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HAMILTONIAN STRUCTURE OF THE BBGKY HIERARCHY

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ABSTRACT. The BBGKY hierarchy equations for the evolution of the i-point functions of a plasma with electrostatic interactions are shown to be Hamiltonian. The Poisson brackets are Lie-Poisson brackets on the dual of a Lie algebra. This algebra is constructed from the algebra of n-point functions under Poisson bracket and the filtration obtained by considering subspaces of i-point functions, $1 \leq i \leq n$.

§1. Introduction

The purpose of this paper is to show that the BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvon) hierarchy equations are Hamiltonian with a Poisson bracket associated to a certain Lie algebra. For background and the original references on the hierarchy, the reader may consult one of the standard texts, such as Clemmow and Dougherty [1969], Ichimaru [1973] or Van Kampen and Felderhof [1967]. For background on Lie-Poisson structures on duals of Lie algebras, see Marsden and Weinstein [1982], Marsden et. al. [1983] and the lectures of Morrison, Ratiu and Weinstein in these proceedings.

In the present paper, we simply exhibit the Hamiltonian structure of the hierarchy equations making use of the theory of momentum mappings. Eventually, we hope to show how this structure is inherited by truncated systems, providing a statistical basis for recently discovered bracket structures for plasma systems (Morrison and Greene [1980], Morrison [1980], Marsden and Weinstein [1982], Morrison [1982] and Marsden, et. al. [1983]).

2. The Hierarchy Equations

Let P be a finite dimensional symplectic manifold; for example, the position-momentum space $\mathbb{R}^6$ for a single particle. Let $P^n = P \times P \times \ldots \times P$ (n times) be thought of as the phase space for n particles. Points in P^n will be denoted $(z_1, \ldots, z_n)$. Consider a Hamiltonian on P^n of the form

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$$H_n(z_1, \ldots, z_n) = \sum_{i=1}^{n} H_1(z_i) + \sum_{i<j} H_2(z_i, z_j)$$

where $H_1 : P \to \mathbb{R}$ and $H_2 : P \times P$ (minus the diagonal) $\to \mathbb{R}$ are given and H_2 is symmetric in its arguments. For example, on $\mathbb{R}^6$ with $z = (q,p)$ and $z' = (q',p')$, the functions

$$H_1(z) = \frac{|p|^2}{2m} \quad \text{and} \quad H_2(z,z') = \frac{e^2}{|q - q'|}$$

describe the dynamics of identical particles of mass m and charge e under electrostatic interaction. (The simple generalization to an arbitrary number of different species is omitted here.)

Hamilton's equations on P^n give the Liouville equation for the evolution of a smooth symmetric function,

$$f_n : P^n \to \mathbb{R},$$

namely

$$\frac{\partial f_n}{\partial t} + \{f_n, H_n\}_n = 0, \tag{L}$$

where $\{\ ,\ \}_n$ denotes the Poisson bracket on P^n, i.e. the n-particle Poisson bracket. The moments of f_n are defined by the following equations

one-point function: $f_1(z;t) = n \int f_n(z, z_2, \ldots, z_n; t)\, dz_2 \ldots dz_n$

two-point function: $f_2(z,z';t) = n(n-1) \int f_n(z, z', z_3, \ldots, z_n; t)\, dz_3 \ldots dz_n$

$$\vdots \qquad\qquad \vdots$$

where dz denotes Liouville measure. The hierarchy equations can be obtained by differentiating these equations in t using the evolution equation for f_n. For example, the first equation is

$$\frac{\partial f_1}{\partial t}(z;t) + \{f_1, \mathcal{H}(f_1)\}(z;t) = \int \{f_1(z;t) f_1(z';t) - f_2(z,z';t),\ H_2(z,z')\}\, dz' \tag{H1}$$

where $\mathcal{H}(f_1)(z) = H_1(z) + \int f(z') H_2(z,z')\, dz'$ and the braces denote the Poisson bracket on P (see the appendix).

§3. Lie-Poisson Equations

A Lie-Poisson bracket is the natural bracket on functions defined on the dual of a Lie algebra. These brackets play a fundamental role in the Hamiltonian description of rigid bodies, fluids, and plasmas, (see the references cited earlier). If G is a Lie group with Lie algebra $\mathfrak{g}$ and dual $\mathfrak{g}^*$, then for $F,G:\mathfrak{g}^* \to \mathbb{R}$, their Lie-Poisson bracket at $\mu \in \mathfrak{g}^*$ is defined by

$$\{F,G\}(\mu) = \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle, \tag{LP}$$

where $\frac{\delta F}{\delta \mu} \in \mathfrak{g}$ is defined by

$$DF(\mu)\cdot\mu' = \left\langle \mu', \frac{\delta F}{\delta \mu}\right\rangle,$$

$DF(\mu)$ is the Frechet derivative, $\langle\ ,\ \rangle$ is the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, and $[\ ,\ ]$ is the Lie bracket on $\mathfrak{g}$.

The Lie-Poisson bracket for the group $\mathrm{Sym}(P)$ of canonical transformations of a symplectic manifold P may now be described as follows. Except for constants, the Lie algebra $\mathrm{sym}(P)$ may be identified with (generating) functions $K:P \to \mathbb{R}$ and its dual $\mathrm{sym}(P)^*$ with densities $f d\mu$, where $f:P \to \mathbb{R}$ and $d\mu$ is Liouville measure on P. Then we set

$$\{F,G\}(f) = \int_P f\left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\} d\mu. \tag{PV}$$

This is the bracket for the Poisson-Vlasov equation; it is also a fundamental ingredient in the Maxwell-Vlasov bracket (Morrison [1980], Marsden and Weinstein [1982]). With P replaced by P^n, it also describes the Liouville equation (L). In fact one can check either by a direct calculation or from considerations of reduction of dynamics on $\mathrm{Sym}(P^n)$ that (L) is equivalent to

$$\dot{F} = \{F,\mathcal{H}_n\}(f_n)$$

where F is a functional of f_n, $\{\quad\}(f_n)$ is given by the bracket (PV) with f_n in place of f, P^n in place of P and

$$\mathcal{H}_n(f_n) = \int_{P^n} H_n(z_1, \ldots, z_n) f_n(z_1,\ldots, z_n) dz_1 \ldots dz_n .$$

Here $\mathrm{Sym}(P^n)$ may be replaced by $\mathrm{Sym}_s(P^n)$, those elements of $\mathrm{Sym}(P^n)$ that commute with permutations and $\mathrm{sym}(P^n)$ by $\mathrm{sym}_s(P^n)$, the symmetric functions on P^n.

4. The Hierarchy Algebra

Suppose that A_n is a real Lie algebra and $A_1 \subset A_2 \subset \dots \subset A_n$ are linear subspaces. Below we shall choose A_n to be the algebra $\mathrm{sym}_s(P^n)$ of symmetric functions under Poisson bracket and A_i to be the space $\mathrm{sym}_s(P^i)$ embedded as a subspace by the map

$$\varepsilon_i : K_i(z_1, \dots, z_i) \mapsto \sum K_i(z_{j_1}, \dots, z_{j_i})$$

where the sum is over distinct subsets of $\{1, \dots, n\}$; i.e. $j_1 \neq j_2 \neq \dots \neq j_i$. For example

$$K_1(z_1) \mapsto \sum_i K_1(z_i) \quad \text{and} \quad K_2(z_1,z_2) \mapsto \sum_{i \neq j} K_2(z_i,z_j).$$

One checks that in this example we have a filtration; i.e.

$$[A_i, A_j] \subset A_{i+j-1} \tag{F}$$

(Note that only A_1 is a Lie subalgebra.) In general given such a filtration there is a Lie algebra structure on

$$\mathcal{A}_n = A_1 \oplus A_2 \oplus \dots \oplus A_n$$

such that the map

$$\alpha_n : \mathcal{A}_n \to A_n$$

defined by

$$\alpha_n(K_1, \dots, K_n) = K_1 + \dots + K_n$$

is a Lie algebra homomorphism. Indeed, set

$$\begin{aligned}[(K_1, \dots, K_n), (L_1, \dots, L_n)] = ([K_1,L_1], [K_1,L_2] + [K_2,L_1],\\ [K_2,L_2] + [K_1,L_3] + [K_3,L_1], \dots)\end{aligned} \tag{HLA}$$

where $[K_i,L_j]$ is to be put in the k^{th} slot if $k = i + j - 1$ and if $i + j - 1 \leqslant n$. If $i + j - 1 > n$, the term is to be put in the last (n^{th}) slot; one has some options here that will be the subject of our work on truncations.

One can check directly that (HLA) defines a Lie algebra structure and that α_n is a Lie algebra homomorphism.

§5. The Moments Comprise a Momentum Map

The dual α_n^* of α_n in our example is determined as follows. We have

$$\alpha_n : A_1 \oplus \ldots \oplus A_n \to A_n \quad \text{and so} \quad \alpha_n^* : A_n^* \to A_1^* \oplus \ldots \oplus A_n^*$$

is given by

$$\int (\alpha_n^* f_n)(K_1, \ldots, K_n) dz_1 \ldots dz_n = \int f_n \alpha_n (K_1, \ldots, K_n) dz_1 \ldots dz_n$$

$$= \sum_{i=1}^{n} \int f_n \varepsilon_i (K_i) dz_1 \ldots dz_n .$$

From the definitions it follows that

$$\alpha_n^* f_n = (f_1, \ldots, f_n)$$

where $f_1, \ldots, f_n$ are the moments of f_n and the embeddings ε_i are suppressed.

Thus, the process of taking moments is given by the dual of a Lie algebra homomorphism and is therefore a momentum map (this is a standard fact; cf. Guillemin and Sternberg [1980] or Marsden et. al. [1983]).

§6. The Hierarchy Equations are Lie-Poisson

Since α_n^* is a momentum map, it is a Poisson map; i.e. it preserves brackets. We have the following maps

$$\begin{array}{ccc} \mathrm{sym}_s^*(P^n) & \xrightarrow{\mathcal{H}_n} & \mathbb{R} \\ \text{taking moments} \Big\downarrow \alpha_n^* & & \\ A_n = \sum_{i=1}^{n} \mathrm{sym}_s^*(P^i) & \xrightarrow{\mathcal{H}} & \mathbb{R} \end{array}$$

Now $\mathcal{H}_n(f_n) = \int_{P^n} H_n(z_1, \ldots, z_n) f_n(z_1, \ldots, z_n) dz_1 \ldots dz_n$

$$= \int_{P^n} \sum H_1(z_i) f_n(z_1, \ldots, z_n) dz_1, \ldots, dz_n$$

$$+ \int_{P^n} \sum_{i<j} H_2(z_i, z_j) f_n(z_1, \ldots, z_n) dz_1 \ldots dz_n$$

$$= \int_P H_1(z)f_1(z)dz + \frac{1}{2}\int_{P^2} H_2(z,z')f_2(z,z')dz\,dz'$$

so that $\mathcal{H}_n$ "collectivizes" in the sense of Guillemin and Sternberg [1980] to a map $\mathcal{H}$ on $A_1^* \oplus \ldots \oplus A_n^*$ that depends only on the first two arguments. From general properties of momentum maps and reduction, it follows that the equations of motion for $\mathcal{H}$ are Lie-Poisson. But these equations are just the equations for f_n written out in terms of the moments; they are thus the hierarchy equations. We summarize:

Theorem. The BBGKY hierarchy equations for the moments $f_1, \ldots, f_n$ of an n particle distribution function $f_n(z_1, \ldots, z_n)$ are equivalent to the Hamiltonian equations

$$\dot{F} = \{F,\mathcal{H}\}_{A_n^*} \tag{LP}$$

where F is a functional of $(f_1, \ldots, f_n)$ (regarded as independent variables), $\mathcal{H}$ is given by

$$\mathcal{H}(f_1, \ldots, f_n) = \int_P H_1(z)f_1(z)dz + \frac{1}{2}\int_{P^2} H_2(z,z')f_2(z,z')dz\,dz'$$

and $\{\ \}_{A_n^*}$ is the Lie-Poisson bracket on the dual of the hierarchy Lie algebra with Lie bracket given by (HLA).

Remark. The present formalism is appropriate for electrostatic interactions and has brackets compatible with those for the Poisson-Vlasov equation. Electromagnetic interactions require a Poisson structure compatible with that for the Maxwell-Vlasov equations, with fully incorporated electromagnetic field variables.

Appendix. Direct Verification of the Main Theorem for the First Two Hierarchy Equations

From (L) we have

$$\frac{\partial}{\partial t} f_n(z_1, \ldots, z_n;t) + \Big\{f_n(z_1, \ldots, z_n;t), \sum_i H_1(z_i) + \sum_{i<j} H_2(z_i,z_j)\Big\}_{z_1,\ldots,z_n} = 0, \tag{L}$$

where we have replaced the brace subscript n by the explicit variable dependence $z_1, \ldots, z_n$. Thus

$$\frac{\partial}{\partial t} f_1(z_1;t) = \frac{\partial}{\partial t}\, n\int f_n(z_1,z_2,\ \ldots,\ z_n;t)dz_2\ \ldots\ dz_n \quad \text{(definition of } f_1\text{)}$$

$$= n\int -\left\{f_n(z_1,\ \ldots,\ z_n;t),\ \sum_i H_1(z_i) + \sum_{i<j} H_2(z_i,z_j)\right\}_{z_1,\ldots,z_n} dz_2..dz_n$$

Using the identity $\int \{f,g\}_{z_k}\, dz_k = 0$, we obtain

$$\frac{\partial}{\partial t} f_1(z;t) = -n\int\left\{f_n(z_1,\ \ldots,\ z_n;t), \sum_i H_1(z_i) + \sum_{i<j} H_2(z_i,z_j)\right\}_{z_1} dz_2\ \ldots\ dz_n$$

$$= -n\int \{f_n(z_1,\ \ldots,\ z_n;t),\ H_1(z_1)\}_{z_1}\ dz_2\ \ldots\ dz_n$$

$$-n\int\left\{f_n(z_1,\ \ldots,\ z_n;t),\ \sum_{1<j} H_2(z_1,z_j)\right\}_{z_1} dz_2\ \ldots\ dz_n$$

$$= -\{f_1(z_1;t),\ H_1(z_1)\}_{z_1} - \int\{f_2(z_1,z_2),\ H_2(z_1,z_2)\}_{z_1}\ dz_2\ . \qquad \text{(H1)}$$

This is equivalent to the first hierarchy equation (H1).

For f_2 we similarly compute (assuming $n \geqslant 3$)

$$\frac{\partial}{\partial t} f_2(z_1,z_2;t) = n(n-1)\frac{\partial}{\partial t}\int f_n(z_1,\ \ldots,\ z_n;t)dz_3\ \ldots\ dz_n$$

$$= -n(n-1)\int\left\{f_n(z_1,\ \ldots,\ z_n;t), \sum_i H_1(z_i)\right.$$

$$\left. + \sum_{i<j} H_2(z_i,z_j)\right\}_{z_1,\ldots,z_n} dz_3\ \ldots\ dz_n$$

$$= -n(n-1)\int\{f_n(z_1,\ \ldots,\ z_n;t),\ H_1(z_1) + H_2(z_2)\}_{z_1,z_2}\ dz_3\ \ldots\ dz_n$$

$$- n(n-1)\int\left\{f_n(z_1,\ \ldots,\ z_n;t),\ H_2(z_1,z_2) + \sum_{j>2} H_2(z_1,z_j)\right.$$

$$\left. + \sum_{k>2} H_2(z_2,z_k)\right\}_{z_1,z_2} dz_3\ \ldots\ dz_n$$

$$= -\{f_2(z_1,z_2;t),\ H_1(z_1) + H_2(z_2)\}_{z_1,z_2}$$

$$- \{f_2(z_1,z_2;t),\ H_2(z_1,z_2)\}_{z_1,z_2}$$

$$- \int \{f_3(z_1,z_2,z_3;t),\ H_2(z_1,z_3) + H_2(z_2,z_3)\}_{z_1,z_2}\, dz_3 \,. \qquad \text{(H2)}$$

Let us now verify that the Lie-Poisson structure also gives (H1) and (H2). Indeed, let $F(f_1,f_2)$ be a functional of f_1 and f_2. Then

$$\dot{F} = DF(f_1,f_2)\cdot(\dot{f}_1,\dot{f}_2)$$

$$= \int \frac{\partial f_1}{\partial t}\frac{\partial F}{\partial f_1}\, dz_1 + \int \frac{\partial f_2}{\partial t}\frac{\partial F}{\partial f_2}\, dz_1 dz_2 \,.$$

Also,

$$\{F,\mathcal{H}\}_{A_n^*}(f_1, \ldots, f_n) = \left\langle (f_1, \ldots, f_n), \left[\left(\frac{\delta F}{\delta f_1}, \frac{\delta F}{\delta f_2}, 0, \ldots, 0\right), \left(\frac{\delta \mathcal{H}}{\delta f_1}, \frac{\delta \mathcal{H}}{\delta f_2}, \ldots\right)\right]\right\rangle$$

Now $\frac{\delta \mathcal{H}}{\delta f_1} = H_1$ and $\frac{\delta \mathcal{H}}{\delta f_2} = \frac{1}{2} H_2$, so

$$\{F,\mathcal{H}\}_{A_n^*}(f_1, \ldots, f_n) = \left\langle (f_1,\ldots, f_n), \left[\left(\frac{\delta F}{\delta f_1}, \frac{\delta F}{\delta f_2}, 0,0, \ldots 0\right), \left(H_1, \frac{1}{2} H_2, 0, \ldots, 0\right)\right]\right\rangle .$$

The bracket is obtained by embedding as functions of $z_1, \ldots, z_n$, taking the Poisson bracket there and then identifying the answer as an embedded function. The embedding introduces various combinatorial factors. For example, we find

$$[(K_1(z_1),\ K_2(z_1,z_2),0,0, \ldots),\ (L_1(z_1),\ L_2(z_1,z_2),0, \ldots)]$$

$$= (\{K_1(z_1),\ L_1(z_1)\}_{z_1},\ 2\{K_1(z_1),\ L_2(z_1,z_2)\}_{z_1}$$

$$+ 2\{K_2(z_1,z_2),\ L_1(z_1)\}_{z_1},\ \frac{2}{n-2}\{K_2(z_1,z_2),\ L_2(z_1,z_2)\}_{z_1,z_2}$$

$$+ 4\{K_2(z_1,z_2),\ L_2(z_2,z_3)\}_{z_1},\ 0,\ 0,\ \ldots).$$

<u>Remark</u>. The term $\{K_2(z_1,z_2),\ L_2(z_1,z_2)\}_{z_1,z_2}$ could have gone into the second slot to give another Lie algebra for which the theorem remains valid.

Thus

$$\{F,\mathcal{H}\}_{A_n^*}(f_1, \ldots, f_n) = \int f_1\left\{\frac{\delta F}{\delta f_1}, H_1\right\}_{z_1} dz_1$$

$$+ 2\int f_2\left(\left\{\frac{\delta F}{\delta f_1}(z_1), \frac{1}{2} H_2(z_1,z_2)\right\}_{z_1}\right.$$

$$\left. + \left\{\frac{\delta F}{\delta f_2}(z_1,z_2), H_1(z_1)\right\}_{z_1}\right) dz_1 dz_2$$

$$+ \int f_3 \cdot \frac{2}{n-2}\left\{\frac{\delta F}{\delta f_2}(z_1,z_2), \frac{1}{2} H_2(z_1,z_2)\right\}_{z_1,z_2} dz_1 dz_2 dz_3$$

$$+ \int f_3 \cdot 4\left\{\frac{\delta F}{\delta f_2}(z_1,z_2), \frac{1}{2} H_2(z_1,z_3)\right\}_{z_1} dz_1 dz_2 dz_3$$

$$= -\left[\int \{f_1(z_1), H_1(z_1)\} \frac{\delta F}{\delta f_1} dz_1\right.$$

$$+ \int \{f_2(z_1,z_2), H_2(z_1,z_2)\}_{z_1,z_2} \frac{\delta F}{\delta f_1} dz_1 dz_2$$

$$+ 2\int \{f_2(z_1,z_2), H_1(z_1)\} \frac{\delta F}{\delta f_2}(z_1,z_2)\, dz_1 dz_2$$

$$+ \int \{f_2(z_1,z_2), H_2(z_1,z_2)\} \frac{\delta F}{\delta f_2} dz_1 dz_2$$

$$\left. + 2\int \{f_3(z_1,z_2,z_3), H_2(z_1,z_3)\}_{z_1} \frac{\delta F}{\delta f_2} dz_1 dz_2 dz_3\right] .$$

Comparing coefficients of $\frac{\delta F}{\delta f_1}$ and $\frac{\delta F}{\delta f_2}$ gives (H1) and (H2).

REFERENCES

1. P.C. Clemmow and J.P. Dougherty, Electrodynamics of Particles and Plasmas, Addison-Wesley, (1969).

2. V. Guillemin and S. Sternberg, The moment map and collective motion, Ann. of Phys. 127, (1980), 220-253.

3. S. Ichimaru, Basic Principles of Plasma Physics, Benjamin, (1973).

4. J.E. Marsden, T. Ratiu and A. Weinstein, Semidirect products and reduction in mechanics, Trans. Amer. Math Soc. (to appear), (1983).

5. J. E. Marsden and A. Weinstein, The Hamiltonian structure of the Maxwell-Vlasov equations, Physica D, 4, (1982), 394-406.

6. J.E. Marsden, A. Weinstein, T. Ratiu, R. Schmid and R.G. Spencer, Hamiltonian systems with symmetry, coadjoint orbits and plasma physics,

Proc. IUTAM-ISIMM Symposium on "Modern Developments in Analytical Mechanics," Torino, June 7-11, (1982).

7. P.J. Morrison, Poisson brackets for fluids and plasmas, in Mathematical Methods in Hydrodynamics and Integrability in Related Dynamical Systems, AIP Conf. Proc., #88 La Jolla, M. Tabor (ed)., (1982).

8. P.J. Morrison, The Maxwell-Vlasov equations as a continuous hamiltonian system, Phys. Lett. 80A, (1980), 383-386.

9. P. J. Morrison and J.M. Greene, Noncanical hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics, Phys. Rev., Letters, 45, (1970), 790-794.

10. N.G. Van Kampen and B.U. Felderhof, Theoretical Methods in Plasma Physics, Wiley-Interscience, (1967).

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