Scaling of the peak magnetic reconnection rate in the inviscid Taylor problem

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A dispute regarding the scaling of the peak magnetic reconnection rate in the inviscid Taylor problem for $ka \ll 1$ is resolved. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2839768]

The well-known "Taylor problem" (so-called since it was first proposed by J. B. Taylor) deals with a magnetized, low-β, tearing-stable, slab plasma with an equilibrium magnetic field of the form

$$\mathbf{B}^{(0)} = [0, B_y^{(0)}(x), 0], \tag{1}$$

where $B_y^{(0)}(-x) = -B_y^{(0)}(x)$. The plasma is bounded by perfectly conducting walls located at $x = \pm a$. At $t=0$, the walls are subject to a sudden deformation (in the x direction), which is such that

$$x_{\rm wall} \to \pm a[1 + \Xi_0 \cos(ky)], \tag{2}$$

where $\Xi_0 \ll 1$. This deformation pushes oppositely directed magnetic field-lines together at the field null ($x=0$), forcing them to reconnect. The essence of the problem is to determine the rate at which this forced reconnection occurs as a function of time.

It is convenient to normalize all lengths to a, all magnetic field-strengths to $B_y^{(0)}(a)$, and all times to the shear-Alfvén time calculated with a and $B_y^{(0)}(a)$. Using this normalization, the standard, low-β, incompressible, inviscid, magnetohydrodynamical model of the plasma dynamics yields the familiar equations

$$\frac{\partial \psi}{\partial t} = [\phi, \psi] + \eta j, \tag{3}$$

$$\frac{\partial U}{\partial t} = [\phi, U] + [j, \psi], \tag{4}$$

where $j = 1 + \nabla^2 \psi$, and $U = \nabla^2 \phi$. Moreover, $\mathbf{B} = \nabla\psi \times \mathbf{e}_z$ is the magnetic field, and $\mathbf{V} = \nabla\phi \times \mathbf{e}_z$ the plasma velocity. Here, η is the (uniform, constant) plasma resistivity, and $[A, B] \equiv A_x B_y - A_y B_x$. Note that we have assumed the existence of a uniform z-directed electric field that maintains the equilibrium plasma current. In the following, this current is taken to be uniform, for the sake of simplicity. We have also assumed that the plasma mass density is uniform.

Since we are dealing with a relatively small perturbation to the plasma (given the relatively small displacement of the walls), we can *linearize* the above equations by writing $\psi = -(1/2)x^2 + \Xi_0 \tilde{\psi}(x,t)\exp(iky)$, $\phi = -i\Xi_0 \tilde{\phi}(x,t)\exp(iky)$, $j = \Xi_0 \tilde{j}(x,t)\exp(iky)$, and $U = -i\Xi_0 \tilde{U}(x,t)\exp(iky)$, where $\tilde{}$ denotes a first-order quantity. Retaining all terms up to first order, we obtain

$$\frac{\partial \tilde{\psi}}{\partial \hat{t}} = x\tilde{\phi} + \hat{\eta}\tilde{j}, \tag{5}$$

$$\frac{\partial \tilde{U}}{\partial \hat{t}} = -x\tilde{j}, \tag{6}$$

where $\tilde{j} = \tilde{\psi}_{xx}$, $\tilde{U} = \tilde{\phi}_{xx}$, $\hat{t} = kt$, and $\hat{\eta} = \eta/k$. Here, we have assumed that $k \ll 1$, for the sake of simplicity. The boundary conditions are $\tilde{\psi} = 1$, and $\tilde{\phi} = \tilde{j} = \tilde{U} = 0$ at $x = \pm 1$. Finally, the initial conditions are $\tilde{\psi} = \tilde{\phi} = \tilde{j} = \tilde{U} = 0$ at $t=0$. Note that the only free parameter remaining in the problem is the normalized plasma resistivity, $\hat{\eta} = \eta/k$.

It is convenient to define $\Psi(\hat{t}) = \tilde{\psi}(0, \hat{t})$ and $J(\hat{t}) = \tilde{j}(0, \hat{t})$. Of course, $\Psi(\hat{t})$ is the *reconnected magnetic flux*. Moreover, it follows from Eq. (5) that $d\Psi/d\hat{t} = \hat{\eta} J$. Hence, $J(\hat{t})$ measures the *magnetic reconnection rate*. The ultimate object of the Taylor problem is to determine the functional form of $J(\hat{t})$.

The Taylor problem was first solved analytically, in the limit $\hat{\eta} \ll 1$, in a classic paper by Hahm and Kulsrud.[1] In this paper, it was demonstrated that the forced magnetic reconnection takes place in two main stages, corresponding to two distinct physical regimes.

The *inertial regime* holds for $1 \ll \hat{t} \ll \tau_1$, where $\tau_1 = \hat{\eta}^{-1/3}$. In this regime,

$$J = J_i \equiv \frac{4}{\pi}\hat{t}. \tag{7}$$

Incidentally, the inertial regime is generally referred to as a *nonconstant-ψ regime*,[2] since it is characterized by a perturbed magnetic flux, $\tilde{\psi}$, which varies strongly (in x) across the reconnecting layer.

The *resistive-inertial regime* holds for $\hat{t} \gg \tau_1$. In this regime,

$$J = J_{ri} \equiv \hat{\eta}^{-2/5} F(\hat{t}/\tau_2), \tag{8}$$

where

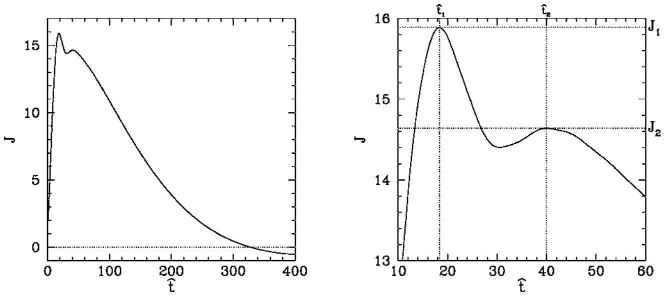

FIG. 1. The magnetic reconnection rate J as a function of normalized time $\hat{t}$, calculated for $\hat{\eta}=5\times 10^{-3}$. The normalized times $\hat{t}_1$ and $\hat{t}_2$ (shown in the inset) correspond to the instances at which the reconnection rate attains its first maximum J_1 and its second maximum J_2, respectively.

$$F(x) = 1.52\left\{ \{\cos(\pi/5)\cos[\sin(\pi/5)x] + \sin(\pi/5)\sin[\sin(\pi/5)x]\}e^{-\cos(\pi/5)x} - \frac{1}{2\sqrt{2}\pi}\int_0^\infty \frac{y^{4/5}e^{-y^{4/5}x}dy}{1-\sqrt{2}y+y^2}\right\}, \tag{9}$$

and $\tau_2=1.049\,\hat{\eta}^{-3/5}$. Incidentally, the resistive-inertial regime is generally referred to as a *constant-ψ regime*,[2] since it is characterized by a perturbed magnetic flux $\tilde{\psi}$, which is approximately constant (in x) across the reconnecting layer. It is easily demonstrated that $F\simeq 1.05x^{1/4}$ for $x\ll 1$. Furthermore, it can be shown that F attains a maximum value of 0.673 at $x=0.367$, passes through zero at $x=3.34$, and asymptotes to zero (from below) as $|x|\to\infty$.

The validity of Hahm and Kulsrud's analytic results was questioned by Ishizawa and Tokuda,[3,4] who argued that Hahm and Kulsrud had illegitimately made use of the so-called *constant-ψ approximation*[2] in the nonconstant-ψ reconnection regime. This dispute was ultimately resolved by Cole and Fitzpatrick,[5] who showed, mostly by means of numerical simulations, that the criticisms of Ishizawa and Tokuda were unfounded.

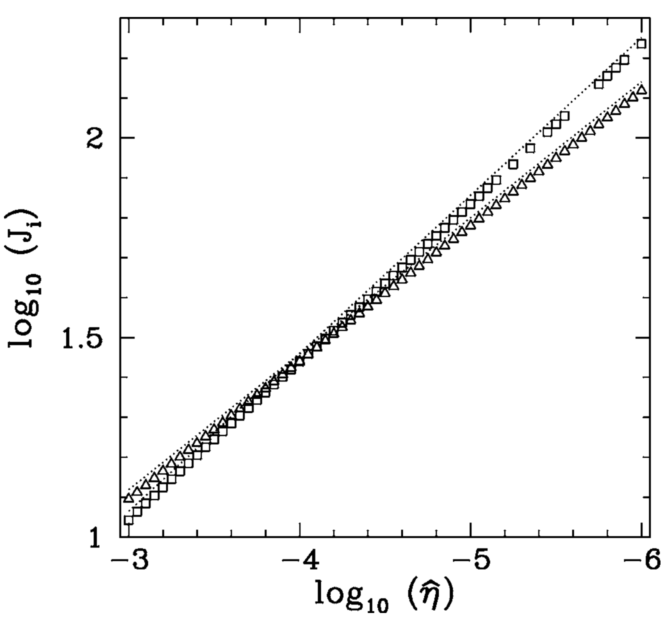

FIG. 2. The scaling of the maximum reconnection rates J_1 (triangles) and J_2 (squares) with $\hat{\eta}$. The corresponding dotted lines (shifted vertically for clarity) show the fits to $J_1\sim\hat{\eta}^{-0.34}$ and $J_2\sim\hat{\eta}^{-0.40}$, respectively.

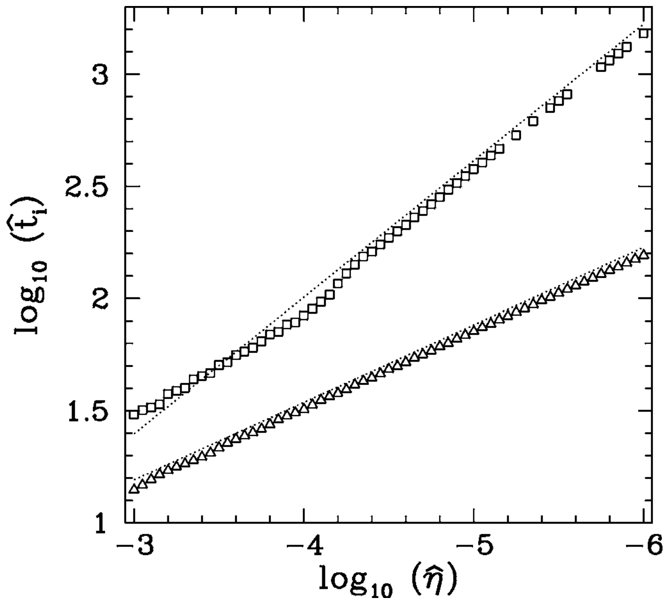

FIG. 3. The scaling of the normalized times $\hat{t}_1$ (triangles) and $\hat{t}_2$ (squares) with $\hat{\eta}$. The corresponding dotted lines (shifted vertically for clarity) show the fits to $\hat{t}_1\sim\hat{\eta}^{-0.35}$ and $\hat{t}_2\sim\hat{\eta}^{-0.61}$, respectively.

Unfortunately, the resolution of this first dispute led immediately to a second. Cole and Fitzpatrick argued, both from physical principles and from the results of numerical simulations, that the *nonconstant-ψ* reconnection regime is generally characterized by a reconnection rate that *increases* in time, whereas the *constant-ψ* regime is characterized by a reconnection rate that *decreases* in time. This observation immediately suggests that the reconnection rate should *peak* at the transition from the nonconstant-ψ to the constant-ψ regime. In other words, a maximum in the reconnection rate should occur at time

$$\hat{t}_1\sim\tau_1\sim\hat{\eta}^{-1/3}, \tag{10}$$

and should scale as

$$J_1\sim J_i(\tau_1)\sim\hat{\eta}^{-1/3}. \tag{11}$$

This result was disputed by Vekstein,[6] who argued that the reconnection rate, in fact, continues to increase well into the constant-ψ regime, and finally attains the peak value specified in Eq. (8). In other words, the maximum reconnection rate should occur at time

$$t_2=0.368\,\tau_2\sim\hat{\eta}^{-3/5}, \tag{12}$$

and should scale as

$$J_2=J_{ri}(\hat{t}=0.368\,\tau_2)\sim\hat{\eta}^{-2/5}. \tag{13}$$

The resolution of the above dispute is quite simple. The direct numerical solution of the inviscid Taylor problem reveals that there are, in fact, *two* maxima in the magnetic reconnection rate as a function of time (see Fig. 1). The first maximum occurs at the transition between the nonconstant-ψ and the constant-ψ regimes, and scales with $\hat{\eta}$ as specified in Eqs. (10) and (11). On the other hand, the second maximum occurs wholly within the constant-ψ regime, and scales with $\hat{\eta}$ as specified in Eqs. (12) and (13). The various scalings are confirmed in Figs. 2 and 3. It can be seen that the absolute maximum in the reconnection rate corresponds to the first maximum when $\hat{\eta} \gtrsim 10^{-4}$, and to the second when $\hat{\eta} \lesssim 10^{-4}$.

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