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STABILIZATION OF AN AXISYMMETRIC TANDEM MIRROR CELL BY A HOT PLASMA COMPONENT

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Abstract:

We use the hot plasma component formalism for a symmetric tandem mirror cell to relate the precessional mode, that is well known from Astron experiments, to more standard MHD modes. The $\ell=1$ mode can be stabilized by conducting walls, while higher ℓ modes can be stabilized by finite Larmor radius. Such a configuration is free of remnant dissipative instabilities because when stability is achieved, all perturbations are positive energy excitations.

I. Introduction

In tandem mirrors it would be advantageous to operate with azimuthally symmetric plugs.[1] However such a system is normally highly unstable to MHD instability.[2] One way to avoid instability is the use of a high energy component in the plasma, as in the EBT[3], to establish a noninteracting in-situ current that would produce favorable MHD properties for the remaining plasma background.[4] However, stability theory has indicated limitations of this method. These limitations include a low stability limit for the beta of the plasma background[5] and the existence of negative energy waves at the curvature drift frequency, that will be destabilized by the excitation of positive energy waves (such as Alfvén waves[6]), or with the dissipative mechanisms of the background plasma and walls.[7,8]

The mode at the curvature drift frequency is quite analogous to the near rigid precessional mode of Astron[9], where it was found that a "forward" precessing mode is a negative energy mode, that is readily destabilized by dissipation or excitation of positive energy waves of the background system.[10] However it was also found that image currents generated in nearby walls could change the sign of the precession, and convert the mode to a stable positive energy mode.[11,12]

In this work we show that a similar stabilization mechanism exists for the $\ell=1$ mode of the hot plasma component system when a disk-like radial pressure profile is considered. The stability mechanism is closely related to conventional MHD stability when a bounding conductor is present. The MHD stability when the pressure is isotropic, has been examined by Haas and Wesson[13] for a theta pinch and D'Ippolito et al.[14] for a tandem mirror. The effect has been overlooked in a recent study of tandem mirror stability of the $\ell=1$ mode[15]. We find that only the external curvature, but not the self-induced curvature of a finite beta plasma, drives $\ell=1$ instability, an observation consistent with the equations found in references (13) and (14). With a conducting shell present, stability is possible, with the stabilization term proportional to the product of the equilibrium's self-induced curvature and the ratio of the magnetic flux enclosed by the plasma and the external conductor. For higher ℓ, the image current mechanism is not effective for stabilization, but one might expect that the finite Larmor radius effects stabilize the shorter wavelength modes by converting all perturbations to positive energy excitations. This is a stronger stability mechanism than conventional FLR stabilization[16], but more stringent requirements are needed to achieve it. Such robust stabilization is already indicated in previously calculated finite Larmor radius calculations in the eikonal approximation.[17] Thus, stability for MHD-like excitations is in principle achievable for a hot plasma component of a few Larmor radii, in a symmetric mirror plug. Unfortunately, this stabilization mechanism for $\ell=1$ is not as applicable to a hot plasma ring configuration, as in EBT, since the stabilizing wall image current is considerably reduced from that of a disc shaped pressure profile.

II. Analysis

To analyze our system we consider a mirror equilibrium in the long thin approximation, where $1 \gg \varepsilon \sim \kappa r \sim r^2/L^2$, with κ the curvature, r the plasma radius, L the axial length of the mirror cells. Further, to obtain stabilization we invoke a subsidiary ordering $\varepsilon \ll \varepsilon_1 \sim \beta_h \sim L^2_h/L^2 \ll 1$ where β_h and L_h are the beta and axial length of the hot plasma. The subscript h will refer to the hot plasma component, and the subscript c to the background plasma core. The self-induced curvature of the equilibrium is competitive with the vacuum field curvature when $\beta_h \sim L_h^2/L^2$.

The equilibrium field is given by $\underset{\sim}{B} = \underset{\sim}{\nabla}\psi \times \underset{\sim}{\nabla}\vartheta$ where ψ is the magnetic flux and ϑ the azimuthal angle about the symmetry axis. The equilibrium is determined from the long-thin equilibrium condition, given by

$$B_v^2(s) = B^2(s,\psi) + 2P_\perp(s,\psi) \tag{1}$$

where $B_v(s)$ is the external vacuum field and s the distance along a field line. We treat the core beta, $\beta_c \sim \kappa r$. Hence β_c is neglected in the following equilibrium calculation. The magnetic flux is given by

$$\psi = \int_0^r Brdr = \int_0^r [B_v^2(s) - 2P_{\perp h}(s,\psi)]^{1/2} rdr$$

$$\doteqdot \frac{B_v(s)r^2}{2} - \int_0^r \frac{P_{\perp h}(s,B_v r^2/2)}{B_v(s)} rdr + \vartheta(\varepsilon_1^2)$$

For simplicity, we take the pressure profile at the midplane as flat to a radius r_0, where the contained magnetic flux is ψ_p^-, and then steeply dropping to zero, with the gradient distance Δ satisfying $\Delta/r << 1$. Just outside the hot region the contained magnetic flux is $\psi_p^+ \doteqdot \psi_p^-$. For this model, the radius of the plasma edge is $r(s) = r_0[B_0/B_v(s)]^{1/2}\{1 + [P_{\perp h}(s) - P_{\perp h}(0)]/2B_v^2(s)\}$ with $B_0 = B_v(s{=}0)$. The curvature, $\kappa \equiv \hat{\underset{\sim}{\psi}}\cdot\underset{\sim}{b}\cdot\underset{\sim}{\nabla}\underset{\sim}{b} \doteqdot d^2r/ds^2$ is then

$$\kappa = r_0 \frac{d^2}{ds^2}\left(\frac{B_0}{B_v(s)}\right)^{1/2} + \frac{d^2}{2ds^2}\left(\frac{B_0^{1/2}P_{\perp h}(s)}{B_v^{5/2}(s)}\right). \tag{2}$$

The first term will be referred to as the external curvature κ_{ext} set up by external currents and the second term is the self curvature κ_s that is induced by the bowing out of a finite beta equilibrium.

To provide $\ell=1$ stability we invoke a conducting shell, with skin time long compared to instability time scales, placed at $\psi = \psi_w(s)$. Between the plasma boundary ψ_p and $\psi_w(s)$ we assume a vacuum. We will consider in detail two limiting cases: the skin time in the conductor short or long compared to the plasma formation time. When the skin time is sufficiently short, the external curvature κ_{ext} is due only to the curvature κ_{extV} produced by distant windings. As complete penetration of the displaced equilibrium magnetic flux through the shell occurs, there is no equilibrium current flowing in the conductor. This is the case of interest when steady-state confinement is achieved. With a long skin time, currents are induced in the bounding conductor which alter the vacuum magnetic field between the plasma and the conductor. This can be the case during plasma formation. Then by demanding that at the conducting wall the magnetic flux of the finite beta equilibrium is the same as the magnetic flux with vacuum fields, we have

$$B_V(s)\left[r_w^2(s) - r_p^2(s)\right] + \left[B_V^2(s) - 2P_{\perp h}(s)\right]^{1/2} r_p^2(s) = B_{V0}(s) r_w^2 ,$$

where $B_V(s)$ is the external magnetic field between the plasma and the walls at finite beta, $B_{V0}(s)$ is the vacuum magnetic field produced by distant windings, $r_w(s)$ is the radius of the conductor, and $r_p(s)$ the radius of the plasma. $B_V(s)$ is then given by

$$B_V(s) = B_{V0}(s) + \frac{\psi_p}{\psi_w(s)} \frac{P_{\perp h}(s)}{B_{V0}(s)} + \vartheta(\beta^2),$$

and from Eq. (2), it then follows that

$$\kappa_{\text{ext}} = \kappa_{\text{extV}} - \kappa_s \frac{\psi_p}{\psi_w} + \vartheta(\beta^2)$$

We take the perturbed fields in the plasma to be derived from a potential $\underset{\sim}{A} = \underset{\sim}{\xi} \times \underset{\sim}{B} = (\phi \underset{\sim}{\nabla} \vartheta - \lambda \underset{\sim}{\nabla} \psi)\exp(i\ell\vartheta)$ where ϕ and λ are two scalars depending on ψ and s. In the plasma $E_{\parallel} = 0$ and it is convenient to introduce the Lagrangian perturbation of the magnetic field, Q_L,

$$\frac{Q_L}{B} = B \frac{\partial}{\partial \psi} (\phi/B) + i\ell\lambda.$$

(We have dropped small terms proportional to κr). The situation of interest to us is a plasma composed of a warm component whose magnetic drift frequencies $|\omega_d|$ are small compared to the mode frequencies $|\omega|$ we wish to consider and a hot component for which $|\omega_d| >> |\omega|$. In this situation, the quadratic energy change δW induced by field perturbations has been given by several authors(18,19). In the limit $\varepsilon \sim \kappa r \sim \frac{r^2}{L^2} \sim \frac{P_c}{P_h}$, where κ is the curvature, r the plasma radius, P_c the core pressure and P_h the hot component pressure, the quadratic form for δW to $\vartheta(\varepsilon^2)$ is:

$$2\ell^2\delta W = \int_{-\infty}^{+\infty} ds \int_0^{\psi_p} d\psi\, W_{plasma}(\phi, Q_L) + \int_{-\infty}^{+\infty} ds \int_{\psi_p}^{\psi_w(s)} d\psi\, \ell^2 \frac{\underset{\sim}{B}_1 \cdot \underset{\sim}{B}_1^*}{B}$$

$$\equiv \int_{-\infty}^{+\infty} ds \int_0^{\psi_p} d\psi\, \Big[\sigma B r^2 \Big(\frac{\partial}{\partial s}\Big(B\frac{\partial}{\partial\psi}\phi/B - \frac{Q_L}{B}\Big)\Big)^2 + \frac{\sigma\ell^2}{r^2 B}\Big(\frac{\partial\phi}{\partial s}\Big)^2 + \frac{\tau Q_L^2}{B}\big(1 + \vartheta(\beta_c)\big)$$

$$- \omega^2\rho r^2\Big(\Big(B\frac{\partial}{\partial\psi}\phi/B - \frac{Q_L}{B}\Big)^2 + \frac{\ell^2}{B^2 r^4}\phi^2\Big) - \frac{\ell^2\kappa}{B^2 r}\phi^2\frac{\partial}{\partial\psi}\big(P_{\perp h} + P_{\parallel h}\big)$$

$$- \frac{1}{B}\int_h d^3v\, \frac{\partial F}{\partial\varepsilon}\overline{K}^2\Big(\frac{\omega^*}{\overline{\omega}_{ph}}\Big(1 - \frac{\omega - \overline{\omega}_d + \overline{\omega}_{ph}}{\overline{\omega}_{ph}}\Big) - \frac{\omega}{\overline{\omega}_{ph}} + \vartheta\Big(\frac{\varepsilon^2}{\beta_h^2}\Big)\Big)\Big]$$

$$+ \int_{-\infty}^{+\infty} ds \int_{\psi_p}^{\psi_w(s)} \frac{d\psi}{B}\,\ell^2\, \underset{\sim}{B}_1 \cdot \underset{\sim}{B}_1^* = 0 \qquad (3)$$

where $\underset{\sim}{B}_1$ is the perturbed magnetic field in the vacuum, $\tau = 1 + \frac{1}{B}\frac{\partial P_{\perp h}}{\partial B}\Big|_\psi$,

$$\omega_d = \frac{\ell\mu}{q}\frac{\partial B}{\partial\psi} + \frac{\ell\kappa v_\parallel^2}{qrB}, \quad \omega^* = \frac{-\ell\frac{\partial F}{\partial\psi}}{q\frac{\partial F}{\partial\varepsilon}}, \quad \omega_{ph} = -\frac{\ell\mu}{qB}\frac{\partial P_{\perp h}}{\partial\psi},$$

$$K = \mu Q_L + \underset{\sim}{\kappa}\cdot\underset{\sim}{\xi}\Big(\frac{\mu B\sigma}{\tau} + mv_\parallel^2\Big), \quad \sigma = 1 + \frac{(P_{\perp h} - P_{\parallel h})}{B^2}, \quad \beta_c = \frac{2P_c}{B^2},$$

$\overline{\alpha}$ refers to the axial bounce average of α, q is the charge, h refers to the hot component, and c to the background core component. In Eq. (3) the terms proportional to σ are field line bending terms, the term proportional to τ is the magnetic compressibility of MHD theory, the terms proportional to ω^2 are inertia terms, the term proportional to κ, the curvature, is the instability drive, and the terms involving $\overline{K}$ are the kinetic terms that alter the quadratic form from conventional MHD theory.

We divide the plasma into two regions. In region I where $0 < \psi < \psi_p^-$, the pressure gradients are assumed negligible and flat pressure profiles in $P_{\perp h0}$, $P_{\perp c0}$, $P_{\parallel h0}$, and $P_{\parallel c0}$ are present. In region II where $\psi_p^- < \psi < \psi_p^+$, there is a steep pressure profile where we assume $\tilde{\beta}_h \equiv \frac{r}{\kappa B} \frac{\partial P_\perp}{\partial \psi} >> 1$. When the pressure goes to zero, we will take the mass density to be zero as well. We denote the region between the plasma interface and the vacuum as region III, and the magnetic field is determined by the vacuum magnetic field equation together with the boundary condition $\underset{\sim}{B}_1 \cdot \hat{\underset{\sim}{\psi}} = 0$ at $\psi = \psi_w(s)$ and the continuity of the normal component of the perturbed magnetic field at the vacuum-plasma interface.

When we construct the Euler-Lagrange equations for the field amplitudes in region I, we find $\frac{Q_L}{\phi B} \sim \frac{r^2}{L^2} \sim \vartheta(\varepsilon)$, and therefore Q_L can be neglected. Then the Euler-Lagrange equation in region I is found to be

$$\frac{\partial}{\partial \psi} \frac{\partial}{\partial s} \sigma B r^2 \frac{\partial}{\partial s} \frac{\partial \phi}{\partial \psi} - \frac{\partial}{\partial s} \sigma \frac{\ell^2}{B r^2} \frac{\partial \phi}{\partial s}$$

$$- \frac{\partial}{\partial \psi} \rho \frac{\omega^2 r^2}{B} \frac{\partial \phi}{\partial \psi} + \ell^2 \frac{\rho \omega^2}{B^3 r^2} \phi = 0 + \vartheta(\varepsilon) \tag{4}$$

where we have neglected $r^2 \frac{\partial B}{\partial \psi} = \vartheta(\varepsilon)$. Note that in this region Br^2 is a constant along a field line. The solution for ϕ is

$$\phi = f^-(s)\left(\frac{\psi}{\psi_p^-}\right)^{\ell/2} + \vartheta(\beta_h \varepsilon^{1/2}), \; \psi < \psi_p^- \tag{5}$$

In region III, the solution for $\underset{\sim}{B}_1$ in the limit $\frac{r}{B}\frac{\partial B}{\partial s} \sim \vartheta(\varepsilon^{1/2})$ and with the boundary condition $\hat{\underset{\sim}{\psi}}\cdot\underset{\sim}{B}_1 = 0$ at $\psi = \psi_w$ is

$$\underset{\sim}{B}_1 = \frac{\partial f^+}{\partial s} \frac{e^{i\ell\vartheta}}{\left[\left(\frac{r_p(s)}{r_w(s)}\right)^{\ell} - \left(\frac{r_w(s)}{r_p(s)}\right)^{\ell}\right]}$$

$$\cdot \left\{\left[\left(\frac{r}{r_w(s)}\right)^{\ell} - \left(\frac{r_w(s)}{r}\right)^{\ell}\right] \frac{\underset{\sim}{\nabla}\psi}{Br^2} + i\left[\left(\frac{r}{r_w(s)}\right)^{\ell} + \left(\frac{r_w(s)}{r}\right)^{\ell}\right]\underset{\sim}{\nabla}\vartheta\right) \qquad (6)$$

where $r_p(s)$ is the radius of the plasma vacuum interface. It can be verified that Eq. (6) satisfies $\underset{\sim}{\nabla}\cdot\underset{\sim}{B} = \underset{\sim}{\nabla}\times\underset{\sim}{B} = 0 + \vartheta(\varepsilon^{1/2})$, and we demand that there be continuity with the normal component of the region II field $\frac{1}{r}\frac{\partial\phi}{\partial s}$. If we substitute Eq. (5) and (6) into Eq. (3) and explicitly integrate in regions I and III, we find

$$2\ell^2\delta W = \int_{II} ds d\psi\, W_{plasma}(\phi, Q_L)$$

$$+ \int ds\left\{|\ell| Z(s)\left(\frac{\partial f^+}{\partial s}\right)^2 + |\ell|\left(\frac{\partial f^-}{\partial s}\right)^2\right\}$$

$$- \int ds\, \rho(\psi_p^-) \frac{\omega^2 r_p^2 |\ell|}{(B_v^2 - 2P_h)^{1/2}} f^-(s) = 0 \qquad (7)$$

where

$$Z(s) = \frac{1 + \left(\frac{r_p}{r_w}\right)^{2\ell}}{1 - \left(\frac{r_p}{r_w}\right)^{2\ell}} .$$

The dominant terms in δW_{II} in our limit of $\beta < 1$ arise from the perturbed magnetic fields. The largest terms involving Q_L, to zero order in ε, are:

$$\int d\psi ds \left\{\tau Q_L^2 - \int_h d^3v \frac{\partial F}{\partial \varepsilon} \frac{\omega^*}{\bar{\omega}_{ph}} \mu^2 \bar{Q}_L^2\right\}/B . \tag{8}$$

From the equilibrium conditions, this is to $\vartheta(\varepsilon)$, positive definite and to this order can be eliminated from δW_{II} if Q_L is chosen to be of the form[18,19]:

$$Q_L = C_0(\psi) \frac{\ell}{B} \frac{\partial P_{\perp h}}{\partial \psi} \tag{9}$$

where the coefficient C_0 is a function of the flux variable ψ only.

The coefficient C_0 which extremizes δW_{II} may then be obtained by substituting this form of Q_L in δW_{II} and minimizing with respect to C_0, leading to

$$C_0 = -\frac{\ell}{\Gamma} \int ds \Big[\frac{1}{\ell^2 B^2} \frac{\partial P_{\perp h}}{\partial \psi} \Big(\frac{\omega^2 \rho r^2}{B} + \frac{\partial}{\partial s} \sigma r^2 B \frac{\partial}{\partial s}\Big) B \frac{\partial}{\partial \psi} (\phi/B)$$

$$+ \phi \frac{\kappa}{rB^2} \frac{\partial}{\partial \psi} (P_{\perp h} + P_{\parallel h})\Big] \tag{10}$$

where

$$\Gamma = \int ds \sigma B r^2 \Big(\frac{\partial}{\partial s} \frac{1}{B^2} \frac{\partial P_{\perp h}}{\partial \psi}\Big)^2 + \Lambda_0 - \int ds \frac{\omega^2 \rho r^2}{B} \Big(\frac{1}{B^2} \frac{\partial P_{\perp h}}{\partial \psi}\Big)^2 \tag{11}$$

$$\Lambda_0 \equiv -\ \ell^2 \int ds\ \frac{\kappa}{rB^2} \frac{\partial}{\partial\psi} (P_{\perp h} + P_{\parallel h}) \{1 - \frac{\omega}{\tilde{\omega}_{\kappa h}} - \tilde{\beta}_c\} \tag{12}$$

$$\frac{\omega}{\tilde{\omega}_{\kappa h}} \equiv \frac{\int ds\ \omega \frac{\partial}{\partial\psi} (\rho_h/B)}{\ell \int ds\ \frac{\kappa}{rB^2} \frac{\partial P_h}{\partial\psi}}\ ,\ \tilde{\beta}_c \equiv \frac{\int \frac{ds}{B} \frac{\partial P_{\perp h}}{\partial\psi} \frac{\partial}{\partial\psi} (P_{\perp c}/B^2)}{\int ds\ \frac{\kappa}{rB^2} \frac{\partial P_h}{\partial\psi}}\ ,\quad P_h = P_{\perp h} + P_{\parallel h}$$

Note that $\tilde{\omega}_{\kappa h}$ is proportional to the distribution averaged curvature drift frequency although the appearance of this term in δW really arises from a charge separation between hot and cold plasmas. $\tilde{\beta}_c$ is the ratio of background density to the Lee-Van Dam density required for decoupling.

For a thin layer, $\kappa_s \sim \kappa$ and $P_{\perp h} >> P_{\perp c}$, the dominant term in Γ is the first one.

Using this result in Eq. (7), we can now write δW_{II} in terms of ϕ:

$$\begin{aligned} 2\ell^2 \delta W_{II} &= 2\ell^2 \int_{II} d\psi ds\ W(\phi) = \int_{II} d\psi ds \Bigg[\sigma B r^2 [\frac{\partial}{\partial s} B \frac{\partial}{\partial\psi} (\phi/B)]^2 + \frac{\sigma\ell^2}{Br^2} (\frac{\partial\phi}{\partial s})^2 \\ &- \frac{\omega^2 \rho r^2}{B} \{[B \frac{\partial}{\partial\psi} (\phi/B)]^2 + \frac{\ell^2\phi^2}{B^2 r^4}\} - \frac{\ell^2\kappa}{B^2 r} \phi^2 \frac{\partial}{\partial\psi} (P_{\perp h} + P_{\parallel h}) \Bigg] \\ &- \int d\psi \frac{1}{\Gamma - \Lambda_0} (1 - \frac{\Lambda_0}{\Gamma}) \{\int ds \sigma B r^2 (\frac{\partial}{\partial s} \frac{1}{B^2} \frac{\partial P_{\perp h}}{\partial\psi}) [\frac{\partial}{\partial s} B \frac{\partial}{\partial\psi} (\phi/B)] \\ &- \int \frac{ds}{B^2} \frac{\partial P_{\perp h}}{\partial\psi} \omega^2 \rho r^2 \frac{\partial}{\partial\psi} (\phi/B) - \ell^2 \int ds \frac{\kappa}{rB^2} \phi \frac{\partial}{\partial\psi} (P_{\perp h} + P_{\parallel h})\}^2\ . \end{aligned} \tag{13}$$

Let us now minimize the variational expression W with respect to ϕ. The volume integral extends over the region where the hot component pressure gradient is finite. The thickness Δ of this region, hereafter called the layer, is considered to be small compared to the plasma radius r. We consider a formal ordering in which $\varepsilon << \frac{\ell\Delta}{r} \sim \varepsilon_1 << 1$ and $\frac{\omega^2\rho r^2}{|\ell|B^2} \sim \beta_h\kappa r$. Furthermore let $\frac{\beta_h r}{L_h^2\kappa} \sim 1$, where L_h is the axial length of the layer. Then

$$\frac{\Lambda_0}{\Gamma} \sim \vartheta(\varepsilon_1) \ .$$

The largest terms in δW_{II} are

$$\int_{II} d\psi \int ds\sigma Br^2\left[\frac{\partial}{\partial s} B \frac{\partial}{\partial\psi} (\phi/B)\right]^2$$

$$- \int_{II} d\psi \frac{\left\{\int ds\sigma Br^2\left(\frac{\partial}{\partial s} \frac{1}{B^2} \frac{\partial P_{\perp h}}{\partial\psi}\right)\left[\frac{\partial}{\partial s} B \frac{\partial}{\partial\psi} (\phi/B)\right]\right\}^2}{\int ds\sigma Br^2\left[\frac{\partial}{\partial s}\left(\frac{1}{B^2} \frac{\partial P_{\perp h}}{\partial\psi}\right)\right]^2}$$

representing the dominant magnetic bending energy.

From the Schwartz inequality this is positive definite and is equal to zero only if

$$B \frac{\partial}{\partial\psi} (\phi/B) = \frac{G(\psi)}{B^2} \frac{\partial P_{\perp h}}{\partial\psi} \tag{14}$$

where G is an arbitrary function of ψ.

This form of $B \frac{\partial}{\partial\psi}(\phi/B)$ minimizes δW_{II}. After substitution in δW_{II}, we obtain:

$$2\ell^2\delta W_{II} = \int_{II} d\psi[G^2\Lambda_0 + 2G\ell^2\int ds\,\frac{\kappa}{rB^2}\,\phi\,\frac{\partial}{\partial\psi}\,(P_{\perp h}+P_{\parallel h})$$

$$- \ell^2\int ds\,\frac{\kappa}{rB^2}\,\phi^2\,\frac{\partial}{\partial\psi}\,(P_{\perp h}+P_{\parallel h})]$$

$$+ \int_{II} d\psi ds\,\frac{\sigma\ell^2}{Br^2}\left(\frac{\partial\phi}{\partial s}\right)^2 . \tag{15}$$

Higher order terms in ε_1 have been neglected.

To proceed further, we note that integration of Eq. (14) and the use of Eqs. (5) and (6) yields:

$$\frac{\partial f^+}{\partial s} - \frac{\partial f^-}{\partial s} = \frac{\partial}{\partial s}\int_{\psi_p^-}^{\psi_p^+} d\psi\,\frac{(G-\phi)}{B^2}\,\frac{\partial P_{\perp h}}{\partial\psi} . \tag{16}$$

In the limit of $\beta_h \sim \varepsilon_1$, Eq. (14) and (16) imply that

$$\phi = \phi^{(0)} + O(\varepsilon_1)$$

$$f^{\pm} = \phi^{(0)} + O(\varepsilon_1)$$

where $\phi^{(0)}$ is a constant which we take to be unity. The first order terms depend on s and contribute only through the boundary terms. Now, using Eq. (16), we minimize the boundary terms in Eq. (7) with respect to $\frac{\partial f^+}{\partial s}$, and we obtain:

$$\int ds|\ell|\{Z(\frac{\partial f^+}{\partial s})^2 + (\frac{\partial f^-}{\partial s})^2\} = \int ds \frac{|\ell|Z}{(1+Z)}(\frac{\partial}{\partial s}\int_{\psi_p^-}^{\psi_p^+} d\psi \frac{(G-1)}{B^2}\frac{\partial P_{\perp h}}{\partial \psi})^2 .$$

Thus, the variational expression δW can be written in terms of G:

$$2\ell^2\delta W \to -\int_{II} d\psi ds \frac{\ell^2\kappa}{rB^2}\frac{\partial}{\partial\psi}(P_{\perp h}+P_{\parallel h})\{(G-1)^2 - G^2(\tilde{\beta}_c + \frac{\omega}{\tilde{\omega}_{\kappa h}})\}$$

$$+\int ds \frac{|\ell|Z}{(1+Z)}\{\frac{\partial}{\partial s}\int_{\psi_p^-}^{\psi_p^+} d\psi \frac{(G-1)}{B^2}\frac{\partial P_{\perp h}}{\partial\psi}\}^2$$

$$-\int ds\, \omega^2 \frac{\rho|\ell|}{B^2}\bigg|_{\psi_p^-} = 0 . \qquad (17)$$

If $\tilde{\beta}_c + \frac{\omega}{\tilde{\omega}_{\kappa h}} < 1$, the term proportional to G^2 is negative (note that $\int ds \frac{\ell^2\kappa}{rB^2}\frac{\partial}{\partial\psi}(P_{\perp h}+P_{\parallel h}) > 0$), and the equilibrium is unstable to localized perturbations where this term dominates W (see the unstable compressional mode of Reference (17)). Such localized modes, however, will be readily stabilized by finite Larmor radius effects.(17)

For nonlocalized modes we vary Eq. (17) with respect to G to extremize δW and we obtain for G the following equation:

$$[G - 1 - G(\tilde{\beta}_c + \frac{\omega}{\tilde{\omega}_{\kappa h}})] \int \frac{ds\ell^2}{B^2 r}\kappa\frac{\partial}{\partial\psi}(P_{\perp h}+P_{\parallel h})$$

$$+\frac{|\ell|}{2}\int\frac{ds}{B^2}\frac{\partial P_{\perp h}}{\partial\psi}\frac{\partial}{\partial s}[1 + (\psi_p/\psi_w)^{|\ell|}]\frac{\partial}{\partial s}\int_{\psi_p^-}^{\psi_p^+} d\psi \frac{(G-1)}{B^2}\frac{\partial P_{\perp h}}{\partial\psi} = 0 . \qquad (18)$$

If parameters are smooth we expect the solution to (18) to depend only on the overall characteristics of the profile. To obtain an explicit solution we assume $P_{\perp h}(\psi,s)$ is separable in ψ and s, that is

$$\frac{\partial}{\partial\psi} P_{\perp h}(\psi,s) = h(\psi)g(s) \quad ,$$

and we take $\frac{\partial P_c}{\partial\psi}$ and $\frac{\partial\rho_h}{\partial\psi}/\frac{\partial P_{\perp h}}{\partial\psi}$ as constant. Then G is found to be a constant given by

$$G = \frac{1 - \frac{1}{|\ell|} <\frac{\kappa_s}{\kappa} [1 + (\psi_p/\psi_w)^{|\ell|}]>}{1 - \tilde{\beta}_c - \frac{\omega}{\tilde{\omega}_{\kappa h}} - \frac{1}{|\ell|} <\frac{\kappa_s}{\kappa} [1 + (\psi_p/\psi_w)^{|\ell|}]>} \tag{19}$$

where

$$<\frac{\kappa_s}{\kappa} [1 + (\frac{\psi_p}{\psi_w})^{|\ell|}]> = \frac{1}{2} \frac{\int ds[\frac{\partial}{\partial s}(\frac{P_{\perp h}}{B^2})]^2 [1 + (\frac{\psi_p}{\psi_w})^{|\ell|}]}{\int ds \frac{\kappa}{B^2 r} (P_{\perp h}+P_{\parallel h})} \tag{20}$$

and $P_\perp$ and $P_\parallel$ are the values of the pressure in region I, the interior plasma.

Substituting for G in Eq. (17), we obtain the dispersion relation:

$$\omega^2 = \frac{\gamma^2_{MHD}(\tilde{\beta}_c + \frac{\omega}{\tilde{\omega}_{\kappa h}})[1 - \frac{1}{|\ell|} <\frac{\kappa_s}{\kappa} [1 + (\frac{\psi_p}{\psi_w})^{|\ell|}]>]}{1 - \tilde{\beta}_c - \frac{\omega}{\tilde{\omega}_{\kappa h}} - \frac{1}{|\ell|} <\frac{\kappa_s}{\kappa} [1 + (\frac{\psi_p}{\psi_w})^{|\ell|}]>} + \vartheta(\varepsilon) \tag{21}$$

where

$$\gamma^2_{MHD} = \frac{|\ell| \int ds \frac{\kappa}{B^2 r} (P_{\perp h}+P_{\parallel h})}{\int ds \ \rho/B^2} \tag{22}$$

and all quantities are evaluated in the interior region (the denominator in Eq. (22) is the integral over the entire flux tube, while all other integrals are integrated only over the region where there are hot particles).

III. Discussion

The dispersion relation, given by Eq. (21), is much like the previously derived form for layer modes[20], except that there is an additional self-curvature term. This term gives rise to the possibility of only positive energy perturbations, and hence complete stability, if

$$\tilde{\beta}_{cr}(\ell) \equiv 1 - \frac{1}{|\ell|} \left\langle \frac{\kappa_s}{\kappa} \left[1 + (\psi_p/\psi_w)^{|\ell|}\right]\right\rangle < 0 \ . \tag{23}$$

In previous analyses only the case $\tilde{\beta}_{cr}(\ell) > 0$ was considered and then MHD instability always occurs if $\tilde{\beta}_c > \tilde{\beta}_{cr}(\ell)$. Even when $\tilde{\beta}_c < \tilde{\beta}_{cr}$ (with $\tilde{\beta}_{cr} > 0$), instability can arise from Eq. (21) if the hot particles are not sufficiently energetic, or if there is positive dissipation[8] or there is excitation of additional positive energy waves (e.g. Alfvén waves) of the background plasma near the frequency $\omega \sim \tilde{\omega}_{\kappa h}$ (which is the negative energy mode of Eq. (21)).[21] For $|\ell| = 1$, and the conducting wall at infinity, $\tilde{\beta}_{cr}(\ell) > 0$ since $-\kappa_s < -\kappa$, but it is only external curvature (not the total curvature) which drives instability. If the walls are close enough and the hot beta sufficiently large so that

$$\tilde{\beta}_{cr}(\ell) \equiv 1 - \left\langle \frac{\kappa_s}{\kappa} \left[1 + \left(\frac{\psi_p}{\psi_w}\right)\right]\right\rangle < 0 \ . \tag{24}$$

MHD stability is achieved for $\ell=1$ and the excitations are positive energy.

When the skin time of the bounding conductor is sufficiently short so that $\kappa_{ext} = \kappa_{extV}$, the stability condition can be roughly written as

$$-\frac{\psi_p}{2\psi_w}\frac{r_p}{B^2}\frac{d^2P_{\perp h}}{ds^2} > -\kappa_{extV} \ . \tag{25}$$

In the opposite limit of an ideal conductor on the equilibrium time scale, we have found that

$$\kappa_{ext} = \kappa_{extV} - \frac{\psi_p r_p}{2\psi_w B^2}\frac{d^2P_{\perp h}}{ds^2} \ ,$$

so that the stability condition roughly becomes

$$-\frac{\psi_p}{\psi_w}\frac{r_p}{B^2}\frac{d^2P_{\perp h}}{ds^2} = -\kappa_{extV} \ . \tag{26}$$

which is more optimistic than Eq. (25), by a factor of 2 and can be useful in establishing stability during plasma buildup. A similar factor of 2 arises in the stabilization of the Astron precessional mode.[10,11] However, now there is an equilibrium beta limit that arises. When

$$-\frac{\psi_p}{2\psi_w}\frac{r_p}{B^2}\frac{d^2P_{\perp h}}{ds^2} > -\kappa_{extV} \cdot$$

κ_{ext} changes sign, which implies that the mirror mode instability criterion, $1 + \frac{\partial P_\perp}{B\partial B} < 0$, is satisfied. This readily follows from the equilibrium condition,

$$P_{\perp h} + \frac{B^2}{2} = \frac{B^2{}_{ext}}{2}$$

and that $\frac{\partial B_{ext}}{\partial B} < 0$ if $\kappa_{ext} > 0$. Thus, the stability window with an ideal conducting shell is

$$-\kappa_{extV} > \frac{-r_p}{2B^2}\frac{d^2P_{\perp h}}{ds^2}\frac{\psi_p}{\psi_w} > -\frac{\kappa_{extV}}{2} . \tag{27}$$

At values of $\tilde{\beta}_c$ where the above dispersion relation predicts stability, there exists a precessional mode with real frequencey, approximately given by

$$\frac{\omega}{\tilde{\omega}_{\kappa h}} = 1 - \tilde{\beta}_c - \frac{1}{|\ell|} <\frac{\kappa_s}{\kappa} [1 + (\psi_p/\psi_w)^{|\ell|}]> . \tag{28}$$

When $\omega/\tilde{\omega}_{\kappa h} > 0$, this mode has negative energy and it is destabilized by dissipation or coupling to positive energy excitations of the plasma.[21] For $\omega/\tilde{\omega}_{\kappa h} < 0$, the precessional mode is positive energy and there are no remnant instabilities.[21] To illustrate this aspect we note that in Ref. 21 the generalization of Eq. (21) with an Alfvén surface wave in the central cell is found to be for $|\ell| = 1$,

$$\omega^2 - (Z+1)k_n^2V_A^2 = \frac{\gamma_{MHD}^2(\tilde{\beta}_c + \frac{\omega}{\tilde{\omega}_{\kappa h}})(1-g)}{(1 - \tilde{\beta}_c - g - \frac{\omega}{\omega_{\kappa h}})} \tag{29}$$

with V_A the Alfvén speed in the central cell, $k_n = \pi n/L_c$ with n an integer, L_c the central cell length and $g = 1 - <\frac{\kappa_s}{\kappa}\,[1 + (\frac{\psi_p}{\psi_w})]>$. If we look for a mode when parameters are such that the surface Alfvén wave and precessional mode are at the same frequency, i.e.

$$(Z+1)^{1/2}\, k_nV_A = (1-g-\tilde{\beta}_c)\omega_\kappa \equiv \omega_0 \ ,$$

we find that Eq. (29) yields with $\omega = \omega_0+\delta\omega$,

$$\delta\omega^2 = \frac{-g^2\gamma_{MHD}^2}{2(g-\tilde{\beta}_c)} \tag{30}$$

which is unstable if $g - \tilde{\beta}_w > 0$, and always stable if $g<0$.

For $|\ell| > 1$, Eq. (21) still indicates the characteristic instability structure even with walls close by. For larger ℓ numbers we can expect the finite Larmor radius of the hot component to cause a similar stabilization structure as in Eq. (21). Proper analysis of this problem has not yet been performed, but we can extrapolate results of eikonal theory(17) and the FLR investigation of the layer mode for the Z-pinch model(22) to conjecture the form of the results. The expected dispersion relation for the layer mode is

$$\omega^2 = \frac{\gamma^2_{MHD}[1 - \frac{1}{|\ell|} <\frac{\kappa_s}{\kappa} [1 + (\frac{\psi_p}{\psi_w})^{|\ell|}> - \frac{\beta_h k^2_{\perp eff} a^2_h}{2\kappa\Delta}](\frac{\omega}{\tilde{\omega}_{\kappa h}} + \tilde{\beta}_c)}{[1 - \frac{\omega}{\tilde{\omega}_{\kappa h}} - \tilde{\beta}_c - \frac{1}{|\ell|} <\frac{\kappa_s}{\kappa} [1 + (\frac{\psi_p}{\psi_w})^{|\ell|}> - \frac{\beta_h k^2_{\perp eff} a^2_h}{2\kappa\Delta}]} \tag{31}$$

where a_h is the Larmor radius of the hot component, $k_{\perp eff}$ the effective perpendicular wave number which for layer modes have the form $k^2_{\perp eff} = (\ell^2-1)/r^2_p$, and Δ the pressure gradient scale length of the hot plasma near $r = r_p$. When $\frac{\omega}{\tilde{\omega}_{\kappa h}} + \tilde{\beta}_c$ are the largest terms of the denominator, Eq. (31) agrees with results obtained from solving Newcomb's FLR equations[23] for our equilibrium.

Equation (31) indicates that if the hot plasma consists of only a few hot gyro-radii, a robust stabilization mechanism occurs for all MHD-like modes and precessional modes become stable positive energy waves. The remaining instability questions in the tandem would then be how to deal with ballooning and trapped particle modes associated with possible loss cone and anisotropy instabilities of the hot component plasma.

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