B}_0)$$

$$Q_{\parallel} = \underline{Q}\cdot\underline{b}$$

$$\underline{Q}_{\perp} = \underline{Q} - \underline{b}Q_{\parallel}$$

$$Q_L = Q_\parallel + \underline{\xi}\cdot\nabla B_0 - \underline{\xi}\cdot\underline{\kappa}\,\frac{B_0\sigma}{\tau}$$

$$= Q_\parallel - \frac{4\pi}{B_0}\,\underline{\xi}\cdot\nabla P_\perp + \underline{\xi}\cdot\underline{\kappa}\,\frac{4\pi\sigma}{\tau}\,\frac{\partial P_\perp}{\partial B_0}$$

$$K = \mu Q_L + \underline{\xi}\cdot\underline{\kappa}\left(\mu\,\frac{B_0\sigma}{\tau} + mv_\parallel^2\right)$$

$$\overline{K} = \oint \frac{ds}{|v_\parallel|}\,K\Big/\oint \frac{ds}{|v_\parallel|}$$

$$\sigma = 1 + \frac{4\pi(P_\perp - P_\parallel)}{B_0^2}$$

$$\tau = 1 + \frac{4\pi}{B_0}\,\frac{\partial P_\perp}{\partial B_0}$$

$$\underline{\kappa} = \kappa\,\frac{\nabla\alpha}{|\nabla\alpha|}$$

$$\kappa = rB_0\,\frac{\partial g_{s\alpha}}{\partial s}$$

$$\mu = \frac{mv_\perp^2}{2B_0}$$

$$\omega_* = -\frac{\ell c}{q}\,\frac{\dfrac{\partial F_0}{\partial \alpha}}{\dfrac{\partial F_0}{\partial E}}$$

$$\omega_D = \frac{\ell c}{qB_0}\,\{\mu\nabla\theta\cdot\underline{b}\times\nabla B_0 + mv_\parallel^2\nabla\theta\cdot\underline{b}\times\underline{\kappa}\}$$

$$= \frac{\ell c}{q}\,\mu\left(\frac{\partial B_0}{\partial \alpha} - g_{s\alpha}\,\frac{\partial B_0}{\partial s}\right) + \frac{\ell c}{qrB_0}\,mv_\parallel^2\kappa$$

$$P_\perp = P_\perp(\alpha, B_0)$$

$$P_\parallel = P_\parallel(\alpha, B_0)$$

$$\hat{\nabla} P_\perp = \nabla P_\perp |_{B_0} \; , \quad \hat{\nabla} P_\parallel = \nabla P_\parallel |_{B_0} \; .$$

N_0 is the plasma density, m and q the mass and charge of each species. Equilibrium electrostatic potentials (ϕ_0=0) and parallel electric field perturbations ($A_\parallel$=0, ϕ=0) are neglected. Q_L is essentially the Lagrangian magnetic field perturbation. $\overline{K}$ implies averaging of K over the periodic orbit of trapped particles. ω_* is the diamagnetic frequency, and ω_D the particle drift frequency.

The equilibrium consists of two components, a main component of warm plasma and a hot component which may be electrons or ions. The kinetic contributions to the variational form are evaluated in the low bounce frequency (ω_b) limit for the warm plasma component ($\omega > \omega_b$, ω_D) and in the high bounce frequency limit for the hot component ($\omega_b > \omega$, ω_D).

Since $\underline{\kappa}$ is in the same direction as $\nabla\alpha$ and $\underline{\xi}\cdot\underline{b} = 0$,

$$\underline{\xi}^+ \cdot \underline{\kappa}\left(\frac{\sigma}{\tau}\, \underline{\xi}\cdot\hat{\nabla} P_\perp + \underline{\xi}\cdot\hat{\nabla} P_\parallel\right) = \frac{\underline{\xi}^+ \cdot \nabla\alpha \;\; \underline{\xi}\cdot\nabla\alpha}{r^2 B_0^2}\, \underline{\kappa} \cdot (\nabla P_\perp + \nabla P_\parallel)$$

$$- \underline{\xi}^+ \cdot \underline{\kappa} \;\; \underline{\xi}\cdot\underline{\kappa} \;\; B_0 \sigma \left(\frac{\partial P_\parallel}{\partial B_0} + \frac{\sigma}{\tau}\frac{\partial P_\perp}{\partial B_0}\right)$$

and the quadratic form is manifestly symmetric. When the subscript ℓ is made explicit:

$$\underline{\xi}^{+}\cdot\underline{\xi} \rightarrow \underline{\xi}^{+}_{-\ell}\cdot\underline{\xi}_{\ell}\ , \text{ etc.},$$

and the symmetry of the quadratic form leads to the adjoint condition

$$\underline{\xi}^{+}_{-\ell}\cdot = \underline{\xi}_{\ell}\cdot$$

Let the covariant components of the perturbed vector potential $\underline{A}_{\perp}$ be denoted by $\alpha_1(\alpha,s)$ and $\beta_1(\alpha,s)$:

$$\underline{A}_{\perp} = (\alpha_1 \nabla\theta - \beta_1 \nabla\alpha) \ . \tag{3}$$

$\underline{\xi}$, $\underline{Q}_{\perp}$, and Q_L can therefore be written in terms of α_1 and β_1:

$$\underline{\xi} = \frac{1}{B_0}\,(\alpha_1\ \underline{b}\times\nabla\theta - \beta_1\ \underline{b}\times\nabla\alpha) \tag{4}$$

$$Q_{\perp} = -r^2 B_0 \frac{\partial\beta_1}{\partial s}\ \nabla\theta + \frac{\partial\alpha_1}{\partial s}\ \underline{b}\times\nabla\theta \tag{5}$$

$$\frac{Q_L}{B_0} = \hat{\chi}_1\alpha_1 + i\ell\beta_1\ , \tag{6}$$

where

$$\hat{\chi}_1\alpha_1 \equiv \left(\frac{\partial}{\partial\alpha} + \frac{4\pi}{B_0^2\tau}\frac{\hat{\partial} P_{\perp}}{\partial\alpha} - g_{s\alpha}\frac{\partial}{\partial s}\right)\alpha_1$$

$$= B_0\frac{\partial}{\partial\alpha}\frac{\alpha_1}{B_0} - B_0 g_{s\alpha}\frac{\partial}{\partial s}\frac{\alpha_1}{B_0} + \frac{\kappa\sigma}{rB_0\tau}\,\alpha_1 \tag{7}$$

$$\frac{\hat{\partial} P_\perp}{\partial \alpha} = \frac{\partial P_\perp}{\partial \alpha} - \frac{\partial P_\perp}{\partial B_0}\frac{\partial B_0}{\partial \alpha}$$

It is more convenient to take α_1 and Q_L to be the field variables rather than α_1 and β_1, where β_1 is related to α_1 and Q_L by Eq. (6). Thus,

$$W(\underline{\xi}^+, \underline{\xi}) = W(\alpha_1^+, \alpha_1, Q_L^+, Q_L) = 0$$

and the eigenmode equations obtained from stationary variation with respect to Q_L^+ and α_1^+ are:

$$\frac{\tau}{4\pi} Q_L - \sum \int_{hot} d^3v \frac{\partial F_0}{\partial E} \frac{(\omega - \omega_*)}{(\omega - \overline{\omega}_D)} \mu \overline{K} - \sum \int_{warm} d^3v \frac{\partial F_0}{\partial E} (1 - \frac{\omega_*}{\omega}) \mu K$$

$$- \frac{1}{\ell^2} \{\frac{\omega^2 r^2 B_0}{4\pi v_A^2} + \frac{\partial}{\partial s} \frac{\sigma r^2 B_0}{4\pi} \frac{\partial}{\partial s}\} (\frac{Q_L}{B_0} - \hat{\chi}_1 \alpha_1) = 0 \qquad (8)$$

$$\hat{\chi}_2 \{\frac{\omega^2 r^2 B_0}{4\pi v_A^2} + \frac{\partial}{\partial s} \frac{\sigma r^2 B_0}{4\pi} \frac{\partial}{\partial s}\} (\hat{\chi}_1 \alpha_1 - \frac{Q_L}{B_0})$$

$$- \ell^2 \{\frac{\omega^2}{4\pi v_A^2 r^2 B_0} + \frac{\partial}{\partial s} \frac{\sigma}{4\pi r^2 B_0} \frac{\partial}{\partial s}\} \alpha_1$$

$$- \frac{\ell^2 \kappa}{r B_0^2} (\frac{\sigma}{\tau} \frac{\hat{\partial} P_\perp}{\partial \alpha} + \frac{\hat{\partial} P_\parallel}{\partial \alpha}) \alpha_1$$

$$+ \frac{\ell^2}{B_0} \sum \int_{hot} d^3v \frac{\partial F_0}{\partial E} \frac{(\omega - \omega_*)}{(\omega - \overline{\omega}_D)} \frac{\kappa}{r B_0} (\frac{B_0 \sigma}{\tau} \mu + m v_\parallel^2) \overline{K}$$

$$+ \frac{\ell^2}{B_0} \sum \int_{warm} d^3v \frac{\partial F_0}{\partial E} \left(1- \frac{\omega_*}{\omega}\right) \frac{\kappa}{rB_0} \left(\frac{B_0 \sigma}{\tau}\mu + mv_\parallel^2\right)K = 0 \; . \qquad (9)$$

The operator $\hat{\chi}_2$ is defined by

$$\hat{\chi}_2 \alpha_1 \equiv \left(\frac{\partial}{\partial\alpha} - \frac{4\pi}{B_0^2 \tau} \frac{\hat{\partial} P_\perp}{\partial \alpha} - \frac{\partial}{\partial s} g_{s\alpha}\right)\alpha_1$$

$$= \frac{1}{B_0} \frac{\partial}{\partial\alpha} B_0 \alpha_1 - \frac{1}{B_0} \frac{\partial}{\partial s} g_{s\alpha} B_0 \alpha_1 - \frac{\kappa\sigma}{\tau r B_0} \alpha_1 \; ,$$

and

$$v_A^2 = \frac{B_0^2}{4\pi N_0 m_i} \; .$$

Equations (8) and (9) are two coupled integro-differential equations in $Q_L(\alpha,s)$ and $\alpha_1(\alpha,s)$.

In conventional MHD analyses of the interchange mode, compressional perturbations involving Q_L is to lowest order decoupled from the perturbation of the variable α_1. However, this is no longer valid in the presence of a hot component with magnetic drift frequencies large compared to frequencies ω of interest. It has previously been shown that there exists a potentially unstable compressional mode associated with the hot component, with growth rates larger than the conventional interchange mode. The interchange mode has a character significantly different from conventional MHD theory.

The eigenmode equation for the coupled compressional and interchange can be derived by first solving Eq. (8) for Q_L in terms of α_1, and then subsituting for Q_L in Eq. (9).

A theory that can be significantly different from MHD theory arises when it is assumed that the hot plasma component has grad-B drift frequencies larger than the characteristic frequency. The latter condition is a fundamental assumption of the forthcoming analysis, that is $\frac{\kappa\Delta}{\beta_H} \sim \varepsilon << 1$, with Δ the radial thickness of the layer where the hot pressure gradient is finite and $\beta_H = \frac{8\pi P_H}{B_0^2}$ the hot component beta. Furthermore, the background plasma beta is also considered to be small $\beta_w = \frac{8\pi P_w}{B_0^2} \sim \varepsilon$. The smallness parameter ε has been introduced as an ordering parameter.

Then writing the particle drift frequency ω_D as:

$$\omega_D = \omega_{DB} + \omega_{D\kappa} ,$$

where

$$\omega_{DB} = - \frac{4\pi\ell c}{\tau q B_0} \mu \frac{\hat{\partial} P_{\perp H}}{\partial\alpha}$$

$$\omega_{D\kappa} = \frac{\ell c}{q} \frac{\kappa}{r B_0}\left(\frac{\sigma}{\tau} \mu B_0 + m v_\parallel^2\right) - \frac{4\pi\ell c}{\tau q B_0} \mu \frac{\hat{\partial} P_{\perp w}}{\partial\alpha}$$

it follows that

$$\frac{\omega_{D\kappa}}{\omega_{DB}} \sim \varepsilon .$$

Thus,

$$\frac{1}{\omega-\bar{\omega}_D} = -\frac{1}{\bar{\omega}_{DB}}\left(1+\frac{\omega-\bar{\omega}_{DK}}{\bar{\omega}_{DB}}+\ldots\right)$$

and the eigenmode equation for Q_L in the hot plasma region may be approximated by

$$\hat{L}^{(0)}Q_L + \hat{L}^{(1)}Q_L + \hat{S}^{(1)}\alpha_1 = 0 \ , \tag{10}$$

where

$$\hat{L}(0)Q_L \equiv \frac{\tau}{4\pi}Q_L - \sum\int_{hot} d^3v\,\frac{\partial F_0}{\partial E}\,\frac{\omega_*}{\bar{\omega}_{DB}}\,\overline{\mu^2 Q_L} \tag{11}$$

$$\hat{L}^{(1)}Q_L \equiv -\sum\int_{hot} d^3v\,\frac{\partial F_0}{\partial E}\,\frac{(\omega\omega_* - \omega\bar{\omega}_{DB} - \omega_*\bar{\omega}_{DK})\,\overline{\mu^2 Q_L}}{\bar{\omega}_{DB}^2}$$

$$-\sum\int_{warm} d^3v\,\frac{\partial F_0}{\partial E}\,\mu^2 Q_L - \frac{1}{\ell^2}\,\hat{D}\,\frac{Q_L}{B_0} \tag{12}$$

$$\hat{S}^{(1)}\alpha_1 = +\sum\int_{hot} d^3v\,\frac{\partial F_0}{\partial E}\,\frac{\omega_*}{\bar{\omega}_{DB}}\,\overline{\mu\,\frac{\kappa}{rB_0}\left(\frac{\sigma}{\tau}B_0\mu + mv_\parallel^2\right)\alpha_1}$$

$$+\frac{1}{\ell^2}\,\hat{D}\,\hat{\chi}_1\alpha_1 \ , \tag{13}$$

where the operator $\hat{D}$ is

$$\hat{D} \equiv \frac{\omega^2 r^2 B_0}{4\pi v_A^2} + \frac{\partial}{\partial s}\frac{\sigma r^2 B_0}{4\pi}\frac{\partial}{\partial s} \,. \tag{14}$$

It should be noted that terms linear in κ in $\hat{L}^{(1)}Q_L$ and $\hat{S}^{(1)}\alpha_1$ are $O(\varepsilon)$, and higher order terms are discarded. Frequency and axial variation terms are ordered to balance the curvature terms.

Let Q_L be expressed as a power series in the smallness parameter ε

$$Q_L = Q_L^{(0)} + Q_L^{(1)} + \, . \, . \, . \, . \, .$$

The lowest order equation is

$$\hat{L}^{(0)}Q_L^{(0)} = 0$$

and the solution is:

$$Q_L^{(0)} = C_0(\alpha)\,\frac{4\pi\ell c}{\tau B_0}\,\frac{\hat{\partial} P_{\perp H}}{\partial\alpha} \,, \tag{15}$$

where $C_0(\alpha)$ is a function of α only.

The first order equation is

$$\hat{L}^{(0)}Q_L^{(1)} + \hat{L}^{(1)}Q_L^{(0)} + \hat{S}^{(1)}\alpha_1 = 0 \,.$$

$C_0(\alpha)$ is determined by multiplying this equation by $(4\pi\ell c\hat{\partial} P_{\perp H}/\partial\alpha)/\tau B_0$ and integrating over a flux tube $\int \frac{ds}{B_0}$. The first term is annihilated and

$$C_0(\alpha) = -\frac{1}{\Gamma}\int \frac{ds}{B_0}\frac{4\pi\ell c}{\tau B_0}\frac{\hat{\partial} P_{\perp H}}{\partial\alpha}\hat{S}^{(1)}\alpha_1$$

$$\approx -\frac{\ell c}{\Gamma}\left[\int \frac{ds}{B_0}\frac{4\pi}{B_0}\frac{\partial P_{\perp H}}{\partial\alpha}\frac{1}{\ell^2}\hat{D}\, B_0\frac{\partial}{\partial\alpha}\frac{\alpha_1}{B_0}\right.$$

$$\left. + \int \frac{ds}{B_0}\frac{\kappa}{rB_0}\frac{\partial P_H}{\partial\alpha}\alpha_1\right] , \qquad (16)$$

where

$$\Gamma = \int \frac{ds}{B_0}\frac{4\pi\ell c}{\tau B_0}\frac{\hat{\partial} P_{\perp H}}{\partial\alpha}\hat{L}^{(1)}\frac{4\pi\ell c}{\tau B_0}\frac{\hat{\partial} P_{\perp H}}{\partial\alpha}$$

$$\approx \int \frac{ds}{B_0}\left[\ell c\omega B_0\frac{\partial}{\partial\alpha}\frac{\rho_H}{B_0} - \ell^2 c^2 B_0\frac{\partial B_0}{\partial\alpha}\frac{\partial}{\partial\alpha}\frac{P_{\perp w}}{B_0^2}\right.$$

$$\left. - \frac{\ell^2 c^2\kappa}{rB_0}\frac{\partial P_{\perp H}}{\partial\alpha} - \frac{4\pi c}{B_0}\frac{\partial P_{\perp H}}{\partial\alpha}\left(\frac{\omega^2 r^2 B_0}{4\pi v_A^2} + \frac{\partial}{\partial s}\frac{\sigma r^2 B_0}{4\pi}\frac{\partial}{\partial s}\right)\frac{4\pi c}{B_0^2}\frac{\partial P_{\perp H}}{\partial\alpha}\right] . \quad (17)$$

ρ_H and P_H are the charge density and $P_H = P_{\perp H} + P_{\parallel H}$ is the total pressure of the hot plasma component. P_w is the pressure of the warm background plasma.

It may be noted that a perturbation $\alpha_1(\alpha,s)$ of the flux surface produces a perturbation Q_L which can be very large if Γ is small.

Equation (15) is applicable in the region of the hot component, bounded by $\alpha^+ > \alpha > \alpha^-$, where the pressure gradient of the hot component is finite. This region is referred to hereafter as the hot component "layer", and its thickness ($\Delta_\alpha = \alpha^+ - \alpha^- << \alpha^-$) is considered to be small compared to the plasma radius.

Inside the "layer", the eigenmode equation for the coupled compressional and interchange mode, obtained by substituting for Q_L in Eq. (9), is:

$$\frac{1}{B_0}\frac{\partial}{\partial\alpha} B_0 \hat{D} \left[B_0 \frac{\partial}{\partial\alpha}\frac{\alpha_1}{B_0} + \frac{4\pi c}{\Gamma B_0^2}\frac{\partial P_\perp}{\partial\alpha}\left(\int \frac{ds}{B_0}\frac{4\pi c}{B_0}\frac{\partial P_{\perp H}}{\partial\alpha}\hat{D} B_0 \frac{\partial}{\partial\alpha}\frac{\alpha_1}{B_0} + \ell^2 \int \frac{ds}{B_0}\frac{c\kappa}{rB_0}\frac{\partial P_H}{\partial\alpha}\alpha_1\right)\right]$$

$$- \frac{\ell^2 c\kappa}{\Gamma r B_0^2}\frac{\partial P_{\perp H}}{\partial\alpha}\left(\int \frac{ds}{B_0}\frac{4\pi c}{B_0}\frac{\partial P_{\perp H}}{\partial\alpha}\hat{D} B_0 \frac{\partial}{\partial\alpha_1}\frac{\alpha_1}{B_0} + \ell^2 \int \frac{ds}{B_0}\frac{c\kappa}{rB_0}\frac{\partial P_H}{\partial\alpha}\alpha_1\right)$$

$$- \ell^2 \left(\frac{\omega^2\alpha_1}{4\pi v_A^2 r^2 B_0} + \frac{\partial}{\partial s}\frac{\sigma}{4\pi r^2 B_0}\frac{\partial}{\partial s}\alpha_1\right) - \frac{\ell^2\kappa}{rB_0^2}\frac{\partial P}{\partial\alpha}\alpha_1 = 0 \; . \qquad (18)$$

A quadratic variational form in $\alpha_1(\alpha,s)$ can be constructed by mulitplying Eq. (18) by α_1 and integrating over the hot component "layer":

$$\overline{W}(\alpha_1,\alpha_1) + \left[\int ds\ \alpha_1 \hat{D}\frac{\partial\alpha_1}{\partial\alpha}\right]_{\alpha^-}^{\alpha^+}$$

$$- \left[\int d\alpha \frac{\partial\alpha_1}{\partial\alpha}\frac{r^2 B_0}{4\pi}\frac{\partial}{\partial s}\frac{\partial\alpha_1}{\partial\alpha}\right]_{s^-}^{s^+}$$

$$- \left[\int d\alpha\ \alpha_1 \frac{\ell^2}{4\pi r^2 B_0}\frac{\partial\alpha_1}{\partial s}\right]_{s^-}^{s^+} = 0 \; , \qquad (19)$$

where

$$\overline{w}(\alpha_1,\alpha_1) = \int \frac{d\alpha ds}{B_0} \left[\frac{\sigma r^2 B_0^2}{4\pi} \left(\frac{\partial}{\partial s} B_0 \frac{\partial}{\partial \alpha} \frac{\alpha_1}{B_0}\right)^2 - \frac{\omega^2 r^2 B_0^2}{4\pi v_A^2} \left(B_0 \frac{\partial}{\partial \alpha} \frac{\alpha_1}{B_0}\right)^2\right]$$

$$- \int \frac{d\alpha}{\Gamma} \left(\int \frac{ds}{B_0} \frac{4\pi c}{B_0} \frac{\partial P_{\perp H}}{\partial \alpha} \hat{D} B_0 \frac{\partial}{\partial \alpha} \frac{\alpha_1}{B_0} + \ell^2 \int \frac{ds}{B_0} \frac{c\kappa}{rB_0} \frac{\partial P_H}{\partial \alpha} \alpha_1\right)^2$$

$$+ \int \frac{d\alpha ds}{B_0} \ell^2 \left\{\frac{\sigma}{4\pi r^2} \left(\frac{\partial \alpha_1}{\partial s}\right)^2 - \frac{\omega^2}{4\pi v_A^2 r^2} \alpha_1^2\right\} - \int \frac{d\alpha ds}{B_0} \frac{\ell^2 \kappa}{rB_0} \frac{\partial P_H}{\partial \alpha} \alpha_1^2 \ . \qquad (20)$$

The surface terms are to be evaluated by matching to the solution outside the "layer" at the boundaries $\alpha=\alpha^+$, $\alpha=\alpha^-$, $s=s^+$, and $s=s^-$. The limits of integration correspond to the boundaries of the "layer" but have been suppressed for convenience.

To obtain the dispersion relation for low frequency perturbations, the eigenmode equation must be solved with the appropriate boundary conditions. In Sec. III, we discuss the dispersion relation, local to a flux surface inside the "layer" for the compressional mode and the interchange mode in the eikonal approximation, while in Sec. IV, we discuss the dispersion relation for the "layer" mode, a global mode extending over the entire equilibrium configuration.

III. DISPERSION RELATION - EIKONAL APPROXIMATION

A. Compressional mode

In the eikonal approximation where the perpendicular wavelength is assumed to be much smaller than the equilibrium scalelengths and parallel wavelengths, the perturbation may be represented by

$$\alpha_1(\alpha,s) = \tilde{\alpha}_1(\alpha,s)e^{i\zeta(\alpha)} \ ,$$

where

$$|\nabla\zeta(\alpha)| \gg |\nabla\tilde{\alpha}_1(\alpha,s)| \ , \ |\nabla B_0| \ .$$

Let

$$\frac{\partial\zeta(\alpha)}{\partial\alpha} = k_\alpha \ ;$$

then

$$\hat{\chi}_1\alpha_1 \approx ik_\alpha\alpha_1$$

and in the limit appropriate for the compressional mode

$$\frac{\omega^2}{v_A^2} \gg \frac{\ell^2}{r^2}\frac{4\pi r\kappa}{B_0}\frac{\partial P_H}{\partial\alpha}\frac{1}{\left(\frac{\ell^2}{r^2}+k_\alpha^2 r^2 B_0^2\right)} \ .$$

Equation (18) may be approximated by

$$\left(k_\alpha^2 + \frac{\ell^2}{r^4 B_0^2}\right)\hat{D}\tilde{\alpha}_1 + k_\alpha^2\,\hat{D}\,\frac{4\pi c}{\Gamma B_0^2}\frac{\partial P_{\perp H}}{\partial\alpha}\int\frac{ds}{B_0}\frac{4\pi c}{B_0}\frac{\partial P_{\perp H}}{\partial\alpha}\hat{D}\tilde{\alpha}_1 = 0 \ , \quad (21)$$

where $\frac{r^2 B_0}{2} \approx \alpha$.

Multiplying Eq. (21) by $\frac{4\pi c}{B_0}\frac{\partial P_{\perp H}}{\partial\alpha}$ and integrating over a flux tube $\int\frac{ds}{B_0}$:

$$[(k_\alpha^2 + \frac{\ell^2}{r^4 B_0^2})\Gamma + k_\alpha^2 \int \frac{ds}{B_0} \frac{4\pi c}{B_0} \frac{\partial P_{\perp H}}{\partial \alpha} \hat{D} \frac{4\pi c}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha}] \int ds \frac{4\pi c}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha} \hat{D}\alpha_1 = 0 \;. \quad (22)$$

Γ is defined by Eq. (17), and it can be expressed as follows:

$$\Gamma = -\ell^2 c^2 A_0 \int \frac{ds}{B_0} \frac{\kappa}{rB_0} \frac{\partial P_H}{\partial \alpha} \quad (23)$$

$$A_0 \equiv 1 - \tilde{\beta}_w - \frac{\omega}{\tilde{\omega}_{\kappa H}} + \frac{1}{2\ell^2} < \frac{\omega^2}{v_A^2} \frac{r^2 \beta_H}{\kappa\Delta} > - \frac{1}{2\ell^2} < \frac{r^2 \beta_H}{L_H^2 \kappa\Delta} > \quad (24)$$

$$\tilde{\beta}_w \equiv + \frac{4\pi \int ds \frac{\partial P_{\perp H}}{B \partial \alpha} \frac{\partial}{\partial \alpha} \frac{P_{\perp w}}{B_0^2}}{\int \frac{ds}{B_0} \frac{\kappa}{rB_0} \frac{\partial P_H}{\partial \alpha}} \quad (25)$$

$$\tilde{\omega}_{\kappa H} \equiv \frac{\ell \int \frac{ds}{B_0} \frac{\kappa}{rB_0} \frac{\partial P_H}{\partial \alpha}}{\int ds \frac{\partial}{\partial \alpha} \frac{\rho_H}{B_0}} \quad (26)$$

$$< \frac{\omega^2}{v_A^2} \frac{r^2 \beta_H}{\kappa\Delta} > \equiv \frac{2 \int \frac{ds}{B_0} \frac{\omega^2 r^2}{B_0^2 v_A^2} (\frac{\partial P_{\perp H}}{\partial \alpha})^2}{\int \frac{ds}{B_0} \frac{\kappa}{rB_0} \frac{\partial P_H}{\partial \alpha}} \quad (27)$$

$$< \frac{r^2 \beta_H}{L_H^2 \kappa\Delta} > \equiv \frac{2 \int \frac{ds}{B_0} 4\pi\sigma r^2 B_0^2 (\frac{\partial}{\partial s} \frac{1}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha})^2}{\int \frac{ds}{B_0} \frac{\kappa}{rB_0} \frac{\partial P_H}{\partial \alpha}} \;. \quad (28)$$

$\tilde{\beta}_w$ depends on the gradient of the warm plasma beta (ratio of the warm plasma pressure to the magnetic pressure). $\tilde{\omega}_{\kappa H}$ is the "precessional" drift frequency of the hot component and is essentially a normalized curvature drift frequency. $\left\langle \frac{r^2\beta_H}{L_H^2\kappa\Delta} \right\rangle$ is associated with field line bending arising from the axial variation of the equilibrium and depends on the hot plasma beta β_H, the field line curvature κ, the square of the ratio of the radius r to the axial extent L_H of the hot plasma component, and the radial thickness Δ of the "layer". If

$$\frac{\beta_H r}{\kappa L_H^2} \sim 1 \ , \qquad \left\langle \frac{r^2\beta_H}{L_H^2\kappa\Delta} \right\rangle \gg 1 \ ,$$

the local dispersion relation for the compressional mode, obtained by equating the sum of terms inside the square bracket of Eq. (22) to zero, is:

$$\frac{\omega^2}{2} \left\langle \frac{r^2\beta_H}{v_A^2\kappa\Delta} \right\rangle - \left\langle \frac{r^2\beta_H}{2L_H^2\kappa\Delta} \right\rangle + (k_\alpha^2 r^4 B_0^2 + \ell^2)\left[1 - \tilde{\beta}_w - \frac{\omega}{\tilde{\omega}_{\kappa H}}\right] = 0 \ . \qquad (29)$$

This dispersion relation for the compressional mode is stable if:

$$1 > \frac{2\tilde{\omega}_{\kappa H}^2}{k_\perp^2 r^2} \left\langle \frac{r^2\beta_H}{v_A^2\kappa\Delta} \right\rangle \left[1 - \tilde{\beta}_w - \frac{\left\langle \frac{r^2\beta_H}{2L_H^2\kappa\Delta} \right\rangle}{k_\perp^2 r^2}\right] ,$$

where

$$k_\perp^2 \equiv k_\alpha^2 r^2 B_0^2 + \frac{\ell^2}{r^2} .$$

Field-line bending is relevant and produces stability at small enough values of $k_\perp^2 r^2$ satisfying

$$k_\perp^2 r^2 < \langle \frac{r^2 \beta_H}{L_H^2 \kappa \Delta} \rangle / (1 - \tilde{\beta}_w) . \tag{30}$$

Short wavelength modes are stabilized by finite Larmor radius effects when[8,9,10]

$$k_\perp^2 r_L^2 > \frac{1}{\tilde{\beta}_H} , \tag{31}$$

where r_L, the Larmor radius of the hot component, and $\tilde{\beta}_H$ are given by

$$r_L^2 = \frac{\int d^3 v v_\perp^4 F_0}{4\omega_c^2 \int d^3 v v_\perp^2 F_0} , \quad \tilde{\beta}_H = \frac{4\pi \int \frac{ds}{B_0} \left(B_0 \frac{\partial P_{\perp H}}{\partial \alpha}\right)^2}{\int \frac{ds}{B_0} \frac{\kappa}{r B_0} \frac{\partial P_H}{\partial \alpha}} , \quad \omega_c = \frac{q_H B}{m_H c} . \tag{32}$$

By combining Eqs. (30) and (31) a window of stability is found and is given by

$$\frac{r_L^2}{2r^2} \tilde{\beta}_H < \langle \frac{\beta_H r^2}{L_H^2 \kappa \Delta} \rangle > 1 - \tilde{\beta}_w . \tag{33}$$

B. Interchange mode

For flute perturbations in which $\tilde{\alpha}_1(\alpha)$ is independent of s, field line bending is minimized, and

$$B_0 \frac{\partial}{\partial \alpha} \frac{\alpha_1}{B_0} \approx ik_\alpha \alpha_1 + \frac{4\pi}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha} \alpha_1$$

$$\frac{\partial}{\partial s} B_0 \frac{\partial}{\partial \alpha} \frac{\alpha_1}{B_0} \approx \alpha_1 \frac{\partial}{\partial s} \left(\frac{4\pi}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha}\right) .$$

In the limit appropriate for the interchange mode

$$\frac{\omega^2}{v_A^2} < \frac{1}{L_H^2}$$

and the quadratic variational form [Eq. (19)] may be approximated by

$$\int d\alpha |\tilde{\alpha}_1|^2 [\omega^2 \int \frac{ds}{B_0} \frac{1}{4\pi v_A^2} \left(\frac{\ell^2}{r^2} + k_\alpha^2 r^2 B_0^2\right)$$

$$- \int \frac{ds}{B_0} 4\pi\sigma r^2 B_0^2 \left(\frac{\partial}{\partial s} \frac{1}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha}\right)^2 + \int \frac{ds}{B_0} \frac{\ell^2 \kappa}{r B_0} \frac{\partial P_H}{\partial \alpha}$$

$$- \frac{c^2}{\Gamma} \left\{\ell^2 \int \frac{ds}{B_0} \frac{\kappa}{r B_0} \frac{\partial P_H}{\partial \alpha} - \int \frac{ds}{B_0} 4\pi\sigma r^2 B_0^2 \left(\frac{\partial}{\partial s} \frac{1}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha}\right)^2\right\}^2]$$

$$= \int d\alpha |\tilde{\alpha}_1|^2 [\omega^2 \int \frac{ds}{B_0} \frac{k_\perp^2}{4\pi v_A^2} + \int \frac{ds}{B_0} \frac{\ell^2 \kappa}{r B_0} \frac{\partial P_W}{\partial \alpha}$$

$$- \int ds \frac{\ell^2 \kappa}{rB_0} \frac{\partial P_H}{\partial \alpha} \frac{\tilde{\beta}_w \left(1 - \frac{1}{2\ell^2} < \frac{r^2 \beta_H}{L_H^2 \kappa \Delta} >\right)}{\left\{1 - \tilde{\beta}_w - \frac{1}{2\ell^2} < \frac{r^2 \beta_H}{L_H^2 \kappa \Delta} >\right\}}] = 0 \; . \tag{34}$$

The perturbation is assumed to be localized on a flux surface inside the layer, and hence the contributions to the surface terms are negligible.

Thus, the local dispersion relation for the interchange mode on a flux surface is obtained by equating the sum of terms inside the square bracket of Eq. (34) to zero.

$$\text{If } |\tilde{\beta}_w| < \left|1 - \frac{1}{2\ell^2} < \frac{r^2 \beta_H}{L_H^2 \kappa \Delta} >\right| \; ,$$

the frequency of the interchange mode is:

$$\omega^2 \int \frac{ds}{B_0} \frac{k_\perp^2}{4\pi v_A^2} + \int \frac{ds}{B_0} \frac{\ell^2 \kappa}{rB_0} \frac{\partial P_w}{\partial \alpha}$$

$$- \int ds \; \ell^2 \frac{\partial P_{\perp H}}{B^2 \partial \alpha} \frac{\partial}{\partial \alpha} \frac{P_{\perp w}}{B_0^2} = 0 \; , \tag{35}$$

and the mode is stable when the magnetic well is deep enough to cause magnetic particle drifts in the opposite direction to the unstable curvature drifts:

$$\frac{\partial B_0}{\partial \alpha} > 0$$

$$\left|\frac{\partial B_0}{\partial \alpha}\right| > \frac{\kappa}{r} \tag{36}$$

$$\text{If } |\tilde{\beta}_w| > \left|1 - \frac{1}{2\ell^2} < \frac{r^2 \beta_H}{L_H^2 \kappa \Delta} >\right|$$

$$\omega^2 \int \frac{ds}{B_0} \frac{k_\perp^2}{4\pi v_A^2} + | \frac{ds}{B_0} \frac{\ell^2 \kappa}{r B_0} \frac{\partial P_w}{\partial \alpha}$$

$$+ \int \frac{ds}{B_0} \frac{\ell^2 \kappa}{r B_0} \frac{\partial P_H}{\partial \alpha} \left(1 - \frac{1}{2\ell^2} < \frac{r^2 \beta_H}{L_H^2 \kappa \Delta} >\right) = 0 \ , \tag{37}$$

and the interchange mode with azimuthal mode numbers satisfying the inequality

$$\ell^2 < < \frac{r^2 \beta_H}{L_H^2 \kappa \Delta} > \tag{38}$$

can be stable.

Since short wavelength interchange modes are also stabilized by finite Larmor radius effects when[8]

$$k_\perp^2 r_L^2 > \frac{1}{\tilde{\beta}_H} \ ,$$

the window of stability in parameter space is determined by

$$\tilde{\beta}_H \frac{r_L^2}{r^2} < \frac{\beta_H r^2}{L_H^2 \kappa \Delta} > > \frac{2\ell^2}{k_\perp^2 r^2} \; . \tag{39}$$

IV. DISPERSION RELATION - "LAYER" MODE

"Layer" modes are characterized by perturbations of the flux surface which are to lowest order constant across the hot component layer. The dispersion relation for these modes is derived by solving the eigenmode equation inside and outside the "layer", and the solutions matched at the boundaries of the "layer," $\alpha=\alpha^+$, $\alpha=\alpha^-$, $s=s^-$, and $s=s^+$. The mirror cell equilibrium is assumed to be bounded radially by conducting walls at $\alpha = \alpha_w(s)$.

Two cases will be discussed: (a) disc-like pressure profile applicable to the end-cell of a tandem mirror where the hot pressure decreases to zero through the layer [see Fig. (1)]; (b) ring-like pressure profile where the hot pressure increases from zero to a maximum and then decreases to zero through the layer [see Fig. (2)]. Such pressure profiles have been produced in the Elmo Bumpy Torus configurations. In this work the bumpy torus is taken in the limit of infinite aspect ratio (bumpy cylinder model).

A. Disc-like Pressure Profile

A solution of Eq. (18) can be obtained for low frequency modes with $\frac{\omega^2 L_H^2}{v_A^2} \sim \varepsilon_1 < 1$ when the layer is thin $\frac{\Delta_\alpha}{\alpha^-} \sim \varepsilon_1$ with ε_1 being introduced as an additional smallness parameter, $\varepsilon_1 >> \kappa r \sim \varepsilon$.

The equation for α_1 up to first order in ε_1 is

$$\frac{1}{B_0}\frac{\partial}{\partial\alpha} B_0 \hat{D}[B_0 \frac{\partial}{\partial\alpha}\frac{\alpha_1}{B_0}$$

$$+ \frac{4\pi c}{\Gamma B_0^2}\frac{\partial P_{\perp H}}{\partial\alpha}(\int \frac{ds}{B_0}\frac{4\pi c}{B_0}\frac{\partial P_{\perp H}}{\partial\alpha}\hat{D}\, B_0 \frac{\partial}{\partial\alpha}\frac{\alpha_1}{B_0}$$

$$+ \ell^2 \int \frac{ds}{B_0}\frac{c\kappa}{rB_0}\frac{\partial P_H}{\partial\alpha}\alpha_1)] = 0 \;. \tag{40}$$

Now integrating from α^+ to α yields

$$\hat{D}\;[B_0 \frac{\partial}{\partial\alpha}\frac{\alpha_1}{B_0} + \frac{4\pi c}{\Gamma B_0^2}\frac{\partial P_{\perp H}}{\partial\alpha}(\int \frac{ds}{B_0}\frac{4\pi c}{B_0}\frac{\partial P_{\perp H}}{\partial\alpha}\hat{D}\, B_0 \frac{\partial}{\partial\alpha}\frac{\alpha_1}{B_0}$$

$$+ \ell^2 \int \frac{ds}{B_0}\frac{c\kappa}{rB_0}\frac{\partial P_H}{\partial\alpha}\alpha_1(0))] = \left(\hat{D}\, B_0 \frac{\partial}{\partial\alpha}\frac{\alpha_1}{B_0}\right)\Big|_{\alpha^+} , \tag{41}$$

where the term to be evaluated on the outer surface $\alpha=\alpha^+$ of the layer is determined by matching to the external solutions.

Multiplying by $\frac{4\pi c}{B_0^2}\frac{\partial P_{\perp H}}{\partial\alpha}$ and integrating in s over the layer $(s^+ > s > s^-)$:

$$\int ds\, \frac{4\pi c}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha} \hat{D}\, B_0 \frac{\partial}{\partial \alpha} \frac{\alpha_1}{B_0} + \frac{(\Lambda_0 - \Gamma)}{\Lambda_0}\, \ell^2 \int \frac{ds}{B_0} \frac{c\kappa}{rB_0} \frac{\partial P_H}{\partial \alpha}\, \alpha_1$$

$$= \frac{\Gamma}{\Lambda_0} \int ds\, \frac{4\pi c}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha} \left(\hat{D}\, B_0 \frac{\partial}{\partial \alpha} \frac{\alpha_1}{B_0}\right)\Big|_{\alpha^+} , \tag{42}$$

where

$$\Lambda_0 \equiv \Gamma + \int ds \frac{4\pi c}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha} \hat{D} \frac{4\pi c}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha}$$

$$= -\ell^2 c^2 \int ds\, \frac{\kappa}{rB_0^2} \frac{\partial P_H}{\partial \alpha}\left(1 - \tilde{\beta}_w - \frac{\omega}{\tilde{\omega}_{\kappa H}}\right) . \tag{43}$$

Thus, Eq. (41) simplifies to:

$$\frac{\partial}{\partial s}\, \sigma r^2 B_0 \frac{\partial}{\partial s}\left[B_0 \frac{\partial}{\partial \alpha} \frac{\alpha_1}{B_0} - \frac{4\pi}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha}\, G\right] = \left(\frac{\partial}{\partial s}\, \sigma r^2 B_0 \frac{\partial}{\partial s}\, B_0 \frac{\partial}{\partial \alpha} \frac{\alpha_1}{B_0}\right)\Big|_{\alpha^+} , \tag{44}$$

where

$$G = - \frac{c^2 \ell^2}{\Lambda_0} \int ds\, \frac{\kappa}{rB_0^2} \frac{\partial P_H}{\partial \alpha}\, \alpha_1$$

$$- \frac{c^2}{\Lambda_0} \int ds\, \frac{1}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha} \left(\frac{\partial}{\partial s}\, \sigma r^2 B_0 \frac{\partial}{\partial s}\, B_0 \frac{\partial}{\partial \alpha} \frac{\alpha_1}{B_0}\right)\Big|_{\alpha^+} . \tag{45}$$

Integrating in s from $s=s^-$ to s:

$$\frac{\partial}{\partial s}\left[B_0 \frac{\partial}{\partial \alpha} \frac{\alpha_1}{B_0} - \frac{4\pi}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha} G\right] = \frac{1}{\sigma r^2 B_0} \left(\sigma r^2 B_0 \frac{\partial}{\partial s} B_0 \frac{\partial}{\partial \alpha} \frac{\alpha_1}{B_0}\right)\Big|_{\alpha^+} . \tag{46}$$

Before proceeding with the solution of Eq. (46), it is convenient at this stage to discuss the external solutions outside the layer.

Outside the layer where the pressure gradient is assumed to be negligible, the equation for α_1 may be approximated by:

$$\frac{\partial}{\partial \alpha} \frac{\partial}{\partial s} r^2 B_0 \frac{\partial}{\partial s} \frac{\partial \alpha_1}{\partial \alpha} - \frac{\partial}{\partial s} \frac{\ell^2}{r^2 B_0} \frac{\partial \alpha_1}{\partial s} = 0 . \tag{47}$$

In the long thin approximate (assumed to be valid) and for small plasma beta, $\alpha \approx \frac{r^2 B_0}{2}$, and the solution for α_1 with $\alpha_1 = 0$ at $\alpha = 0$ is

$$\alpha_1 = f^-(s) \left(\frac{\alpha}{\alpha^-}\right)^{|\ell|/2} , \qquad \alpha^- > \alpha . \tag{48}$$

In the vacuum region $\alpha_w > \alpha > \alpha^+$, and with the boundary condition $Q_{\perp\alpha} = \frac{\nabla\alpha}{|\nabla\alpha|} \cdot \underline{Q}_\perp = 0$ at the surface of the conducting wall located at $\alpha = \alpha_w$, the solution for $Q_{\perp\alpha}$ is

$$rQ_{\perp\alpha} = - \frac{\partial f^+}{\partial s} g(\alpha,s) , \tag{49}$$

where

$$g(\alpha,s) = \frac{\left(\frac{\alpha}{\alpha_w}\right)^{\ell/2} - \left(\frac{\alpha_w}{\alpha}\right)^{\ell/2}}{\left(\frac{\alpha^+}{\alpha_w}\right)^{\ell/2} - \left(\frac{\alpha_w}{\alpha^+}\right)^{\ell/2}} , \tag{50}$$

and for $Q_{\perp\theta} = \frac{\nabla\theta}{|\nabla\theta|} \cdot \underline{Q}_\perp$:

$$Q_{\perp\theta} = - \frac{rB}{i\ell} \frac{\partial}{\partial\alpha} rQ_{\perp\alpha} .$$

From the continuity of $\underline{Q}_\perp$ at $\alpha = \alpha^+$:

$$\left.\frac{\partial\alpha_1}{\partial s}\right|_{\alpha=\alpha^+} = -rQ_{\perp\alpha}\Big|_{\alpha=\alpha^+} = \frac{\partial f^+}{\partial s} \tag{51a}$$

$$\frac{\partial}{\partial s} \left.\frac{\partial\alpha_1}{\partial\alpha}\right|_{\alpha=\alpha^+} = \frac{i\ell}{B} \nabla\theta \cdot \underline{Q}_\perp\Big|_{\alpha=\alpha^+} = - \frac{\ell}{2\alpha^+} \frac{\partial f^+}{\partial s} Z_\ell , \tag{51b}$$

where

$$Z_\ell = \frac{1 + \left(\frac{\alpha^+}{\alpha_w}\right)^\ell}{1 - \left(\frac{\alpha^+}{\alpha_w}\right)^\ell} .$$

If the external solutions are substituted in Eq. (47):

$$\frac{\partial}{\partial s} \left[B_0 \frac{\partial}{\partial\alpha} \frac{\alpha_1}{B_0} - \frac{4\pi}{B^2} \frac{\partial P_{\perp H}}{\partial\alpha} G\right] = - \frac{\ell}{2\sigma\alpha} Z_\ell \frac{\partial f^+}{\partial s} , \tag{52}$$

and from Eq. (46):

$$G = -\frac{c^2}{\Lambda_0}\{\ell^2 \int ds \frac{\kappa}{rB_0^2}\frac{\partial P_H}{\partial\alpha}\alpha_1 - \ell\int ds \frac{1}{B_0^2}\frac{\partial P_{\perp H}}{\partial\alpha}\frac{\partial}{\partial s} Z_\ell \frac{\partial f^+}{\partial s}\} \qquad (53)$$

Now, $\frac{\partial}{\partial s}\frac{\partial\alpha_1}{\partial\alpha}$ is continuous at the boundaries. To satisfy continuity of $\frac{\partial}{\partial s}\frac{\partial\alpha_1}{\partial\alpha}$ at $\alpha=\alpha^-$, it is required that

$$Z_\ell \frac{\partial f^+}{\partial s} + \frac{\partial f^-}{\partial s} = 0 \ . \qquad (55)$$

Integrating Eq. (52) from $\alpha=\alpha^-$ to $\alpha=\alpha^+$:

$$\left[\frac{\partial\alpha_1}{\partial s}\right]_{\alpha^-}^{\alpha^+} = \frac{\partial f^+}{\partial s} - \frac{\partial f^-}{\partial s}$$

$$= \frac{\partial}{\partial s}\int_{\alpha^-}^{\alpha^+} d\alpha \frac{4\pi}{B_0^2}\frac{\partial P_{\perp H}}{\partial\alpha}(G-\alpha_1) - \frac{\ell}{2} Z_\ell \frac{\partial f^+}{\partial s} \ln\frac{\alpha^+}{\alpha^-} \ . \qquad (56)$$

Now using that $\frac{\partial\alpha_1}{\partial s}$ is continuous at the boundaries, we have from Eqs. (55) and (56):

$$\frac{\partial f^+}{\partial s} \approx \frac{1}{(1+Z_\ell)}\frac{\partial}{\partial s}\int_{\alpha^-}^{\alpha^+} d\alpha \frac{4\pi}{B_0^2}\frac{\partial P_{\perp H}}{\partial\alpha}(G-\alpha_1) \ . \qquad (57)$$

Furthermore, from Eq. (53):

$$G = -\frac{c^2\ell^2}{\Lambda_0}\int ds\,\frac{\kappa}{rB_0^2}\frac{\partial P_H}{\partial\alpha}\,\alpha_1$$

$$+\frac{c^2|\ell|}{\Lambda_0}\int ds\,\frac{1}{B_0^2}\frac{\partial P_{\perp H}}{\partial\alpha}\frac{\partial}{\partial s}\left[1+\left(\frac{\alpha^+}{\alpha_w}\right)^{|\ell|}\right]\frac{\partial}{\partial s}\int_{\alpha^-}^{\alpha^+}d\alpha\,\frac{2\pi}{B_0^2}\frac{\partial P_{\perp H}}{\partial\alpha}(G-\alpha_1)\ . \quad (58)$$

In order to proceed further with the analysis, the hot plasma beta is considered to be small, $\beta_h \sim \varepsilon_1$. In this case, Eq. (52) implies that

$$\alpha_1 = \alpha_1^{(0)} + 0(\varepsilon_1)\ ,$$

where $\alpha_1^{(0)}$ is a constant which is taken to be unity. Continuity of α_1 at $\alpha=\alpha^-$ and $\alpha=\alpha^+$ then requires that

$$f^{(0)\pm} = \alpha_1^{(0)} = 1\ .$$

If $\frac{\partial P_\perp}{\partial\alpha}$ and Λ_0 is approximately constant inside the layer, so also is G, and G may then be approximated by:

$$G = \frac{1 - \left\langle \frac{\kappa_s}{|\ell|\kappa}\left(1+\left(\frac{\alpha^+}{\alpha_w}\right)^{|\ell|}\right)\right\rangle}{A_1}\ , \quad (59)$$

where

$$A_1 = 1 - \tilde{\beta}_w - \frac{\omega}{\tilde{\omega}_{\kappa H}} - \left\langle \frac{\kappa_s}{|\ell|\kappa}\left(1+\left(\frac{\alpha^+}{\alpha_w}\right)^{|\ell|}\right)\right\rangle \quad (60)$$

$$\left\langle \frac{\kappa_s}{|\ell|\kappa}\left(1+\left(\frac{\alpha^+}{\alpha_w}\right)^{|\ell|}\right)\right\rangle = \frac{\int ds\, \frac{2\pi}{B_0^2}\frac{\partial P_{\perp H}}{\partial \alpha}\frac{\partial}{\partial s}\left[1+\left(\frac{\alpha^+}{\alpha_w}\right)^{|\ell|}\right]\frac{\partial}{\partial s}\left(\frac{P_{\perp H}(\alpha^-)}{B_0^2}\right)}{|\ell| \int ds\, \frac{\kappa}{rB_0^2}\frac{\partial P_H}{\partial \alpha}} \;. \quad (61)$$

The symbol κ_s is introduced to indicate dependence on the self-curvature, that is, the curvature introduced by the finite beta of the plasma

$$\kappa_s = \frac{2\pi r_0}{B_0^2}\frac{\partial^2 P_\perp}{\partial s^2}\,, \qquad r_0 = (2\alpha^+/B_0)^{1/2} \;. \qquad (62)$$

This is the self-consistent expression for the self-curvature in the limit of $\beta_H < 1$ and small mirror ratio.

These solutions can now be used to evaluate the quadratic variational form [Eq. (19)] correct to second order in ε_1 and thereby obtain the dispersion relation. The substitution of Eq. (52) for $\frac{\partial}{\partial s} B_0 \frac{\partial}{\partial \alpha}\frac{\alpha_1}{B_0}$ in Eq. (20) yields for the volume integral $\overline{w}(\alpha_1,\alpha_1)$:

$$\overline{w}(\alpha_1,\alpha_1) = \int d\alpha ds\, \frac{\ell^2\kappa}{rB_0^2}\frac{\partial P_H}{\partial \alpha}\left[G^2\left(\tilde{\beta}_w + \frac{\omega}{\tilde{\omega}_{\kappa H}}\right) - (G-1)^2\right] \;.$$

The surface terms are determined by matching to the external solutions. The boundary terms at $\alpha=\alpha^+$ and $\alpha=\alpha^-$ are:

$$\left[\int_{s^-}^{s^+} ds \ \alpha_1 \ \hat{D} \frac{\partial \alpha_1}{\partial \alpha}\right]_{\alpha^-}^{\alpha^+} = - \int ds \frac{\omega^2}{4\pi v_A^2} \left\{|\ell| + \ell Z_\ell \frac{N_0(\alpha^+,s)}{N_0(\alpha^-,s)}\right\}$$

$$+ \int ds \frac{|\ell|}{4\pi} \left\{Z_\ell \left(\frac{\partial f^+}{\partial s}\right)^2 + \left(\frac{\partial f^-}{\partial s}\right)^2\right\} , \qquad (63)$$

where

$$v_A^2 = \frac{B_0^2}{4\pi N_0(\alpha^-,s) m_i} .$$

The second integral in s may be approximated by

$$\int ds \frac{|\ell|}{4\pi} \left\{Z_\ell \left(\frac{\partial f^+}{\partial s}\right)^2 + \left(\frac{\partial f^-}{\partial s}\right)^2\right\}$$

$$= \int d\alpha \int ds \ |\ell| \frac{2\pi}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha} (G-1)^2 \frac{\partial}{\partial s} \nu_\ell \frac{\partial}{\partial s} \frac{P_{\perp H}}{B^2}$$

$$= \int d\alpha \int ds \ \ell^2 \frac{\kappa}{r B_0^2} \frac{\partial P_H}{\partial \alpha} (G-1)^2 < |\ell| \frac{\kappa_s}{\kappa} \nu_\ell > , \qquad (64)$$

with $\nu_\ell = 1 + (\alpha^+/\alpha_w)^{|\ell|}$.

The quadratic variational form is then given by:

$$\int d\alpha \int ds \frac{\ell^2}{r B_0^2} \frac{\partial P_H}{\partial \alpha} \frac{\left(\tilde{\beta}_w + \frac{\omega}{\tilde{\omega}_{\kappa H}}\right)\left(1 - < \frac{\kappa_s \nu_\ell}{|\ell| \kappa} >\right)}{\left(1 - \tilde{\beta}_w - \frac{\omega}{\tilde{\omega}_{\kappa H}} - < \frac{\kappa_s \nu_\ell}{|\ell| \kappa} >\right)}$$

$$- \int_{s^-}^{s^+} ds \frac{\omega^2}{4\pi v_A^2} |\ell| \left(1 + z_{|\ell|} \frac{N_0^+}{N_0^-}\right)$$

$$= \int_{\alpha^-}^{\alpha^+} d\alpha \left[\frac{\partial \alpha_1}{\partial \alpha} \frac{r^2 B_0}{4\pi} \frac{\partial}{\partial s} \frac{\partial \alpha_1}{\partial \alpha} + \alpha_1 \frac{\ell^2}{4\pi r^2 B_0} \frac{\partial \alpha_1}{\partial s}\right]_{s^-}^{s^+} . \tag{65}$$

There remains the boundary terms at $s=s^-$ and $s=s^+$, which are determined by the solutions of the outer region MHD equations. The boundary conditions to be satisfied at $s=s^-$ and $s=s^+$ by the outer solutions are obtained by integrating Eq. (18) in α from α^+ to α and in s from s^- to s^+. One finds

$$\left[\frac{\sigma r^2 B_0}{4\pi} \left(\frac{\partial}{\partial s} \frac{\partial \alpha_1}{\partial \alpha}\right)\right]_{s^-}^{s^+} - \left[\frac{\sigma r^2 B_0}{4\pi} \frac{\partial}{\partial s} \frac{\partial \alpha_1}{\partial \alpha}\right]_{s^-(\alpha=\alpha^+)}^{s^+}$$

$$\simeq - \int_{\alpha^+}^{\alpha} d\alpha \int_{s^-}^{s^+} ds \frac{\ell^2 \kappa}{r B_0^2} \frac{\partial P_H}{\partial \alpha} (G-\alpha_1)\left(1 - \left\langle \frac{\kappa_s}{|\ell|\kappa} \nu_\ell \right\rangle\right)$$

$$\approx - \int_{\alpha^+}^{\alpha} d\alpha \int_{s^-}^{s^+} ds \frac{\ell^2 \kappa}{r B_0^2} \frac{\partial P_H}{\partial \alpha} \frac{\left(\tilde{\beta}_w + \frac{\omega}{\tilde{\omega}_{\kappa H}}\right)\left(1 - \left\langle \frac{\kappa_s}{|\ell|\kappa} \nu_\ell \right\rangle\right)}{\left(1 - \tilde{\beta}_w - \frac{\omega}{\tilde{\omega}_{\kappa H}} - \left\langle \frac{\kappa_s}{|\ell|\kappa} \nu_\ell \right\rangle\right)} .$$

This constitutes a jump condition in the outer solutions across the hot particle region. This jump condition has been employed in Ref. (15) to investigate the coupling of the mirror cell to surface Alfvén waves propogating in the central cell of tandem mirror configurations.

In this analysis we assume $\omega L_W/v_A < 1$, where L_W is the axial extent of the background plasma. In this case the solutions for $s > s^+$ and $s < s^-$ (the outer solutions) are flute-like, and we evaluate the boundary term in Eq. (65) for such a flute solution. The boundary term in Eq. (65) can be related to integrals of the outer solutions over the outer region by multiplying Eq. (18) by α_1, and then integrating in α from $\alpha = \alpha^-$ to α^+ and in s in the range $s < s^-$, $s > s^+$ outside the hot plasma component region. We then obtain

$$\int d\alpha \left(\frac{\partial\alpha_1}{\partial\alpha} \frac{r^2 B_0}{4\pi} \frac{\partial}{\partial s} \frac{\partial\alpha_1}{\partial\alpha}\right)\Big|_{s^+}^{s^-}$$

$$= \left\{\int_{-\infty}^{s^-} ds + \int_{s^+}^{\infty} ds\right\}\left[\alpha_1 \left(\frac{\omega^2 r^2 B_0}{4\pi v_A^2} + \frac{\partial}{\partial s} \frac{r^2 B_0}{4\pi} \frac{\partial}{\partial s}\right) \frac{\partial\alpha_1}{\partial\alpha}\right]\Big|_{\alpha^-}^{\alpha^+}$$

$$+ \theta\left(\frac{\ell\Delta\alpha}{\alpha^-}\right) . \tag{66}$$

For a flute-like solution α_1 is independent of s, and we need $\frac{\partial\alpha_1}{\partial\alpha}$ to determine a dispersion relation in Eq. (66). The solutions in the region $\alpha < \alpha^-$ and $\alpha_p^+ < \alpha < \alpha_w(s)$ can be determined straight-forwardly in a manner similar to Eqs. (49)-(51). We find

$$\frac{1}{\alpha_1} \frac{\partial\alpha_1}{\partial\alpha}\Big|_{\alpha^-} = \frac{|\ell|}{2\alpha^-} \doteq \frac{|\ell|}{2\alpha_0} \tag{67}$$

$$\frac{1}{\alpha_1} \frac{\partial\alpha_1}{\partial\alpha}\Big|_{\alpha^+} = \frac{\ell}{2\alpha^+} Z_\ell \doteq \frac{\ell Z_\ell}{2\alpha_0} .$$

Equation (66) then becomes

$$\int d\alpha \, \frac{\partial\alpha_1}{\partial s} \frac{r^2 B_0}{4\pi} \frac{\partial}{\partial s} \frac{\partial\alpha_1}{\partial\alpha} \Big|_{s^+}^{s^-}$$

$$= [\int_{-\infty}^{s^-} ds + \int_{s^+}^{\infty} ds][\frac{\omega^2}{4\pi v_A^2} \, |\ell| \, (1 + Z_{|\ell|} \frac{N_0^+}{N_0^-})] \; .$$

Substituting this expression into Eq. (65) then yields the dispersion relation for the layer mode,

$$\omega^2 = \gamma_{MHD}^2 \frac{(\tilde{\beta}_w + \frac{\omega}{\tilde{\omega}_{\kappa H}})(1 - \langle \frac{\kappa_s \nu_\ell}{|\ell| \kappa} \rangle)}{\{1 - \tilde{\beta}_w - \frac{\omega}{\tilde{\omega}_{\kappa H}} - \langle \frac{\kappa_s \nu_\ell}{|\ell| \kappa} \rangle\}} \; , \tag{68}$$

where

$$\gamma_{MHD}^2 = \frac{|\ell| \int_{\alpha^-}^{\alpha^+} d\alpha \int_{-\infty}^{\infty} \frac{ds}{B_0^2} \frac{\kappa}{r} \frac{\partial P_H}{\partial \alpha}}{\int_{-\infty}^{+\infty} ds \, \frac{1}{4\pi v_A^2} \, (1 + Z_{|\ell|} \frac{N_0^+}{N_0^-})} \doteq \frac{-|\ell| \int_{-\infty}^{\infty} \frac{ds}{B_0^2} \frac{\kappa}{r_0} P_H}{\int_{-\infty}^{\infty} \frac{ds}{4\pi v_A^2} \, (1 + Z_{|\ell|} \frac{N_0^+}{N_0^-})} \tag{69}$$

and terms of higher order in ε have been neglected.

Eq. (68) has only real roots when

$$g_{|\ell|} \equiv 1 - \langle \frac{\nu_\ell \kappa_s}{|\ell| \kappa} \rangle < 0 \tag{70}$$

and when this condition is satisfied the layer modes have a robust stability. However, as $\kappa = \kappa_s + \kappa_v$, with κ_v the correction vacuum curvature due to external currents, we have

$$g_{|\ell|} \doteq 1 - \frac{1 + (\alpha^+/\alpha_w)^{|\ell|}}{|\ell|(1 + \kappa_v/\kappa_s)} = \frac{\kappa_v/\kappa_s + 1 - [1 + (\alpha^+/\alpha_w)^{|\ell|}]/|\ell|}{1 + \kappa_v/\kappa_s}$$

$$= \frac{|\ell| - 2 + |\ell|[\frac{\kappa_v}{\kappa_s}(1 + \frac{1}{|Z_\ell|}) + \frac{1}{|Z_\ell|}]}{|\ell|[1 + (1 + \frac{\kappa_v}{\kappa_s})/Z_{|\ell|}]} \tag{71}$$

and we notice that since $\kappa_v/\kappa_s > 0$ only $|\ell| = 1$ can satisfy the robust stability condition when the conducting wall is close enough to the plasma surface. Other investigations, to be presented elsewhere[16] and discussed in Ref. (13), have shown that finite Larmor effects can introduce a similar robust stability for higher $|\ell|$-numbers.

When $g_{|\ell|} > 0$, stability from Eq. (68) is possible if $\varepsilon_v \equiv \gamma^2_{MHD}/\tilde{\omega}^2_{\kappa H}$ and $\tilde{\beta}_w$ are below certain critical threshold values. These threshold values shall be referred to as the decoupling conditions.

Let

$$\varepsilon_\ell = \varepsilon_v/g_\ell \ , \quad \tilde{\beta}_\ell = \tilde{\beta}_w/g_\ell \ , \quad \tilde{\omega}_{\kappa\ell} = g_\ell\tilde{\omega}_{\kappa H} \ .$$

Eq. (68) can then be written as

$$y^2 \equiv \frac{\omega^2}{\tilde{\omega}^2_{\kappa\ell}} = \frac{\varepsilon_\ell (y + \tilde{\beta}_\ell)}{1 - \tilde{\beta}_\ell - y} \; . \tag{72}$$

This is a prototype form of the dispersion elation discussed in Ref. (7) for eikonal modes. However, with a reinterpretation of parameters, the dispersion relation applies to layer modes. Notice that ε_ℓ and $\tilde{\beta}_\ell$ is largest for smaller ℓ.

First of all we investigate stability for $\tilde{\beta}_\ell << 1$, so that we can neglect $\tilde{\beta}_\ell$ in the numerator. The stability condition is then

$$\varepsilon_\ell < \frac{1}{4} - \frac{\tilde{\beta}_\ell}{2} \; , \quad \text{if } \tilde{\beta}_\ell << 1$$

or

$$\frac{\varepsilon_V}{g_\ell} < \frac{1}{4} \left(1 - \frac{2\tilde{\beta}_W}{g_\ell}\right) \; . \tag{73}$$

Because $g_\ell < g_{\ell+1}$, the decoupling condition is hardest to achieve for the low ℓ-modes if $g_\ell > 0$ (it is of course automatically satisfied if $g_\ell < 0$). We also observe that as $\tilde{\beta}_W$ increases, a smaller value of ε_V is needed to satisfy the decoupling condition (physically this means higher hot particle energies).

If we consider $\varepsilon_\ell \to 0$ (the limit of arbitrarily high hot-particle energy), the decoupling condition becomes

$$\tilde{\beta}_\ell \equiv \frac{\tilde{\beta}_W}{g_\ell} < 1 \; . \tag{74}$$

This is the appropriate ideal Lee-Van Dam-Nelson core beta limit, but it is now a function of ℓ-number. The critical core beta threshold is determined by the ℓ=1 mode.

For a finite ε_v, the critical core beta limit is below the upper limit given by Eq. (74). If $\varepsilon_\ell << 1$, the shift below the ideal limit can be determined from Eq. (72) by setting $\tilde{\beta}_\ell = 1 - \delta\tilde{\beta}_\ell$ and assuming $1 >> y_1 \delta\tilde{\beta}_\ell$, Eq. (72) then becomes

$$y^2 = - \frac{\varepsilon_\ell}{y - \delta\beta_\ell}$$

and stability requires

$$\tilde{\beta}_\ell < 1 - 3\left(\frac{\varepsilon_\ell}{4}\right)^{1/3} , \tag{75}$$

and when $\tilde{\beta}_\ell = 1$, the growth rate is

$$\frac{(3)^{1/2} \tilde{\omega}_{\kappa H} \varepsilon_v^{1/3} g_\ell^{2/3}}{2} .$$

An interpolation formula can be obtained from Eqs. (73) and (75) for the stable range of $\tilde{\beta}_w$. With theassumption $g_\ell > 0$, we find

$$\tilde{\beta}_w < \tilde{\beta}_{wcr} \equiv \mathrm{Min}\left[\frac{g_\ell}{2} - 2\varepsilon_v,\ g_\ell - 3\left(\frac{g_\ell^2 \varepsilon_v}{4}\right)^{1/3}\right] \tag{76}$$

and it is clear that the $\ell=1$ supplies the most restrictive conditions for the stability window.

Above marginal stability the dispersion relation approaches the predictions of MHD theory, with the growth rate, γ, given by

$$\gamma = g_\ell^{1/2} \gamma_{MHD} \ .$$

At higher frequencies where $\omega \sim \tilde{\omega}_{\kappa H} >> \gamma_{MHD}$, there exists a precessional mode with frequency

$$\omega \approx \tilde{\omega}_{\kappa H} \left(1 - \left< \frac{\kappa_s \nu_\ell}{|\ell| \kappa} \right> - \tilde{\beta}_w\right) \ . \tag{77}$$

When $\omega > 0$, it is a negative energy mode. It can be destabilized by dissipation effects or by coupling to positive energy modes (for example, Alfvén modes[15,18] which can be excited in regions outside the hot plasma component layer). In this case, Eq. (67) is no longer appropriate to represent the perturbations in the region $s < s^-$, $s > s^+$. This problem has been considered elsewhere[15]. Eq. (65) with the boundary term at $s=s^-$ and $s=s^+$ determined by Eq. (66) provides the framework with which to discuss the coupling of the precessional mode to other modes excited outside the layer.

Throughout, we have neglected finite Larmor radius (FLR) effects. However, we note that if FLR effects of the hot particles are retained in the layer mode analysis, a recent investigation[16] has shown that a dispersion is obtained similar to what we have presented, except that $g_{|\ell|}$, given in Eq. (70) needs to be redefined as

$$g_{|\ell|} \to 1 - \left\langle \frac{\kappa_s}{|\ell|\kappa}\left[1 + \left(\frac{\alpha^+}{\alpha_w}\right)^{|\ell|}\right]\right\rangle - (\ell^2 - 1)\left\langle \frac{r_L^2 \beta_{\perp H}}{r^2 \kappa\Delta}\right\rangle ,$$

where

$$\left\langle \frac{r_L^2}{2r^2}\,\frac{\beta_{\perp H}}{\kappa\Delta}\right\rangle = \frac{4\pi \int_{\alpha^-}^{\alpha^+} d\alpha \int \frac{ds}{B_0^3 r^2}\,\frac{\partial}{\partial\alpha}\,(P_{\perp H} r_L^2)\,\frac{\partial P_{\perp H}}{\partial\alpha}}{\int ds\,\frac{\kappa}{B_0^2 r}\,\frac{\partial P_H}{\partial\alpha}} .$$

Robust stability to all modes is then achieved if $g_\ell < 0$. Roughly, this is fulfilled for all ℓ if

$$\frac{r_L^2 \beta_{\perp H}}{2 r_0^2 \kappa\Delta} > 1 , \qquad \frac{\kappa_s}{\kappa}\left(1 + \frac{\alpha^+}{\alpha_w}\right) > 1 .$$

(B) Ring-like Pressure Profile

In the case of a ring-like pressure profile in a single cell where the pressure maximum is at α_0, $\alpha^+ > \alpha_0 > \alpha^-$, Eq. (41) is solved separately in the two regions of the layer, $\alpha_0 > \alpha > \alpha^-$ and $\alpha^+ > \alpha > \alpha_0$.

By repeating the manipulations described above and making similar approximations, similar solutions are obtained, and it is readily verified that from the continuity of $\frac{\partial}{\partial s}\frac{\partial\alpha_1}{\partial\alpha}$ and $\frac{\partial\alpha_1}{\partial s}$ at $\alpha = \alpha^+, \alpha^-$:

$$Z_\ell \frac{\partial f^+}{\partial s} + \frac{\partial f^-}{\partial s} = 0$$

$$(1 + Z_\ell) \frac{\partial f^+}{\partial s} = (G^+ - 1) \frac{\partial}{\partial s} \int_{\alpha_0}^{\alpha^+} d\alpha \frac{4\pi}{B^2} \frac{\partial P_{\perp H}}{\partial \alpha}$$

$$+ (G^- - 1) \frac{\partial}{\partial s} \int_{\alpha^-}^{\alpha_0} d\alpha \frac{4\pi}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha}$$

$$\approx (G^- - G^+) \frac{\partial}{\partial s} \frac{4\pi P_{\perp H}^*}{B_0^2} ,$$

where

$$G^+\left(1 - \tilde{\beta}_w^+ - \frac{\omega}{\tilde{\omega}_{\kappa H}^+}\right) = 1 + (G^+ - G^-) \left\langle \frac{r\beta_{\perp H}}{|\ell| \kappa L_H^2} \right\rangle^+$$

$$G^-\left(1 - \frac{\omega}{\tilde{\omega}_{\kappa H}^-}\right) = 1 + (G^+ - G^-) \left\langle \frac{r\beta_{\perp H}}{|\ell| \kappa L_H^2} \right\rangle^-$$

$$\left\langle \frac{r\beta_{\perp H}}{4\kappa L_H^2} \right\rangle = \frac{2\pi \int \frac{ds}{B_0^2} \frac{\partial P_{\perp H}}{\partial \alpha} \frac{\partial}{\partial s} \nu_\ell \frac{\partial}{\partial s} \frac{P_{\perp H}^*}{B_0^2}}{\int ds \frac{\kappa}{r B_0^2} \frac{\partial P_H}{\partial \alpha}} .$$

The superscripts '+' and '−' are used to differentiate the two regions $\alpha^+ > \alpha > \alpha_0$ and $\alpha_0 > \alpha > \alpha^-$ respectively. $\frac{\partial P_{\perp H}}{\partial \alpha}$ is considered to be constant over most of each region, and approximate solutions for G^+ and G^- determined as before. $P_{\perp H}^*$ is the maximum of $P_{\perp H}$ which occurs at $\alpha = \alpha_0$.

If the following equalities are assumed

$$\left\langle \frac{r\beta_{\perp H}}{\kappa L_H^2} \right\rangle^{+} = \left\langle \frac{r\beta_{\perp}}{\kappa L_H^2} \right\rangle^{-}$$

$$\tilde{\omega}^{+}_{\kappa H} = \tilde{\omega}^{-}_{\kappa H} .$$

The solution for G^+ and G^- simplifies to:

$$G^{+} = \frac{\left(1 - \frac{\omega}{\tilde{\omega}_{\kappa H}}\right)}{\left[\left(1 - \tilde{\beta}_w - \frac{\omega}{\tilde{\omega}_{\kappa H}}\right)\left(1 - \frac{\omega}{\tilde{\omega}_{\kappa H}}\right) - \tilde{\beta}_w \left\langle \frac{r\beta_{\perp H}\nu_\ell}{4|\ell|\kappa L_H^2} \right\rangle\right]}$$

$$G^{-} = \frac{\left(1 - \tilde{\beta}_w - \frac{\omega}{\tilde{\omega}_{\kappa H}}\right)}{\left[\left(1 - \tilde{\beta}_w - \frac{\omega}{\tilde{\omega}_{\kappa H}}\right)\left(1 - \frac{\omega}{\tilde{\omega}_{\kappa H}}\right) - \tilde{\beta}_w \left\langle \frac{r\beta_{\perp H}\nu_\ell}{4|\ell|\kappa L_H^2} \right\rangle\right]} ,$$

As before, the solutions for α_1 are substituted in the quadratic forms of Eq. (19). The volume integral is:

$$\overline{W}(\alpha_1,\alpha_1)$$

$$= \int_{\alpha_0}^{\alpha^+} d\alpha \int ds \, \frac{\ell^2\kappa}{rB_0^2} \frac{\partial P_H}{\partial\alpha} \left[G^{+2}\left(\tilde{\beta}_w + \frac{\omega}{\tilde{\omega}_{\kappa H}}\right) - (G^{+}-1)^2 - G^{-2} \frac{\omega}{\tilde{\omega}_{\kappa H}} + (G^{-}-1)^2\right] , \quad (78)$$

where the additional assumption

$$\int_{\alpha_0}^{\alpha^+} d\alpha \int ds \, \frac{\ell^2 \kappa}{r B_0^2} \frac{\partial P_H}{\partial \alpha} = - \int_{\alpha^-}^{\alpha_0} d\alpha \int ds \, \frac{\ell^2 \kappa}{r B_0^2} \frac{\partial P_H}{\partial \alpha}$$

has been made. The surface integrals [Eqs. (63) and (66)] are:

$$\left[\int_{s^-}^{s^+} ds \; \alpha_1 \hat{D} \frac{\partial \alpha_1}{\partial \alpha}\right]_{\alpha^-}^{\alpha^+} = - \int_{s^-}^{s^+} ds \, \frac{\omega^2}{4\pi v_A^2} \, |\ell| \left(1 + Z_{|\ell|} \frac{N_0^+}{N_0^-}\right)$$

$$+ \int_{\alpha_0}^{\alpha^+} d\alpha \int_{s^-}^{s^+} ds \, \frac{\ell^2 \kappa}{r B_0^2} \frac{\partial P_H}{\partial \alpha} \, (G^+ - G^-)^2 \left\langle \frac{r \beta_{\perp H} \nu_\ell}{4 |\ell| \kappa L_H^2} \right\rangle , \tag{79}$$

$$\int ds \left(\frac{\partial \alpha_1}{\partial \alpha} \frac{r^2 B_0}{4\pi} \frac{\partial}{\partial s} \frac{\partial \alpha_1}{\partial \alpha} + \alpha_1 \frac{\ell^2}{4\pi r^2 B_0} \frac{\partial \alpha_1}{\partial s}\right)\Bigg|_{s^-}^{s^+}$$

$$= \left(\int_{-\infty}^{s^-} ds + \int_{s^+}^{\infty} ds\right) \left[\frac{\omega^2}{4\pi v_A^2} \, |\ell| \left(1 + Z_{|\ell|} \frac{N_0^+}{N_0^-}\right)\right] . \tag{80}$$

Flute-like perturbations are assumed to extend into the range $s > s^+$, $s < s^-$.

The dispersion relation obtained from Eq. (19) [analogous to Eq. (68)] is:

$$\omega^2 = \frac{\gamma_{MHD}^2 \tilde{\beta}_W}{\left[\left(1 - \frac{\omega}{\tilde{\omega}_{\kappa H}}\right)\left(1 - \tilde{\beta}_W - \frac{\omega}{\tilde{\omega}_{\kappa H}}\right) - \tilde{\beta}_W \left\langle \frac{r \beta_{\perp H}}{4 |\ell| \kappa L_H^2} \right\rangle\right]} , \tag{81}$$

where

$$\gamma_{MHD}^2 = - \frac{\int ds \, \dfrac{\ell^2 \kappa P_H^*}{rB_0^2}}{|\ell| \int_{-\infty}^{+\infty} ds \, \left(1 + Z_{|\ell|} \dfrac{N_0^+}{N_0^-}\right)/4\pi v_A^2}$$

To analyze Eq. (81), let

$$x \equiv \frac{\omega}{\tilde{\omega}_{\kappa H}} , \qquad b \equiv \frac{\tilde{\beta}_w \gamma_{MHD}^2}{\tilde{\omega}_{\kappa H}^2} , \tag{82}$$

$$a_1 = \frac{1}{2} \left[(2 - \tilde{\beta}_w) - \left(\tilde{\beta}_w^2 + \tilde{\beta}_w \left< \frac{r\beta_{\perp H} \nu_\ell}{|\ell| \kappa L_H^2} \right>\right)^{1/2}\right] ,$$

$$a_2 = \frac{1}{2} \left[(2 - \tilde{\beta}_w) + \left(\tilde{\beta}_w^2 + \tilde{\beta}_w \left< \frac{r\beta_{\perp H} \nu_\ell}{|\ell| \kappa L_H^2} \right>\right)^{1/2}\right] .$$

The dispersion relation can be written as follows:

$$x^2 = \frac{b}{(x-a_2)(x-a_1)}$$

In the limit of $\tilde{\beta}_w \to 0$, such that $a_2 = 1$ and $a_1 = 1$, the dispersion relation has four real roots if

$$b = \frac{\tilde{\beta}_w \gamma^2_{MHD}}{\tilde{\omega}^2_{\kappa H}} < \frac{1}{16} , \tag{83}$$

which implies that there is always stability as $\tilde{\beta}_w \to 0$. If $b \ll \frac{1}{16}$, the roots are

$$x = x_p^{\pm} \equiv 1 - \frac{\tilde{\beta}_w}{2} \pm \frac{1}{2} \left(\tilde{\beta}_w^2 + \tilde{\beta}_w < \frac{r\beta_{\perp H}\nu_\ell}{|\ell| \kappa L_H^2} >\right)^{1/2}$$

$$x = x_i^{\pm} = \pm b^{1/2} .$$

The $x_i^{\pm}$ roots are the interchange modes of the background plasma stabilized by the diamagnetic well. The $x_p^{\pm}$ roots are the precessional modes associated with the inner (the x_p^+ mode) and outer (the x_p^- mode) parts of the pressure gradient.

However, a more careful analysis[11] of the variation of the α_1 eigenfunction in the plasma ring to higher order in $\ell\Delta_\alpha/\alpha_0$ shows that a finite upper bound on $\gamma^2_{MHD}/\tilde{\omega}^2_{\kappa H}$ does exist as $\tilde{\beta}_w \to 0$. The validity of Eq. (81) requires $\tilde{\beta}_w > (\ell\Delta_\alpha/\alpha_0)^{1/2}$.

In the limit of small $\gamma^2_{MHD}/\tilde{\omega}^2_{\kappa H} \to 0$ (arbitrarily hot particles), the dispersion relation for the interchange mode is

$$\omega^2 = \frac{\gamma^2_{MHD}\tilde{\beta}_w}{1 - \tilde{\beta}_w - \frac{\tilde{\beta}_w}{4} < \frac{r\beta_{\perp H}\nu_\ell}{|\ell|\kappa L^2_H} >} \tag{84}$$

The mode is stable only if

$$\tilde{\beta}_w < \tilde{\beta}_{wcr} \equiv \frac{1}{1 + < \frac{r\beta_{\perp H}\nu_\ell}{4|\ell|\kappa L^2_H} >} . \tag{85}$$

Thus the bending energy term lowers the threshold for the Lee Van Dam-Nelson beta limit.

In order to obtain the stability threshold somewhat more accurately, consider $b << \frac{1}{16}$ and take $\tilde{\beta}_w = \tilde{\beta}_{wcr} - \delta\tilde{\beta}_w$ with $\delta\tilde{\beta}_w << \tilde{\beta}_{wcr}$. The dispersion relation then becomes:

$$x^2 = - \frac{\tilde{\beta}_{wcr}\gamma^2_{MHD}}{\tilde{\omega}^2{}_{\kappa H}(2-\tilde{\beta}_{wcr})\left(x - \frac{\delta\tilde{\beta}_w}{\tilde{\beta}_{wcr}} \frac{1}{(2-\tilde{\beta}_{wcr})}\right)} \tag{86}$$

and the stability criterion is:

$$\tilde{\beta}_w < \tilde{\beta}_{wcr}\left\{1 - \frac{3}{2}\left[\frac{2\gamma^2_{MHD}}{\tilde{\omega}^2_{\kappa H}} \tilde{\beta}_{wcr}(2 - \tilde{\beta}_{wcr})^2\right]^{1/3}\right\} . \tag{87}$$

When $\delta\breve{\beta}_w = 0$, the growth rate γ is

$$\gamma = \frac{\sqrt{3}}{2}\left\{\frac{\breve{\beta}_{wcr}\gamma^2_{MHD}\breve{\omega}_{\kappa H}}{(2-\breve{\beta}_{wcr})}\right\}^{1/3} . \tag{88}$$

When $\breve{\beta}_w \gg \breve{\beta}_{wcr}$, the growth rate is

$$\gamma \rightarrow \frac{\gamma_{MHD}}{\left[1 + \left< \frac{r\beta_{\perp H}\nu_\ell}{4|\ell|\kappa L_H^2} \right>\right]^{1/2}} \tag{89}$$

and the bending term substantially lowers the growth rate from that obtained in a previous calculation.[11]

The dispersion relation given in Eq. (81) can be generalized to discuss EBT configuration in which N mirror cells are connected to form a torus. The flux surface perturbations are then of the form

$$\alpha_1 = \phi(\alpha,s)\ \cos ks$$

or

$$\alpha_1 = \phi(\alpha_1 s)\ \sin ks ,$$

where ϕ is a periodic function of s with periodic length equal to the length of the mirror cell L, and

$$k = \frac{2\pi n}{NL} , \quad n \equiv \text{integer} .$$

For simplicity we deal with the cosks term, and the sinks term can be treated in exactly the same way (except that there is then no nontrivial n=0 term). If n is sufficiently small, cosks is nearly constant over the region occupied by the hot particles in each cell. Thus, the lowest order solutions in each cell are essentially the same as those determined for a single cell. However, in evaluating the quadratic forms to determine the dispersion relation, the additional factor (namely cosks) introduces terms which take into account the presence of long wavelength Alfvén waves. Thus Eqs. (79) and (80) for the surface integrals are modified as follows:

$$\sum_{p=0}^{N-1} \left[\int_{s^-}^{s^+} ds\,\alpha_1 \hat{D} \frac{\partial \alpha_1}{\partial \alpha}\right]_{\alpha^-}^{\alpha^+} = -\sum_{p=0}^{N-1} \int_{s^-}^{s^+} ds \left\{\frac{\omega^2}{v_A^2}\left(1 + Z_{|\ell|}\frac{N_0^+}{N_0^-}\right)\cos^2 k(s+pL)\right.$$

$$\left. - k^2(1 + Z_{|\ell|})\sin^2 k(s+pL)\right\}\frac{|\ell|}{4\pi}$$

$$-\sum_{p=0}^{N-1} \int_{s^-}^{s^+} ds\, \frac{\ell^2\kappa}{rB_0^2} P_H^*(G^+ - G^-)^2 \left\langle \frac{r\beta_{\perp H}\nu_\ell}{4|\ell|\kappa L_H^2}\right\rangle \cos^2 k(s+pL) \tag{90}$$

and

$$\sum_{p=0}^{n-1} \int d\alpha\left(\frac{\partial\alpha_1}{\partial\alpha}\frac{r^2B_0}{4\pi}\frac{\partial}{\partial s}\frac{\partial\alpha_1}{\partial\alpha} + \alpha_1\frac{\ell^2}{4\pi r^2 B_0}\frac{\partial\alpha_1}{\partial s}\right)\Big|_{s^-}^{s^+}$$

$$= -\sum_{p=0}^{N-1}\left\{\int_0^{s^-} ds + \int_{s^+}^{L} ds\right\}\left\{\frac{\omega^2}{v_A^2}\left(1 + Z_{|\ell|}\frac{N_0^+}{N_0^-}\right)\cos^2 k(s+pL) - k^2(1 + Z_{|\ell|})\sin^2 k(s+pL)\right\}\frac{|\ell|}{4\pi}\,. \tag{91}$$

The cells are all assumed to be the same and the summation is over the N cells.

Thus the dispersion relation determined from Eq. (19) is:

$$\omega^2 - k^2 \langle v_A^2 \rangle = \frac{\gamma_{MHD}^2 \tilde{\beta}_W}{\left[\left(1 - \frac{\omega}{\tilde{\omega}_{\kappa H}}\right)\left(1 - \tilde{\beta}_W - \frac{\omega}{\tilde{\omega}_{\kappa H}}\right) - \tilde{\beta}_W \left\langle \frac{r\beta_{\perp H}\nu_\ell}{4|\ell|\kappa L_H^2} \right\rangle\right]} ,$$

where

$$\gamma_{MHD}^2 = \frac{-\sum_{p=0}^{N-1} \int_0^L ds \frac{|\ell|\kappa}{rB_0^2} P_H^* \cos^2 k(s+pL)}{\sum_{p=0}^{N-1} \int_0^L \frac{ds}{4\pi v_A^2} \left(1 + Z_{|\ell|} \frac{N_0^+}{N_0^-}\right) \cos^2 k(s+pL)}$$

$$= \frac{-|\ell| \int_{s^-}^{s^+} ds \frac{\kappa}{rB_0^2} P_H^*}{\int_0^L ds\ 4\pi v_A^2 \left(1 + Z_{|\ell|} \frac{N_0^+}{N_0^-}\right)}$$

$$\langle v^2{}_A \rangle = \frac{\int_0^L ds(1 + Z_{|\ell|})}{\int_0^L ds \frac{1}{v_A^2}\left(1 + Z_{|\ell|} \frac{N_0^+}{N_0^-}\right)} .$$

When k=0, the dispersion relation is the same as that previously discussed for a single cell.

Even if $k=0$ modes are stable, $k\neq 0$ modes can be unstable due to coupling of Alfvén modes $\omega = k \langle v_A^2\rangle^{1/2}$ to the precessional modes $\omega \sim \tilde{\omega}_{\kappa H}$.

Let $d^2 = \langle v_A^2\rangle/\tilde{\omega}_{\kappa H}^2$. The dispersion relation can then be written as

$$x^2 - k^2 d^2 = \frac{b}{[x - (1-\tilde{\beta}_w/2)]^2 - \tilde{\beta}_w^2 \lambda^2} \tag{92}$$

with x and b defined in Eq. (82) and

$$\lambda^2 = \frac{1}{4}\left(1 + \left\langle \frac{r\beta_{\perp H}\nu_\ell}{4\ell\kappa L_H^2} \right\rangle / \tilde{\beta}_w\right)$$

When $\tilde{\beta}_w \lambda / b^{1/3} < 1$, one finds a band of instability given by

$$-\frac{2}{3} b^{1/3} < k^2 d^2 - (1 - \beta_w/2)^2 < b^{1/3}\left(\frac{b^{1/3}}{\tilde{\beta}_w \lambda}\right)^2 . \tag{93}$$

This band of instability is broad band (the width of stable k comparable or larger than k itself) if

$$b^{1/3}\left(\frac{b^{1/3}}{\tilde{\beta}_w \lambda}\right)^2 > 1 .$$

the mode of instability if interpreted as the interaction of the two precessional modes associated with each side of the hot pressure gradient being destabilized by the additional coupling with a backgound Alfvén wave. The growth γ is

$$\frac{\gamma}{\tilde{\omega}_{\kappa H}} = [\frac{b}{k^2d^2 - (1 - \tilde{\beta}_w/2)^2} - \tilde{\beta}_w^2\lambda^2]^{1/2} \tag{94}$$

if $k^2d^2 - (1 - \tilde{\beta}_w^2/2)^2 >> b^{1/3}$. Near resonance, we have a direct three-wave interaction of the Alfvén wave and the two precessional modes with a maximum growth rate

$$\gamma_{max} = \frac{3^{1/2}}{2} (\frac{b}{2-\tilde{\beta}_w})^{1/3} \tilde{\omega}_{\kappa H}$$

arising when $kv_A = \tilde{\omega}_{\kappa H}(1 - \tilde{\beta}_w/2)$.

As $b^{1/3}/\tilde{\beta}_w\lambda$ decreases, the two precessional modes begin to be mismatched and the unstable band in k narrows. When $\tilde{\beta}_w\lambda < b^{1/2}$, the unstable spectrum of k is less than k itself with $k \sim (1-\tilde{\beta}_w)/d$. When $\tilde{\beta}_w\lambda \gtrsim b^{1/3}$, the instability is due to the direct interaction of the Alfvén wave and the precessional mode associated with the outer edge of the plasma. Hence, we look for frequencies near

$$x = \frac{\omega}{\tilde{\omega}_{\kappa H}} = 1 - \frac{\tilde{\beta}_w}{2} - \tilde{\beta}_w\lambda$$

and wave numbers near

$$\frac{kv_A}{\tilde{\omega}_{\kappa H}} = 1 - \frac{\tilde{\beta}_w}{2} - \tilde{\beta}_w\lambda \ .$$

We find that the unstable k-spectrum is given by

$$-2\left(\frac{b}{\tilde{\beta}_w\lambda}\right)^{1/2}\left(1 - \frac{\tilde{\beta}_w}{2} - \tilde{\beta}_w\lambda\right)^{1/2} < \frac{k^2v_A^2}{\tilde{\omega}_{\kappa H}^2} - \left(1 - \frac{\tilde{\beta}_w}{2} - \tilde{\beta}_w\lambda\right)^2 < 2\left(\frac{b}{\tilde{\beta}_w\lambda}\right)^{1/2}\left(1 - \frac{\tilde{\beta}_w}{2} - \tilde{\beta}_w\lambda\right)^{1/2} , \tag{95}$$

and when

$$\frac{k^2v_A^2}{\tilde{\omega}_{\kappa H}^2} = \left(1 - \frac{\tilde{\beta}_w}{2} - \tilde{\beta}_w\lambda\right)^2 ,$$

the growth rate is given by

$$\gamma = \frac{\tilde{\omega}_{\kappa H}}{2}\left[\frac{b}{\tilde{\beta}_w\lambda(1 - \tilde{\beta}_{w/2} - \lambda\tilde{\beta}_w)}\right]^{1/2} .$$

In an EBT, the k-modes are quantitized with a spacing $2\pi/NL$. If

$$\frac{2\pi}{NL} > \frac{2\frac{\tilde{\omega}_{\kappa H}}{v_A}\left(\frac{b}{\tilde{\beta}_w\lambda}\right)^{1/2}}{\left(1 - \frac{\tilde{\beta}_w}{2} - \tilde{\beta}_w\lambda\right)^{1/2}} \equiv \Delta k ,$$

then the quantitization interval is larger than the instability spectrum, and the instability may be avoided. However, when $2\pi/NL \lesssim \Delta k$, unstable modes will exist if $2\pi n v_A/NL \approx \tilde{\omega}_{\kappa H}(1 - \tilde{\beta}_w/2 - \tilde{\beta}_w\lambda)$ with n an integer. Another way to avoid the precessional mode interaction is to have $k_{min}v_A = 2\pi v_A/NL > \tilde{\omega}_{\kappa H}\left(1 - \frac{\tilde{\beta}_w}{2} - \tilde{\beta}_w\lambda\right)$ so that resonance with the Alfvén wave is avoided. However, this condition is generally incompatible with establishing a decoupling condition when $N \gg 1$.

The highest frequency for the shear Alfvén wave is approximately $\omega \approx v_A/L_H$ and if $\omega_{\kappa H} > v_A/L_H$ the precessional-shear Alfvén wave coupling may be avoided. The decoupling condition requires at sufficiently small $\tilde{\beta}_w$

$$\omega_{\kappa H}^2 > 4\gamma_{MHD}^2 \approx 2v_A^2 \frac{\kappa}{r} \beta_{\perp H} \frac{|\ell|}{(1 + \langle \nu_\ell \beta_{\perp H} r/4|\ell|\kappa L_H^2 \rangle)} . \tag{96}$$

Thus, the combined condition for decoupling from the MHD interchange and not having a shear Alfvén resonance is

$$\frac{\omega_{\kappa H}^2 L^2}{\ell^2 v_A^2} > \max\left[\frac{L^2}{\ell^2 L_H^2} , \frac{2\beta_{\perp H}\kappa L^2}{\left[|\ell| + \langle \frac{r\beta_{\perp H}\nu_\ell}{4L_H^2\kappa} \rangle\right]}\right] , \tag{97}$$

where we assume $k_{max} \approx 1/L_H$. This condition is most difficult to satisfy for the ℓ=1 mode. It indicates that when $\tilde{\beta}_w \ll 1$, the hot particle energy needs to be a factor $L/\beta_{\perp H}^{1/2} L_H$ larger than the energy needed to establish decoupling from the MHD interchange mode.

One should note that dissipation mechanisms in the background plasma can also destabilize the precessional mode as discussed in Ref. (14).

V. SUMMARY OF RESULTS

We have analyzed the hot particle stabilization of a symmetric mirror system which can either be a tandem mirror, or a multiple mirror system (the mathematical limit of large aspect ratio bumpy torus). Throughout this work it is assumed that the diamagnetic well of the hot particles produces strong grad-B drift reversal, the linear frequency response is well below the ion cyclotron frequency, and that the hot particle beta is low even though there is strong drift reversal. The work differs from most past studies in that it systematically investigates the effect of low ℓ-numbers, both in the eikonal approximation (the mode can still be radially localized) and for long wavelength layer modes. For such modes the positive energy bending term of the quadratic form is important and can lead to significant consequences. Finite Larmor radius effects are not treated in this work, but results from other investigations are used to obtain robust stability criteria.

Without finite Larmor radius effects, hot particles change the character of MHD modes, but the negative energy curvature drive that can cause instability still exists.[19] The stability criteria for hot particles differs from MHD predictions because the negative energy drives are so strong that negative energy modes can decouple from positive energy modes, and instead of an unstable mode one can find a stable negative energy wave if positive dissipation mechanisms are neglected (or overcome by negative energy dissipation). As coupling to positive energy waves is increased by increasing positive energy coupling, the negative energy waves are <u>destabilized</u>, Thus one finds that a small increase of a positive energy source that arises from line bending energy and magnetic compressional finite Larmor radius effects is at first destabilizing. However, with enough line bending and finite Larmor radius effects, all perturbations can be converted to positive energy and robust stability can be found in some configurations.

One finds that a disc-shaped hot particle pressure profile achieves robust stabilization if

$$\frac{r_L^2 \beta_{\perp H}}{2 r_0^2 \kappa \Delta} > 1 \ , \quad \frac{\kappa_s}{\kappa}\left(1 + \frac{\alpha_p}{\alpha_w}\right) > 1 \ , \tag{98}$$

where r_L is the Larmor radius of the hot plasma, r_0 the radius of the plasma edge, $\beta_{\perp H}$ the perpendicular beta of the hot component, κ is the total curvature, κ_s is defined in Eq. (62) and for a disc-shaped plasma at low beta it is self-induced curvature from the equilibrium currents, Δ is the scale length of the pressure gradient of the hot component, α_p is the magnetic flux enclosed by the hot plasma, and α_w is the magnetic flux enclosed by the conducting wall.

For a thin ring shaped plasma, wall stabilization of the ℓ=1 layer mode is not obtained. However, line bending does cause a decrease in the core beta limit of layer modes. The critical beta limit, β_{wcr}, of the background plasma is roughly given by (we assume $P_{\parallel}/P_{\perp} \ll 1$)

$$\beta_{wcr} \equiv \frac{2\kappa\Delta}{[1 + \kappa_s(1 + \alpha_p/\alpha_w)/\kappa\}} \ , \tag{99}$$

where κ_s is given by Eq. (62) but it is not the induced self-curvature for a ring-like pressure profile.

The higher ℓ-layer modes is stabilized by FLR effects if the first expression in Eq. (98) is satisfied.

Complete stabilization of eikonal modes can be obtained by combining the positive energy effects of line bending and FLR. The stability condition is

$$\frac{r_L^2 \kappa_s \beta_H}{r_0 \Delta^2 \kappa^2} > 1 , \tag{100}$$

which is easier to fulfill than the layer mode criteria given in Eq. (98).

We note that the finite Larmor radius robust stability conditions are not likely to be satisfied with hot electrons except at extremely high energies or low magnetic fields. However, hot ions can satisfy the stability conditions at reasonable energies for reactor plasmas (typically $\sim 1/2$ MeV). If robust stability conditions are not achieved, it appears difficult to find a window of stability. This is because (1) the negative energy spectrum of the precessional modes is broad band and there are likely to be modes of excitation to match those of the background plasma, and (2) there is enough positive energy bending and FLR sources so that the critical core beta limit is reduced over a broad spectrum if $\beta_W/2\kappa\Delta$ is not too small. As an example of the first type of destabilization, the excitation of surface Alfvén waves in a bumpy torus configuration has been discussed in this paper. Other examples exist in the literature[14,18,20,21,22] (these citations also describe the destabilization of the precessional mode by positive energy dissipation).

ACKNOWLEDGMENTS

We would like to thank Marshall Rosenbluth and Tom Antonsen for useful discussions. This work was supported by the U.S. Department of Energy under contract #DE FG05-80ET-53088

REFERENCES

1. J. Kesner, "Axisymmetric, Wall-Stabilized Tandem Mirrors," Report PFC/JA-84-29, Massachusetts Institute of Technology, Plasma Fusion Center (to be published, Nuclear Fusion), August (1984).

2. R.A. Dandl, H.O. Eason, P.H. Edmonds, A.C. England, G.E. Guest, C.L. Hedrick, J.T. Hogan, and J.C. Spott, Proceedings of Plasma Physics and Controlled Fusion (IAEA, Vienna, 1971) Vol. II, p. 607.

3. N.A. Krall, Phys. Fluids 9, 820 (1966).

4. R.R. Dominguez and H.L. Berk, Phys. Fluids 21, 827 (1978).

5. J.W. Van Dam and Y.C. Lee in "Proceedings of the EBT Ring Workshop (Oak Ridge National Laboratory, Oak Ridge, 1979) Conference 791228, p. 471.

6. D.B. Nelson, Phys. Fluids 23, 1850 (1980).

7. H.L. Berk, J.W. Van Dam, M.N. Rosenbluth, and D.A. Spong, Phys. Fluids 26, 201 (1983).

8. H.L. Berk, C.Z. Cheng, M.N. Rosenbluth, and J.W. Van Dam, Phys. Fluids 26, 2642 (1983).

9. K.T. Tsang, X.S. Lee, and P.J. Catto, Phys. Fluids 26, 3079 (1983).

10. T.M. Antonsen, Jr., Y.C. Lee, H.L. Berk, M.N. Rosenbluth, and J.W. Van Dam, Phys. Fluids 26, 3580 (1983).

11. H.L. Berk, J.W. Van Dam, and D.A. Spong, Phys. Fluids 26, 606 (1983).

12. H. Abe, T.M. Antonsen, JR., Y.C. Lee, et al., Proceedings of Plasma Physics and Controlled Fusion (IAEA, Vienna, 1983) Vol. III, p. 427.

13. H.L. Berk, M.N. Rosenbluth, H.V. Wong, and T.M. Antonsen, Phys. Fluids 27, ??? (1984).

14. H.L. Berk and T.B. Kaiser, Phys. Fluids 28, ??? (1985).

15. T.B. Kaiser and L.D. Pearlstein, Lawrence Livermore Laboratory Report UCBL-91320 (July 1984) (to be published, Phys. Fluids).

16. H.L. Berk and J.H. Hammer, Bulletin of the American Physical Society 29, 1417 (9Q7) (1984).

17. T.M. Antonsen, Jr. and Y.C. Lee, Phys. Fluids 25, 132 (1982).

18. D.E. Baldwin and H.L. Berk, Phys. Fluids 26, 3595 (1984).

19. J.W. Van Dam, M.N. Rosenbluth, and Y.C. Lee, Phys. Fluids 24, 1349 (1982).

20. T.M. Antonsen, Jr., and Y.C. Lee, "Proceedings of Hot Electron Workshop" (Oak Ridge National Laboratory, Oak Ridge, 1981) p. 191.

21. J.M. Greene and B. Coppi, Phys. Fluids 8, 1745 (1965).

22. H.P. Furth, Phys. Fluids 8, 2020 (1965).

FIGURE CAPTIONS

Fig. 1. Disc-like pressure profile of the hot component.

Fig. 2. Ring-like pressure profile of the hot component.

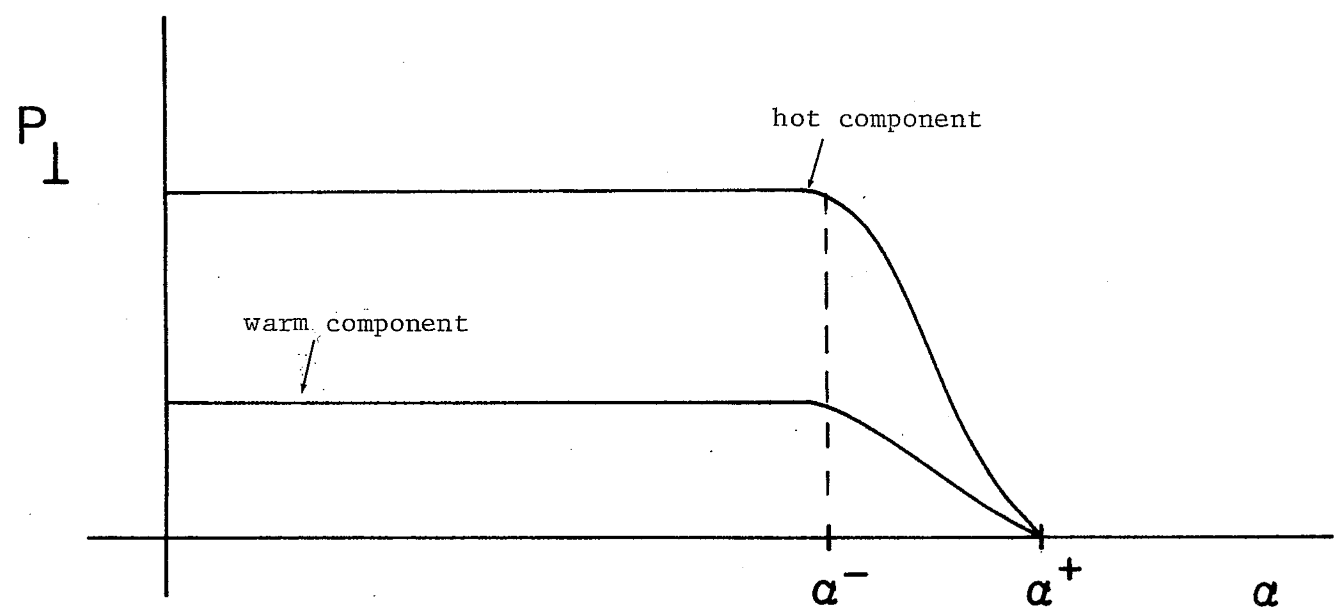
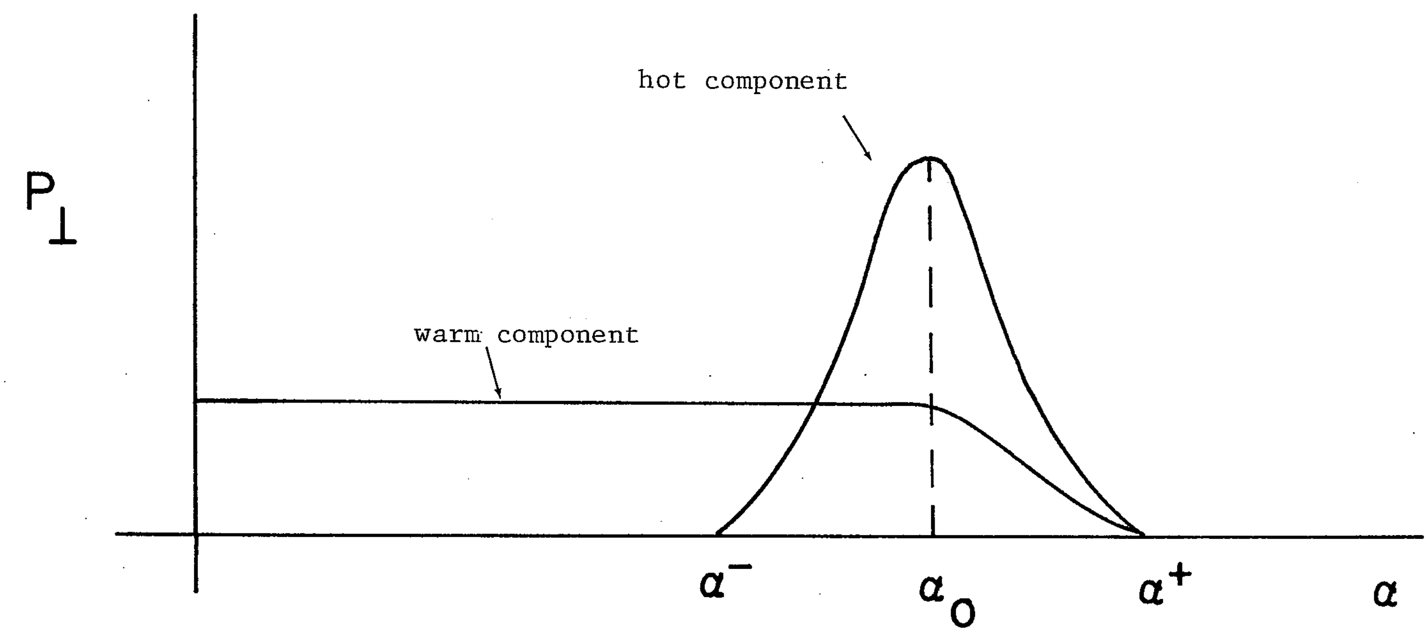


Fig. 1



$P_\perp$
hot component
warm component
α^-
α_0
α^+
α

Fig. 2