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A GENERALIZED KINETIC ENERGY PRINCIPLE

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Abstract:

Using three single-particle adiabatic invariants, we derive an energy principle which generalizes that for a usual guiding center plasma in order to describe the low-frequency stability of a plasma containing an energetic non-hydromagnetic component (such as the annular electrons in an Elmo bumpy torus device).

I. Introduction

The energy principle for an anisotropic, guiding center plasma was originally obtained by Kruskal and Oberman.[1] Its isotropic form was simultaneously derived by Rosenbluth and Rostoker[2] from a kinetic approach. This energy principle has been subsequently studied by Taylor and Hastie[3] and by Grad.[4]

The derivation presented by Kruskal and Oberman[1] was a thermodynamic argument. That is, they constructed a general macroscopic constant of the motion, namely, the entropy, and varied it so as to remove any variation in the microscopic distribution function. (A detailed account of this derivation has been provided by Guest and Hedrick.[5])

It has been pointed out,[2,6,7] however, that this energy principle is more simply equivalent to requiring that both the single particle magnetic moment and the longitudinal action be adiabatically conserved. In addition, the magnetic lines of force are assumed to be frozen into the plasma.

The assumption that the plasma displacement is the same as that for the magnetic field is invalid, however, for a rather energetic plasma component. For example, the plasma in an Elmo bumpy torus confinement device[8] contains poloidal rings of microwave-heated electrons, with temperatures in the 150-500 keV range, which precess about the toroidal core plasma. For this hot species, the normal hydromagnetic response is negligible, the $E \times B$ displacement being inversely proportional to the frequency which in this case is strongly Doppler shifted by the large precessional magnetic drift.

Here, we obtain a kinetic energy principle which is able to describe the low-frequency stability of a plasma containing such energetic particles. In place of the frozen-in constraint, for the energetic species we employ another condition on the particle behavior that allows us to relate it to a specified magnetic field perturbation. The condition is that the magnetic flux passing through the precessional drift orbit be adiabatically conserved, which is ensured if the hot particles complete many such drift orbits in the period of a typical low-frequency plasma fluctuation. In the Elmo bumpy torus EBT-I, for example, the ring electron drift frequency is more than comparable to the ion cyclotron frequency ($\cong$ 9 GHz at the annulus location). Study of the bumpy torus thus furnishes one of the rare applications of the so-called third adiabatic invariant.[9]

In the next section, we illustrate how to use adiabatic invariants by reproducing the guiding center energy principle with this method. Sec. III then derives the generalized energy principle for a nonhydromagnetic plasma. The last section discusses this new principle, with particular application to the bumpy torus device. Sufficient conditions for its plasma to be unstable with respect to the interchange mode and to a novel form of the mirror mode are presented.

II. Method of Invariants: Guiding Center Plasma

In order to illustrate the technique of using single-particle adiabatic invariants to obtain the appropriate energy principle, let us first consider an anisotropic guiding center plasma whose behavior is entirely hydromagnetic in the sense of being frozen-in with the magnetic field. This limit will allow us to reproduce the usual kinetic energy principle.[1,2] The modification introduced by the presence of non-hydromagnetic plasma species will thereafter be more easily explained.

For this guiding center plasma, then, there are two invariants: the magnetic moment $\mu = Mv_\perp^2/2B$, where M is the particle mass, $v_\perp$ its transverse speed, and B the field strength, and also the action invariant $J=M\oint d\ell v_\parallel$, where $v_\parallel$ is the parallel speed and ℓ the arc length along a field line. The integral defining J is taken between turning points for trapped particles or over one period of the axial motion for circulating particles. The existence of these two invariants is well known to correspond to having the gyration and bounce (or transit) periods be much shorter than the period of a characteristic fluctuation, which is certainly valid in a marginal stability analysis.

The potential energy of the plasma may be written as

$$W_p = \int d^3x \int \frac{d\varepsilon d\mu B}{v_\parallel} (\varepsilon F) = \int d\alpha d\beta d\mu dJ \; (\varepsilon F), \qquad (1)$$

where F is the gyro- and bounce-averaged guiding center distribution function and $\varepsilon=Mv_\parallel^2/2 + \mu B$ is the particle energy. The Clebsch coordinates α and β (which reduce to the flux and the azimuthal angle, respectively, for an axisymmetric system) define the magnetic field $\underset{\sim}{B}=\underset{\sim}{\nabla}\alpha \times \underset{\sim}{\nabla}\beta$. The velocity space integration in Eq. (1) is understood to include a summation over plasma species and one over the two directions of the parallel motion; the latter disappears for completely trapped populations.

The electrostatic potential, as well as finite Larmor radius and relativistic effects, will be neglected in this discussion.

We wish to calculate the second-order variation of the plasma energy for a given displacement ξ of the magnetic field. For this purpose, we adopt a purely Lagrangian point of view, moving with the local velocity of the magnetic lines. In this picture, the number of particles in an infinitesmal phase space volume, $F\, d\alpha d\beta d\mu dJ$, is conserved by virtue of the Liouville theorem for (α,β,μ,J) space.[9] Hence, to obtain the variation of W_p, we need calculate only the change in the single-particle energy ε . This is accomplished by invoking the invariance of μ and J.

If, during the displacement of the field, the energy ε were held fixed, the action integral J would change by an amount

$$J_1 = M \oint \left[\left(\frac{d\ell}{B}\right)_1 Bv_{\|} + \frac{d\ell}{B}\left(Bv_{\|}\right)_1 \right] = \oint \frac{d\ell}{v_{\|}}\left(Mv_{\|}^2 q - \mu B_1\right) \qquad (2)$$

to first order (indicated by the subscript) in the displacement. In Eq. (2), $B_1 = -B(\nabla\cdot\xi - q)$, with $q = \hat{b}\hat{b}:\nabla\xi$ and $\hat{b} = B/B$, is the first-order Lagrangian change in the magnetic field strength. To calculate the change in the element of arc length, we have used the information that the volumetric quantity $B^{-1}d\ell$ has the same transformation properties under a displacement as does the Jacobian relating x to the new position $x + \xi$, namely, $B^{-1}d\ell \rightarrow B^{-1}d\ell\{1+\nabla\cdot\xi+\frac{1}{2}[(\nabla\cdot\xi)^2-(\nabla\xi):(\nabla\xi)]+\ldots\}$.

However, since J is invariant, there must be a change in the particle energy, ε_1, to offset the change J_1. The total first-order change of J should therefore be given by

$$J^{(1)} = 0 = J_1 + \varepsilon_1 \left(\frac{\partial J}{\partial \varepsilon}\right)_{\alpha,\beta,\mu} \tag{3}$$

from which ε_1 is immediately obtained. Defining the bounce average $\langle \cdots \rangle = (\frac{\partial J}{\partial \varepsilon})^{-1} \oint \frac{d\ell}{v_\|}(\ldots)$, where $\partial J/\partial \varepsilon$ is the bounce period, we find that

$$\varepsilon_1 = \langle H \rangle . \tag{4}$$

The quantity $H = -Mv_\|^2 q + \mu B_1$ may be identified as the gyro-averaged increment in the particle kinetic energy. Inserting Eq. (4) into Eq. (1) yields the plasma energy variation:

$$W_p^{(1)} = \int d\alpha d\beta d\mu dJ\ F \left\langle H \right\rangle = \int d^3x\ (-p_\| q + p_\perp B_1/B) \tag{5}$$

with $p_{\|,\perp}$ the longitudinal and transverse pressure components.

When the change in the magnetic field energy $W_m = \frac{1}{2}\int d^3x\ B^2$, namely,

$$W_m^{(1)} = \int d^3x\ B(B_1 - \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} B) \tag{6}$$

where the displacement is presumed to vanish at the boundary of integration, is now added to the plasma energy variation, the change in the total potential energy is found to vanish to first order as long as the equilibrium condition

$$\underset{\sim}{B} \times (\underset{\sim}{\nabla} \times \underset{\sim}{B}) + \underset{\sim}{\nabla} \cdot [(p_\| - p_\perp)\hat{b}\hat{b} + p_\perp \underset{\approx}{I}] = 0 \tag{7}$$

is satisfied.

Continuing to next order, we have the invariance condition

$$J^{(2)} = 0 = J_2 + \varepsilon_2\left(\frac{\partial J}{\partial \varepsilon}\right) + \frac{1}{2}\,\varepsilon_1{}^2\left(\frac{\partial^2 J}{\partial \varepsilon^2}\right) + \varepsilon_1\left(\frac{\partial J_1}{\partial \varepsilon}\right). \qquad (8)$$

Eq. (8) yields the energy,

$$\varepsilon_2 = -\left(\frac{\partial J}{\partial \varepsilon}\right)^{-1}\left[J_2 - \frac{\partial}{\partial \varepsilon}\left(\frac{1}{2}\,\varepsilon_1{}^2\,\frac{\partial J}{\partial \varepsilon}\right)\right] \qquad (9)$$

with the quantity J_2 given by

$$J_2 = \frac{1}{2}\left(\frac{\partial J}{\partial \varepsilon}\right)\Big\langle Mv_\parallel{}^2[(\hat{b}\cdot\nabla\xi)^2 - q^2] - \mu B\Big[(\hat{b}\cdot\nabla\xi)^2 - q^2$$

$$- (\nabla\cdot\xi)^2 + (\nabla\xi):(\nabla\xi) + 2\left(\frac{B_1}{B}\right)^2\Big] - \frac{1}{M}\left(\frac{\mu B_1}{v_\parallel}\right)^2\Big\rangle. \qquad (10)$$

The expression for the second-order Lagrangian field change, $B_2=\frac{1}{2}B[(\nabla\cdot\xi)^2-2q(\nabla\cdot\xi)-q^2 + (\nabla\xi):(\nabla\xi) + (\hat{b}\cdot\nabla\xi)^2]$, used in writing Eq. (10), is developed in Ref. 5.

Calculating the second-order variation of the magnetic energy is somewhat easier in the Eulerian frame of reference, where the real space volume element does not change, than in the Lagrangian picture. Either approach yields the same result[5]:

$$W_m{}^{(2)} = \frac{1}{2}\int d^3x\,\Big\{Q^2 - (\xi\cdot\nabla\xi)\cdot[B\times(\nabla\times B)] + (\xi\times Q)\cdot(\nabla\times B)\Big\} \qquad (11)$$

with $Q=\nabla\times(\xi\times B)$ the Eulerian magnetic field perturbation.

Finally, combine the plasma and field energies to form the variation in the total energy:

$$W = \frac{1}{2}\int d^3x\,[\sigma Q_\perp{}^2 + \tau Q_\parallel{}^2 + \sigma j_\parallel \hat{b}\cdot(\xi\times Q_\perp) + q\xi\cdot\nabla' p_\parallel$$

$$- \frac{1}{B}(2Q_{\parallel} + \underset{\sim}{\xi}\cdot\nabla\underset{\sim}{B})\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}' p_{\perp}] + \delta W_k \quad . \tag{12}$$

The local part of δW in Eq. (12) has been expressed in the Taylor-Hastie form,[3] in terms of the parallel and perpendicular components of the perturbed field. (Details of the manipulation may be found in either Ref. 3 or 5.) The two coefficients $\sigma = 1 - B^{-1}(\partial p_{\parallel}/\partial B)$ and $\tau = 1 - (\partial^2 p_{\parallel}/\partial B^2)$ are familiar measures of stability against firehose and mirror anisotropy modes, respectively. Also, $j_{\parallel}$ is the parallel equilibrium current, and $\underset{\sim}{\nabla}' = \underset{\sim}{\nabla} - (\underset{\sim}{\nabla}B)\,\partial/\partial B$.

The kinetic contribution to the energy variation, δW_k , arises from the second term for ε_2 in Eq. (9). Integrating by parts on F with respect to the particle energy, we find

$$\delta W_k = - \frac{1}{2} \int d\alpha d\beta d\mu dJ \left(\frac{\partial F}{\partial \varepsilon}\right) \left\langle H \right\rangle^2 \quad . \tag{13}$$

This represents the nonlocal part of the plasma compression. It is identical to that derived by Kruskal and Oberman.[1]

This procedure has conformed that the usual energy principle for a guiding center plasma is equivalent to μ,J invariance.

Incidentally, note that for a hydromagnetic plasma with the magnetic moment μ as its only invariant, the potential energy is again given by Eq. (12) but with the kinetic part in this case simply

$$\delta W_k = - \frac{1}{2} \int d\alpha d\beta d\mu dJ \left(\frac{\partial F}{\partial \varepsilon}\right) H^2 \tag{14}$$

Here the velocity space integration can be done explicitly, since there are no bounce averages. We recognize that the energy variational for this case is the same as that for a double adiabatic plasma[10,11] with $\xi_{\parallel} = 0$ and, therefore, is at least as stable as the usual double adiabatic δW. It is also an upper bound on the guiding center plasma energy, as is obvious from a comparison of Eqs. (13) and (14), though less stringent than the well-known double adiabatic bound.[1,2]

III. Generalized Energy Principle

Now consider the case of actual interest, a plasma so energetic as not to behave in a hydromagnetic fashion. For simplicity, we here take the entire plasma to be non-hydromagnetic and later make distinctions through the summation over species.

To calculate the new form of the energy variation for the non-hydromagnetic case, we repeat the previous argument, but with the essential difference that the frozen-in constraint is replaced by conservation of the magnetic flux through a guiding center drift trajectory, which is expressed as $\Phi = \oint \alpha \, d\beta$. As before, the invariance of J implies that the particle energy must change when the magnetic field is displaced. During the displacement, the coordinates α and β continue to label the same field lines. However, since the energetic particles are not frozen to the magnetic field, the surfaces of constant J may move in (α, β) space. The original contours $\alpha = \alpha_0(\beta, \mu, \varepsilon, J)$, obtained by inverting the equation $J(\alpha, \beta, \mu, \varepsilon) = \text{constant}$, are displaced into new contours $\alpha = \alpha_0 + \alpha_1(\alpha_0, \beta, \mu, \varepsilon, J)$. Therefore, the total first-order change in J is now given by

$$J^{(1)} = 0 = J_1 + \varepsilon_1\left(\frac{\partial J}{\partial \varepsilon}\right)_{\alpha,\beta,\mu} + \alpha_1\left(\frac{\partial J}{\partial \alpha}\right)_{\varepsilon,\beta,\mu} . \tag{15}$$

The quantity $\partial J/\partial\alpha$ corresponds to the drift away from a line of force. Eq. (15) may thus be interpreted as first making the displacement as if the particles and field lines do stick together and then, because they actually do not, allowing the constant-J surfaces to be perturbed. The secondary displacement α_1 is determined by the additional condition of flux invariance:

$$\Phi^{(1)} = 0 = \oint \alpha_1 \, d\beta \quad , \tag{16}$$

where the integration is around a constant-J contour (i.e., in toroidal geometry, once around the short way for precessing particles or between end points for trapped-particle banana drift orbits).

Eqs. (15) and (16) can be easily solved for α_1 and ε_1. Define the double average over both bounce and drift motion, $\langle\langle \ldots \rangle\rangle = (\partial\Phi/\partial\varepsilon)^{-1} \oint d\beta (\partial\varepsilon/\partial\alpha)^{-1} \langle \ldots \rangle$, where we note that $\partial\Phi/\partial\varepsilon$ is the precessional drift period and $\partial\varepsilon/\partial\alpha = \langle d\beta/dt \rangle$ the rate of precession.[9]

Then we have

$$\varepsilon_1 = \langle\langle H \rangle\rangle \tag{17}$$

$$\alpha_1 = (\partial\varepsilon/\partial\alpha)^{-1} \left[\langle\langle H \rangle\rangle - \langle H \rangle \right] , \tag{18}$$

with H the same as before. The first-order energy variation again reduces to the condition for having an equilibrium.

The second-order invariance conditions are

$$J^{(2)} = 0 = J_2 + \varepsilon_2\left(\frac{\partial J}{\partial \varepsilon}\right) + \alpha_2\left(\frac{\partial J}{\partial \alpha}\right) + \frac{1}{2}\,\varepsilon_1{}^2\left(\frac{\partial^2 J}{\partial \varepsilon^2}\right)$$

$$+ \frac{1}{2}\,\alpha_1{}^2\left(\frac{\partial^2 J}{\partial \alpha^2}\right) + \alpha_1\varepsilon_1\left(\frac{\partial^2 J}{\partial\alpha\partial\varepsilon}\right) + \left(\varepsilon_1\frac{\partial}{\partial\varepsilon} + \alpha_1\frac{\partial}{\partial\alpha}\right)J_1 \quad , \quad (19)$$

$$\Phi^{(2)} = 0 = \oint \alpha_2 \, d\beta \qquad (20)$$

By combining terms in Eq. (19) and using the trivial information $(\partial\Phi^{(1)}/\partial\varepsilon)_J = 0$ with $\Phi^{(1)}$ given by Eq. (16), we obtain

$$\varepsilon_2 = \left(\frac{\partial\Phi}{\partial\varepsilon}\right)^{-1} \oint \frac{d\beta}{(\partial J/\partial\alpha)} \left\{ J_2 + \frac{\partial}{\partial\alpha}\left[\frac{1}{2}\,\alpha_1\left(\varepsilon_1\frac{\partial J}{\partial\varepsilon} - J_1\right)\right] - \frac{\partial}{\partial\varepsilon}\left(\frac{1}{2}\,\varepsilon_1{}^2\,\frac{\partial J}{\partial\varepsilon}\right)\right\} \qquad (21)$$

with J_2 given earlier in Eq. (10).

When we now calculate the plasma energy with the expression $W_p^{(2)} = \int d\alpha \, d\beta \, d\mu \, dJ (F\varepsilon_2)$, we must be careful to convert from using α as a description of a particular drift orbit back to using α as a field line coordinate. That is, in Eq. (21), the derivative with respect to ε is performed for a fixed drift orbit $\alpha = \alpha_0(\beta,\mu,\varepsilon,J)$, and the derivative with respect to α indicates a deviation from this orbit at fixed energy ε . To effect this conversion, we translate $(\partial/\partial\varepsilon)_{\alpha_0}$ into $(\partial/\partial\varepsilon)_J$ and $(\partial J/\partial\alpha_0)^{-1}(\partial/\partial\alpha_0)_\varepsilon$ into $(\partial/\partial J)_\varepsilon$ as we integrate by parts on the distribution function F. This is appropriate since a given orbit is specified by the values of μ,ε, and J and since the equilibrium distribution $F(\alpha,\beta,\mu,\varepsilon)=F[J(\alpha,\beta,\mu,\varepsilon),\mu,\varepsilon]$ is itself properly such a function.

In calculating the total energy, we note that the magnetic energy and the J_2 term of the plasma energy make the same contribution as before in Eq. (12), while the new form of the kinetic contribution δW_k is derived from the last two terms of Eq. (21). Doing the indicated partial integrations and using Eqs. (17) and (18), we finally obtain

$$W = \frac{1}{2}\int d^3x \Big[\sigma Q_\perp{}^2 + \tau Q_\parallel{}^2 + \sigma j_\parallel \hat{b}\cdot(\underset{\sim}{\xi}\times\underset{\sim}{Q}) + q\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}' p_\parallel - \frac{1}{B}(2Q_\parallel + \underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}B)\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}' p_\perp \Big] - \frac{1}{2}\int d\alpha d\beta d\mu dJ \left\{ \left(\frac{\partial F}{\partial \varepsilon}\right)\langle\langle H\rangle\rangle^2 + \left(\frac{\partial F}{\partial J}\right)\left(\frac{\partial J}{\partial \varepsilon}\right)\left[\langle H\rangle^2 - \langle\langle H\rangle\rangle^2\right]\right\} \tag{22}$$

The local part of this δW is the same as before in Eq. (12). The kinetic part, however, is quite different for the non-hydromagnetic case. Only its first term, apart from the additional drift average, resembles the guiding center result of Eq. (13). The second term, proportional to $\partial F/\partial J$, is dominant for most cases of interest, although it vanishes identically for distributions $F(\varepsilon,\mu)$ which yield $p_{\parallel,\perp}$ as functions of $|B|$ only.

Since δW of Eq. (22) is manifestly the energy of the system, $\delta W>0$ for all allowable nontrivial $\underset{\sim}{\xi}$ is sufficient for stability. Furthermore, the condition that δW be minimum with respect to variations in $\underset{\sim}{\xi}$ corresponds to satisfying the zero-frequency equation of motion (i.e., the momentum transfer equation with pressure calculated from the distribution function which obeys a drift kinetic equation). Since this is self-adjoint in form, $\delta W = 0$ corresponds to marginal stability. Hence, $\delta W>0$ is both necessary as well as sufficient for stability and constitutes a genuine energy principle, though only for low-frequency modes which satisfy the conditions of this theory (viz., growth time less than the bounce and drift periods).

IV. Discussion

In an axisymmetric device (i.e., no azimuthal dependence), the drift-averaged terms in Eq. (22) disappear. They survive only for the $m = 0$ breather mode (where m is the azimuthal mode number of the perturbation), but for this mode δW reduces to the result of Eq. (12). Also, in a nonsymmetric system, if we consider high-m modes, these terms are of order m^{-1} and may be dropped.

Employing the Schwartz inequality, we see that the kinetic part of our new δW of Eq. (22) is positive definite if two local conditions,

$$\left(\frac{\partial F}{\partial \varepsilon}\right)_{J,\mu} < 0 \qquad (23)$$

$$\left(\frac{\partial F}{\partial J}\right)_{\varepsilon,\mu} < 0 \qquad (24)$$

are satisfied. The latter is related to the direction of particle drift and has been discussed in a different context by Rutherford and Frieman.[10]

In the Elmo bumpy torus, the warm toroidal core plasma ($T_i \cong 100$ ev, $T_e \cong 500$ eV) could be described by a collisionless guiding center plasma model. The effect of the extremely high-energy annulus electrons on the low-frequency plasma stability may be treated by means of the theory developed in the preceding section. Adding and subtracting a term like that of Eq. (13) for the hot electrons, we may write the energy variation for the system as

$$\delta W = \delta W^{(GC)} + \frac{1}{2}\int d\alpha d\beta d\mu dJ \left(\frac{\partial F_h}{\partial \varepsilon}\right)\left(1 - \frac{\omega_{*h}}{\langle\omega_{dh}\rangle}\right)\left[\langle H\rangle^2 - \langle\langle H\rangle\rangle^2\right] \quad (25)$$

The total plasma, core and rings, enters the complete guiding center plasma variational form $\delta W^{(GC)}$, given by Eq. (12), whereas the explicit difference term in Eq. (25) involves only the non-hydromagnetic hot electrons (labeled h), with ω_{*h} and $\langle\omega_{dh}\rangle$ their diamagnetic and bounce-averaged magnetic drift frequencies. For normal bumpy torus operation[8], the ring pressure is sufficiently large to produce a local minimum in B and reverse the magnetic drift. Then, since the plasma pressure decreases outward, the two drifts are opposite in direction and, because Eq. (23) is generally valid, the difference term in Eq. (25) is negative. Our new energy principle, in this case, predicts the plasma to be less stable than would be determined by the guiding center plasma energy principle.

Beyond this qualitative statement, the new energy principle can be shown to lead to an important pressure limitation. Again take the ring electron pressure large enough so that the particle magnetic drifts are reversed. Consider the terms (call them U) in δW which involve the perturbed parallel field, $Q_{\parallel}$, quadratically:

$$U = \oint \frac{d\ell}{B}(\tau Q_{\parallel})^2 - \int d\varepsilon\, d\mu \left(\frac{\partial F_c}{\partial \varepsilon}\right) \frac{\left[\oint \frac{d\ell}{v_{\parallel}}(\mu Q_{\parallel})\right]^2}{\oint \frac{d\ell}{v_{\parallel}}}$$

$$+ \int d\varepsilon d\mu \left(\frac{\partial F_h}{\partial \alpha}\right) \frac{\left[\oint \frac{d\ell}{v_\parallel}(\mu Q_\parallel)\right]^2}{\oint \frac{d\ell}{v_\parallel}(v_\parallel{}^2 k + \mu B')} \quad . \tag{26}$$

We restrict the discussion to axisymmetric systems and high-m modes, for which δW may be minimized on each field line. In Eq. (26), subscripts c and h designate core plasma and hot electron ring quantities, respectively; also, the magnetic drift has been expressed in terms of $B' = \partial B/\partial\alpha$ and the scaled curvature $k = (\hat{b}\cdot\nabla\hat{b})\cdot\nabla\alpha/|\nabla\alpha|^2$. By the Schwartz inequality, we have

$$U \leqslant \oint \frac{d\ell}{B}\left(1+\frac{\partial p_{\perp h}}{B\partial B}+\frac{2p_{\perp c}}{B^2}\right)Q_\parallel^2 + \frac{\left[\oint \frac{d\ell}{B^2}Q_\parallel\left(p_{\perp h}' - B'\frac{\partial p_{\perp h}}{\partial B}\right)\right]^2}{\oint \frac{d\ell}{B}\left[k\left(p_{\parallel h}'-B'\frac{\partial p_{\parallel h}}{\partial B}\right)+\frac{B'}{B}\left(p_{\perp h}'-B'\frac{\partial p_{\perp h}}{\partial B}\right)\right]} \tag{27}$$

It is possible to choose a perturbation which emphasizes these terms in δW such that the sign of U can determine stability. In particular, take $Q_\parallel \propto B^{-1}$ and look at the isotropic case for simplicity; then the following condition,

$$- p_c' \oint \frac{d\ell}{B^3} + 2\oint \frac{d\ell}{B}k + \frac{\oint \frac{d\ell}{B^3}\left(\frac{2p_c}{B^2}\right)\oint \frac{d\ell}{B}\left(2k - \frac{(p_h+p_c)'}{B^2}\right)}{\oint \frac{d\ell}{B^3}} < 0 \tag{28}$$

is sufficient for instability. This may be termed a modified mirror instability by analogy with the usual mirror mode argument.[3]

Suppose, on the other hand, that the inequality of Eq. (28) is not satisfied. Then the terms quadratic in the field perturbation are positive definite, and δW may be minimized with respect to the magnitude of $BQ_{\|}$, which was taken to be constant along a line of force. Let us also consider a flute-like displacement, i.e., $\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}\alpha$ constant along a field line. By virtue of the same Schwartz inequality as before, we find $\delta W \leq \frac{1}{2}\int d\alpha(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}\alpha)^2 V(\alpha)$, with stability on each line determined by the sign of the quantity

$$V(\alpha) = -p_c' \oint \frac{d\ell}{B}\left(2k - \frac{p'}{B^2}\right) + 2p_c \oint \frac{d\ell}{B}\left[\frac{7}{2}k^2 - 3k\left(\frac{p'}{B^2}\right) + \left(\frac{p'}{B^2}\right)^2\right] - \frac{\left[p_c' \oint \frac{d\ell}{B^3} - p_c \oint \frac{d\ell}{B^3}\left(2k - \frac{p'}{B^2}\right)\right]^2}{\oint \frac{d\ell}{B^3}\left(1 + \frac{2p_c}{B^2}\right) + \left[p_h' \left(\oint \frac{d\ell}{B^3}\right)^2 \Big/ \oint \frac{d\ell}{B}\left(2k - \frac{p'}{B^2}\right)\right]} . \qquad (29)$$

Equilibrium pressure balance, $BB' + p' = B^2 k$ with $p = p_c + p_h$, has been used in Eq. (29) to eliminate field gradients. The first term of V is recognized to be the driving term for the electrostatic interchange mode. The second term, typically stabilizing but smaller than the first one by the factor p_c/B^2 , represents compressional effects. With respect to the first term, interchange modes would be stable, since we have all along required the ring electron pressure to be large enough to reverse the magnetic drifts. The lower limit imposed by this requirement may be written in terms of the ring beta (ratio of thermal and magnetic energy densities) as

$$\beta_h \equiv \frac{2p_h}{B^2} > \frac{4 \oint \frac{d\ell}{B} k}{\left(B^2 \oint \frac{d\ell}{B^3}\right)\left(p_h'/p_h\right)} . \qquad (30)$$

The onset of stable operation of the Elmo bumpy torus has been associated with surpassing this threshold.[8] The last term in Eq. (29), however, is destabilizing, since we have also assumed that the core plasma pressure is high enough to satisfy the reverse of Eq. (28). Indeed, where this condition is but marginally satisfied, the last term of $V(\alpha)$ clearly overwhelms the first two, and the interchange mode is unstable. The upper limit on the core pressure for interchange stability that we have thus obtained from a δW approach is the same one that has been pointed out by other means recently.[13,14]

Our energy principle confronts us with the interesting dilemma that the plasma can be mirror unstable where the core plasma pressure is lower than a certain value described by Eq. (28) and interchange unstable when it exceeds the same value. We should be careful to mention, however, that the former conclusion is mitigated by the knowledge that effects which do not enter an energy principle analysis, such as finite gyroradius and non-adiabatic behavior for the hot species, finite gyrofrequency for the ions, etc., can have a significant effect on bumpy torus stability. For instance, the hot electron response to an electromagnetic perturbation may not be instantaneous (i.e., $\omega/\overline{\omega}_d \neq 0$, with $\overline{\omega}_d$ the magnetic drift frequency evaluated at a typical speed), so that the third adiabatic invariant is violated. A normal mode analysis of the compressional Alfven wave shows that this has the effect of introducing a frequency shift. Then, the modified mirror mode can be stabilized if the following condition[15] (written for the isotropic case) obtains:

$$N_i M_i p_h'^{\,3} \oint \frac{d\ell}{B^7} |\nabla\alpha|^2 \oint \frac{d\ell}{B}\left(2k - \frac{p_c'}{B^2}\right) < N_h'^{\,2} \left(\oint \frac{d\ell}{B}\right)^2 \quad , \qquad (31)$$

with N_i, M_i the ion density and mass and N_h the hot electron density. This condition is nontrivial when $\oint \frac{d\ell}{B}(2k - p_c'/B^2) < 0$, implying that even when Eq. (28) is (approximately) satisfied, the modified mirror and interchange modes can both be stable.

On the other hand, it appears that violation of Eq. (29) should indeed lead to interchange instability, thereby imposing an upper limit on the core plasma pressure in the bumpy torus device.

In summary, we propose Eq. (22) as a new kinetic energy principle valid in the limit of very low frequency $\omega \lesssim \bar{\omega}_d$.[16]

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[16] After this work was completed, it came to our attention that a variational principle similar to Eq. (22) has recently been derived by a completely different technique by T.M. Antonsen, Jr., B. Lane, and J.J. Ramos (to be published, Phys. Fluids), who found certain necessary conditions for stability. However, since scalar pressure was assumed, their work is limited to the azimuthally symmetric, high mode number limit. For a marginal analysis, our δW reduces to their result in this limit. For the special case of equal-temperature Maxwellian ions and electrons, they further showed that Eq. (22) is indeed necessary and sufficient for stability, whereas in this paper we have only been able to demonstrate that it applies near marginal stability ($\omega \approx 0$).