QUASILINEAR EVOLUTION OF COLLISIONAL TEARING MODES

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ABSTRACT

A theoretical model is developed to describe the quasilinear evolution of tearing modes in configurations where many modes are unstable at a particular resonant surface. An expression for the positive definite anomalous resistivity is derived and used to study the self-consistent diffusion of the equilibrium magnetic field. It is shown that the anomalous diffusion can be much faster than the collisional diffusion at a low level of magnetic fluctuations.

I. INTRODUCTION

Tearing modes are driven unstable by the gradient of the current density in magnetized plasmas and lead to the formation of magnetic islands which can destroy the equilibrium configuration or enhance the transport of particles and energy.[1-3] The nonlinear evolution of tearing modes has been studied in two different regimes. In the Rutherford regime[4], the initial exponential growth of a single mode saturates when the width of magnetic island w becomes of the order of the width Δ of the tearing layer, i.e., $w \sim x_o \varepsilon^{2/5}$. Here x_o is the width x_o of the current layer, $\varepsilon = \tau_A/\tau_r$, where $\tau_A = x_o/v_A$ is the Alfvén characteristic time, $\tau_r = \mu_o x_o^2/\eta$ is the resistive diffusion time, and η is the plasma resistivity. In the nonlinear stage, w increases linearly in time modifying the equilibrium field until the available magnetic free energy vanishes.[5] A different regime can occur when two or more modes at different rational surfaces grow simultaneously due to linear coupling introduced by the equilibrium geometry.[6] Initially, each individual mode grows exponentially, reaches saturation, and enters the Rutherford regime. However, when the separated islands grow sufficiently large to touch each other, there occurs a destabilization of the interacting modes and the growth again becomes fast leading to the formation of regions with stochastic magnetic field lines.

These two nonlinear regimes are relevant to confinement configurations where the mode rational surfaces, given by $\underset{\sim}{k} \cdot \underset{\sim}{B} = 0$, occur at well separated locations inside the plasma column for different values of the wavenumber $\underset{\sim}{k}$. This may be the typical situation in tokamaks where the toroidal component B_t of the confining magnetic field $\vec{B}$ is approximately constant and the value of the poloidal component B_p increases monotonically from the magnetic axis to the plasma boundary. The mode rational surface for each $\underset{\sim}{k}$ occurs where the pitch of the helical perturbation matches the pitch of the helical magnetic field line. However, in confinement configurations where one component of the magnetic field vanishes at some position other than the magnetic axis, the resonance surfaces for all modes with $\underset{\sim}{k}$ parallel to the vanishing component occur at the same position. Typical examples are the current sheet with a guide field[7], the field configuration in the current penetration phase of tokamaks[8], and the reversed-field pinch.[9] In this case, a one-dimensional spectrum of modes with different values of k can grow simultaneously and interfere strongly with each other. In this regime, it is not clear whether the nonlinear evolution of each mode will proceed into the Rutherford regime. In this paper we study this problem by developing a quasilinear theory appropriate to the many mode case.

The physical mechanisms involved in the collisional tearing instabilities are well known.[1-12] Let us consider a plane current sheet parallel to a guide magnetic field $\underset{\sim}{B}_o$. The magnetic field produced by the plasma current has opposite signs on the two sides of the sheet. Around the point where this field vanishes, magnetic perturbations induce an electric field parallel to $\underset{\sim}{B}_o$. This field drives a large parallel current due to the electron motion which is, however, limited by the resistivity. The ion response to the electric field is due to the $\underset{\sim}{E}\times\underset{\sim}{B}$ and polarization drifts. The polarization current is perpendicular to the magnetic field $\underset{\sim}{B}_o$ and combines with the perturbed electron current parallel to $\underset{\sim}{B}_o$ to provide charge neutrality. The resulting perturbed field configuration is formed by a chain of magnetic islands. The overall fluid motion is made of convective cells with fluid entering the islands at the X-points and exiting them at the O-points. We expect that the interaction of many modes will modify the growth of an individual island through a diffusion process that changes the equilibrium configuration.

We discuss the basic equations that describe the evolution of tearing modes in Sec. II. In Sec. III we derive the quasilinear equations for the many mode case and obtain a diffusion equation for the magnetic flux involving an anomalous resistivity. The explicit form for the anomalous resistivity is derived in Sec. IV from the solution of the linear stability problem. It is then shown that the anomalous resistivity is always positive for a quasilinear spectrum of tearing modes. The positive resistivity of tearing modes derived here is in contrast to the negative resistivity derived by Biskamp and Welter[13] from a secondary spectrum of incompressible two-dimensional resistive magnetohydrodynamic turbulence. In Sec. V we analyze further the nature of the anomalous transport by considering the motion of test particles in the quasilinear spectrum. The discussion and conclusions are presented in Sec. VI.

II. NONLINEAR TEARING MODE EQUATIONS

We consider a two-dimensional equilibrium configuration with the magnetic field given by $B_o\hat{z} + B_y(x)\hat{y}$, where B_o = const and $B_y(x)$ reverses sign somewhere inside the plasma as shown in Fig. 1. The electromagnetic perturbations associated with tearing modes are conveniently described in terms of the magnetic flux function ψ($\hat{z}$-component of the vector potential) and the electrostatic potential ϕ.[10-11] The perturbed electric field $\underset{\sim}{E}$ and the total magnetic field $\underset{\sim}{B}$ (equilibrium plus perturbed) are given in terms of ψ and ϕ by

$$\underset{\sim}{E} = -\nabla\phi + \frac{\partial\psi}{\partial t}\hat{z} \tag{1}$$

and

$$\underset{\sim}{B} = \hat{z} \times \nabla\psi + \underset{\sim}{B}_o\hat{z} \; . \tag{2}$$

According to the physical mechanisms discussed in the introduction, the basic equations governing the evolution of tearing modes can be simply derived from Ampère's law, $\nabla \times \underset{\sim}{B} = \mu_o\underset{\sim}{j}$, Ohm's law, $\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} = \eta_c\underset{\sim}{j}$, and the quasineutrality condition, $\nabla \cdot \vec{j} = 0$. Here $\vec{j}$ is the plasma current density and η_c is the plasma collisional resistivity which is assumed constant. Substituting Eq. (2) into Ampère's law, we obtain

$$\nabla^2\psi = \mu_o j_{\parallel} \; , \tag{3}$$

where $j_{\parallel}$ is the component of the current density parallel to the magnetic field. The parallel component of the electric field is given by $E_{\parallel} = \underset{\sim}{E} \cdot \underset{\sim}{B}/B$; substituting this into Ohm's law and using Eq. (1), it follows that

$$-\frac{\partial\phi}{\partial z} + \frac{1}{B_o}\,[\phi,\psi] + \frac{\partial\psi}{\partial t} = \eta_c j_{\parallel} \,, \tag{4}$$

where we have defined the Poisson brackets

$$[A,B] = \frac{\partial A}{\partial x}\,\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\,\frac{\partial B}{\partial x} \,.$$

Finally, we recall that the current density perpendicular to the field lines is given by the ion polarization current,

$$\underset{\sim}{j}_{\perp} = \frac{m_i n}{B_o^2}\,\frac{\partial \underset{\sim}{E}_{\perp}}{\partial t} = -\frac{1}{\mu_o v_A^2}\,\nabla_{\perp}\frac{\partial\phi}{\partial t} \,, \tag{5}$$

where m_i is the ion mass, n is the plasma density, and $v_A = B_o/\sqrt{\mu_o n m_i}$ is the Alfvén velocity. Thus, the conservation of charge in the quasineutral approximation,

$$\nabla \cdot \vec{j} = \nabla \cdot \vec{j}_{\perp} + \vec{B} \cdot \nabla(j_{\parallel}/B_o) = 0 \,,$$

can be written

$$\frac{1}{\mu_o v_A^2}\,\nabla^2\,\frac{\partial\phi}{\partial t} = \frac{\partial j_{\parallel}}{\partial z} + \frac{1}{B_o}\,[\psi,\nabla^2\psi] \,, \tag{6}$$

where we have used Eq. (3).

Equations (4) and (6) are the basic equations describing the nonlinear evolution of the ψ and ϕ fields. In the following, we will consider only two-dimensional perturbations. In this case, using Eq. (3) and $\partial/\partial z = 0$, Eqs. (4) and (6) reduce to

$$\frac{\partial\psi}{\partial t} + \frac{1}{B_o}[\phi,\psi] = \frac{\eta_c}{\mu_o}\nabla^2\psi \tag{7}$$

and

$$\frac{1}{\mu_o v_A^2}\nabla^2\frac{\partial\phi}{\partial t} = \frac{1}{B_o}[\psi,\nabla^2\psi] \ , \tag{8}$$

respectively. The term $[\phi,\psi]/B_o$ in Eq. (7) is the convective part $\underset{\sim}{v}\cdot\nabla\psi$ of the total time derivative of ψ. To see this, we recall that the main fluid motion is given by the $\vec{E}\times\vec{B}$ drift. Using Eq. (1), it follows that the drift velocity is given to the lowest order by

$$\underset{\sim}{v} = \frac{\hat{z}\times\nabla\phi}{B_o} \tag{9}$$

and thus $[\phi,\psi]/B_o = \vec{v}\cdot\nabla\psi$.

Equations (7) and (8) plus the appropriate boundary conditions constitute a mixed initial value and boundary value nonlinear problem which cannot be solved analytically. In the following we will solve this problem under the condition of weak nonlinearity or, equivalently, in the quasilinear approximation. Before introducing the quaslinear approximation we derive the basic conservation laws that follow from Eqs. (7) and (8).

Integrating ψ times Eq. (7) over the entire plasma cross-section and using the following property of the Poisson bracket $\psi[\phi,\psi] = [\phi,\psi^2]/2$, we obtain

$$\frac{\partial}{\partial t} \int dxdy \frac{\psi^2}{2} = - \frac{\eta_c}{\mu_o} \int dxdy(\nabla\psi)^2 \quad . \tag{10}$$

This equation governs the resistive diffusion of the poloidal flux which occurs on the resistive time scale τ_r.

To obtain the rate of energy dissipation, we first define the magnetic energy as

$$W_B = \frac{1}{2\mu_o} \int dxdy(\nabla\psi)^2 \tag{11}$$

and the fluid kinetic energy as

$$T = \int dxdy \frac{nm_i}{2} v^2 = \frac{1}{2\mu_o v_A^2} \int dxdy(\nabla\phi)^2 \quad . \tag{12}$$

Then, multiplying Eq. (7) by $\nabla^2\psi$ and Eq. (8) by ϕ, integrating the resulting equations over the entire plasma cross-section, and adding the results, we get the energy conservation equation

$$\frac{d}{dt}(W_B + T) = - \frac{\eta_c}{\mu_o^2} \int dxdy(\nabla^2\psi)^2 + \oint \frac{\partial\psi}{\partial t} \nabla\psi \cdot \hat{n} \, d\ell \ , \tag{13}$$

where the last term represents the energy transported away from the plasma. For instability, it is necessary that $dT/dt > 0$; it is easy to see that this corresponds to $\int dxdy j_{\parallel} E_{\parallel}^{\phi} > 0$, where $E_{\parallel}^{\phi}$ is the electrostatic part of the parallel component of the electric field. If the system is isolated, the last term in Eq. (13) vanishes and the instability can grow only if the magnetic energy can decrease.

III. QUASILINEAR EQUATIONS

To study the evolution of tearing modes in the weak nonlinear regime, let us derive the quasilinear equations from Eqs. (7) and (8) following an approach similar to the one described in Ref. (14). Any quantity A is split in an average plus a perturbation, $A(x,y,t) = \langle A\rangle + \delta A$, where

$$\langle A\rangle = \frac{1}{L_y} \int dyA \ , \tag{14}$$

$$\delta A = \sum_k A_k(x) \exp\left[iky + \int^t \gamma_k(t')dt'\right] \ , \tag{15}$$

and L_y is the periodic length in the $\hat{y}$-direction. Substituting $\psi = \langle\psi\rangle + \delta\psi$ and $\phi = \langle\phi\rangle + \delta\phi$ into Eqs. (7) and (8) and taking the average of the resulting equations, we obtain the equations for $\langle\psi\rangle$ and $\langle\phi\rangle$,

$$\frac{\partial\langle\psi\rangle}{\partial t} - \frac{1}{B_o} \frac{\partial}{\partial x} \langle\frac{\partial\delta\phi}{\partial y} \delta\psi\rangle = \frac{\eta_c}{\mu_o} \frac{\partial^2\langle\psi\rangle}{\partial x^2} \tag{16}$$

and

$$\frac{1}{\mu_o v_A^2} \frac{\partial^2}{\partial x^2} \frac{\partial\langle\phi\rangle}{\partial t} = - \frac{\partial}{\partial x} \langle\frac{\partial\,\delta\psi}{\partial y} \nabla^2\,\delta\psi\rangle \ , \tag{17}$$

where we have used the following property of the Poisson brackets

$$\langle[f,g]\rangle = - \frac{\partial}{\partial x} \langle\frac{\partial f}{\partial y}\, g\rangle = \frac{\partial}{\partial x} \langle f\, \frac{\partial g}{\partial y}\rangle \ . \tag{18}$$

To derive the quasilinear equations for $\delta\psi$ and $\delta\phi$, we subtract the averaged eqs. (16) and (17) from the original equations for ψ and ϕ, respectively, and neglect high-order mode coupling terms. We obtain

$$\frac{\partial\,\delta\psi}{\partial t} + \frac{1}{B_o} [\,\delta\phi,\langle\psi\rangle] + \frac{1}{B_o} [\langle\phi\rangle,\delta\psi] = \frac{\eta_c}{\mu_o} \nabla^2\,\delta\psi \tag{19}$$

and

$$\frac{1}{\mu_o v_A^2} \nabla^2 \frac{\partial\,\delta\phi}{\partial t} = \frac{1}{B_o} [\,\delta\psi,\nabla^2\langle\psi\rangle] + \frac{1}{B_o} [\langle\psi\rangle,\nabla^2\,\delta\psi] \ . \tag{20}$$

The term $[\langle\phi\rangle,\delta\psi]$ represents the influence of equilibrium flows on the linear behavior of tearing modes. Although this term is usually neglected, it can have a stabilizing or destabilizing effect on tearing modes and also influence its quasilinear evolution.[14-16] In particular, in the presence of a background turbulence of ideal magnetohydrodynamic waves with $\langle\phi\rangle \neq 0$, the tearing modes may become more unstable.[17-19] Here we consider the quasilinear evolution of a spectrum of pure tearing modes, neglecting the equilibrium diffusive flow. Accordingly, the effect of $\langle\phi\rangle$ vanishes, as will be shown in Sec. IV.

A. Evolution equation of the poloidal flux $\langle\psi\rangle$

In the nonlinear stage of a single tearing mode, the width w_k of a magnetic island increases in time, modifying the equilibrium field until the available magnetic energy vanishes.[10] In the presence of a spectrum of tearing modes, the rate of change of $\langle\psi\rangle$ in Eq. (16) depends not only on the collisional resistivity η_c but also on the anomalous flux $F_\psi^a = \langle v_x \psi\rangle$ caused by the nonlinear interaction of the modes. To show the effect of the anomalous transport of flux F_ψ^a explicitly, let us evaluate the nonlinear term in Eq. (16). From the general expression for the perturbations $\delta\psi$ and $\delta\phi$ given by Eq. (15) and the definition of the average in Eq. (14), we have

$$F_\psi^a = -\frac{1}{B_o}\left\langle\frac{\partial\delta\phi}{\partial y}\,\delta\psi\right\rangle = \frac{i}{B_o}\sum_k k(\phi_k^*\psi_k - \phi_k\psi_k^*)\,\exp(2\int^t \gamma_k dt') \; . \tag{21}$$

Substituting this into Eq. (16), we obtain

$$\frac{\partial\langle\psi\rangle}{\partial t} = -\frac{\partial F_\psi}{\partial x} \qquad \text{with} \qquad F_\psi = F_\psi^a + F_\psi^c \; , \tag{22}$$

where the fluxes F_ψ^a and F_ψ^c are given by

$$F_{\psi}^{a} = -2\sum_{k} \frac{k}{B_o} \mathrm{Im}(\phi_k^* \psi_k) \exp(2\int^t \gamma_k dt') \tag{23}$$

and

$$F_{\psi}^{c} = -\frac{\eta_c}{\mu_o} \frac{\partial\langle\psi\rangle}{\partial x} . \tag{24}$$

The cross correlation $\sum k \, \mathrm{Im}(\phi_k^* \psi_k)$ must be calculated from the quasilinear system of Eqs. (19) and (20). Making the long wavelength approximation, $\partial^2\psi_k/\partial x^2 >> k^2\psi_k$, which is appropriate for incompressible tearing modes[1], Eq. (19) reduces to

$$\left(\gamma - \frac{\eta_c}{\mu_o} \frac{\partial^2}{\partial x^2}\right)\psi_k = i \frac{k}{B_o} \phi_k \frac{\partial\langle\psi\rangle}{\partial x} - i \frac{k}{B_o} \psi_k \frac{\partial\langle\phi\rangle}{\partial x} . \tag{25}$$

Using the general expression for $\delta\psi$, Eq. (15), in Eq. (17), we obtain

$$\frac{1}{\mu v_A^2} \frac{\partial^2\langle\phi\rangle}{\partial x \partial t} = \mathscr{C}(t) , \tag{26}$$

where

$$\mathscr{C}(t) = i\sum_{k} k\left(\psi_k^* \frac{\partial^2\psi_k}{\partial x^2} - \psi_k \frac{\partial^2\psi_k^*}{\partial x^2}\right) , \tag{27}$$

where $\psi_k(x,t) = \psi_k(x) \exp(\int^t \gamma_k dt')$. Substituting Eq. (26) into

Eq. (25), multiplying the resulting equation by ϕ_k^*, and calculating F_ψ^a, the diffusion equation (22) for $\langle\psi\rangle$ becomes

$$\frac{\partial\langle\psi\rangle}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\eta(x,t)}{\mu_o}\frac{\partial}{\partial x}\langle\psi\rangle\right] = -\frac{\partial F_\psi(x,t)}{\partial x}\,, \tag{28}$$

where

$$\eta(x,t) = \eta_c + \eta_a(x,t) = \eta_c + 2\sum_k \frac{\mu_o k^2 |\phi_k(x,t)|^2}{B_o^2\left[\gamma_k + \frac{ikv_A^2}{B_o}\int^t \mathscr{C} dt' + \frac{\eta_c}{\mu_o}k_\perp^2\right]}\,, \tag{29}$$

and

$$k_\perp^2 = -\frac{\partial^2\psi_k/\partial x^2}{\psi_k}\,. \tag{30}$$

The second term on the right-hand side of Eq. (29) is the anomalous resistivity η_a due to the electromagnetic perturbations associated with the tearing modes. Because these perturbations are localized in a narrow layer around the singular surface, η_a represents an increase in the value of the resistivity in that layer and thus causes an enhanced diffusion of $\langle\psi\rangle$ over the classical value. The magnitude of η_a depends on the amplitude of the spectrum $|\phi_k|^2$.

B. Solution of the linear equations

To calculate the anomalous resistivity η_a, we have to solve the linear Eqs. (19) and (20) to self-consistently determine the growth rate γ_k and the "perpendicular wavenumber $k_\perp$". Evaluating the four Poisson brackets appearing in Eqs. (19) and (20), we find the following set of equations relating the Fourier amplitudes ψ_k and ϕ_k:

$$\gamma\psi_k = \frac{ik}{B_o}\frac{\partial\langle\psi\rangle}{\partial x}\phi_k - i\frac{k}{B_o}\frac{\partial\langle\phi\rangle}{\partial x}\psi_k + \frac{\eta_c}{\mu_o}\left(\frac{\partial^2}{\partial x^2} - k^2\right)\psi_k \qquad (31)$$

and

$$\frac{1}{\mu_o v_A^2}\gamma\left(\frac{\partial^2}{\partial x^2} - k^2\right)\phi_k = -i\frac{k}{B_o}\frac{\partial^3\langle\psi\rangle}{\partial x^3}\psi_k + i\frac{k}{B_o}\frac{\partial\langle\psi\rangle}{\partial x}\left(\frac{\partial^2}{\partial x^2} - k^2\right)\psi_k . \qquad (32)$$

The term $\partial\langle\phi\rangle/\partial x$ in Eq. (31) represents the effect of equilibrium flows. When this term is kept, the eigenfunctions ϕ_k and ψ_k do not have a definite parity.[14] In this case, the quantity $\mathscr{C}(t)$ defined by Eq. (26) can be nonzero. As mentioned earlier, we assume here that $\langle\phi\rangle = 0$ at t=0 and this leads to the vanishing of $\mathscr{C}(t)$.

The solutions of Eqs. (31) and (32) are well-known.[1,21] In this section we reproduce the basic results required to obtain the relationships needed for the anomalous flux F^a_ψ. The boundary layer procedure for solving Eqs. (31) and (32) is to consider a resistive layer around the point where $d\langle\psi\rangle/dx$ vanishes surrounded by an ideal region where resistive and inertial effects are negligible. The eigenvalue γ_k is determined by the asymptotic matching of the solutions of the relevant equations in the two regions.

1. External MHD region

The equation to be solved in the ideal region is obtained by taking $\gamma = \eta_c = \phi_k = 0$. Then, from Eq. (32) we obtain a simplified version of Newcomb's equation[20],

$$\frac{d^2\psi_k}{dx^2} - \left(k^2 + \frac{\partial^3\langle\psi\rangle/\partial x^3}{\partial\langle\psi\rangle/\partial x}\right)\psi_k = 0 . \tag{33}$$

For tearing modes, the solution of Eq. (33) is continuous at the singular point and has discontinuous first derivatives. The amount of discontinuity is specified by the parameter

$$\Delta'(k) = \frac{\left.\frac{d\psi_k}{dx}\right|_+ - \left.\frac{d\psi_k}{dx}\right|_-}{\psi_{ko}} , \tag{34}$$

where ψ_{ko} is the value of the ψ_k at the singular point. The value of $\Delta'(k)$ depends on the profile of the equilibrium current density. The solution of Eq. (33) for a simple equilibrium model is presented in Sec. VI.

2. Resistive layer

We assume that the equilibrium profile is such that $B_y(x)$ vanishes linearly at the singular point x=0. Then, in the resistive layer around the origin, the equilibrium flux function $\langle\psi\rangle$ can be represented by $\langle\psi\rangle = B_y^o x^2/2x_o$, where B_y^o is a constant. Following boundary layer theory, we rescale the variables inside the resistive layer to make the small terms in Eqs. (31) and (32) that are proportional to η_c and γ of the order of the other terms. The relevant small parameter in these equations is $\varepsilon = \tau_A/\tau_r$, where the Alfvén time and the resistive time are defined by

$$\tau_A = \frac{x_o\sqrt{\mu_o m_i n}}{B_y^o} \qquad \text{and} \qquad \tau_r = \frac{\mu_o x_o^2}{\eta_c} . \tag{35}$$

The rescaling of the inner layer variables is carried out by defining the following order one normalized quantities:

$$\Lambda = \varepsilon^{-a}\tau_A\gamma , \qquad \Phi_k = i\ \varepsilon^{-c}\frac{B_y^o}{B_o}\ \frac{k}{\gamma}\ \phi_k ,$$

$$\xi = \varepsilon^{-b}\frac{x}{x_o} , \qquad \psi_{k1} = \varepsilon^{-d}\ (\psi_k - \psi_{ko}) , \tag{36}$$

where a, b, c, and d are constants to be determined and ψ_{ko} is the amplitude of ψ_k at x=0. Substituting the normalized quantities just defined into Eqs. (31) and (32), we obtain

$$\psi_{ko} = \varepsilon^{b+c}\xi\Phi_k + \frac{\varepsilon^{1-a-2b+d}}{\Lambda}\frac{d^2\psi_{k1}}{d\xi^2} \tag{37}$$

and

$$\frac{d^2\Phi_k}{d\xi^2} = -\frac{k^2x_o^2}{\Lambda^2}\varepsilon^{b-c-2a+d}\xi\frac{d^2\psi_{k1}}{d\xi^2}\ , \tag{38}$$

to lowest order in the parameter ε. To have all terms of the same order, we require that $a = 3/5$ and $b = d = -c = 2/5$.[21] Then Eqs. (37) and (38) reduce to

$$\frac{d^2\psi_{k1}}{d\xi^2} = \Lambda\psi_{ko}\left(1 - \frac{\xi\Phi_k}{\psi_{ko}}\right) \tag{39}$$

and

$$\frac{d^2\Phi_k}{d\xi^2} - \frac{k^2x_o^2}{\Lambda}\xi^2\Phi_k = -\frac{k^2x_o^2}{\Lambda}\xi\psi_{ko}\ . \tag{40}$$

The solution[1,21] of Eq. (40) is given by

$$\Phi_k(\xi) = \frac{\psi_{ko}}{(\Lambda/k^2x_o^2)^{1/4}}\chi(z)\ , \tag{41}$$

where

$$\chi(z) = z\int_0^{1/2}(1-4t^2)^{-1/4}\, e^{-z^2 t}\, dt \qquad \text{with} \qquad z = \left(\frac{k^2 x_o^2}{\Lambda}\right)^{1/4} \xi \ . \tag{42}$$

We note that $\chi(z) \geqslant 0$ for $z \geqslant 0$ and the limiting forms of $\chi(z)$ are

$$\chi(z) \simeq 0.6z - \frac{z^3}{6} - 0.06z^5 + \ldots \qquad \text{for } |z| \ll 1$$

and

$$\chi(z) \simeq \frac{1}{z} + \frac{2}{z^5} + \ldots \qquad \text{for } |z| \gg 1 \ . \tag{43}$$

The eigenvalue $\Lambda(k)$ is obtained by equating the discontinuity in the logarithmic derivative of ψ_{k1} for $\xi \to \pm\infty$ to the external value of $\Delta'(k)$ properly normalized to the inner layer variables. The result is[1,10,21]

$$\Lambda(k) = \left[\frac{\Gamma(1/4)}{\pi\Gamma(3/4)}\, x_o \Delta'(k)\right]^{4/5} (k^2 x_o^2)^{1/5} \ , \tag{44}$$

and thus, $\gamma(k) = \varepsilon^{3/5}\Lambda(k)/\tau_A$ from Eq. (36) with $a = 3/5$.

IV. ANOMALOUS RESISTIVITY

We have now all the quantities required to calculate the anomalous resistivity given by Eq. (29). Using Eqs. (36) and (41), we obtain

$$\frac{k^2|\phi_k|^2}{B_o^2} = v_{Ay}^2 \frac{|\psi_{ko}|^2}{(B_y^o x_o)^2} \Lambda_k^{3/2} (kx_o)\ \chi^2(z) \ . \tag{45}$$

Substituting Eq. (41) into Eq. (39) and using the definition of the normalized quantities, Eq. (36), the second derivative of ψ_k and $k_\perp^2$ [Eq. (30)] can be calculated giving

$$k_\perp^2 = -\frac{\Lambda}{x_o^2}[1 - z\chi(z)]\varepsilon^{-2/5} \ ,$$

where z is given by Eq. (42). Calculating the fluctuation propagator $(\gamma_k + \eta_c k_\perp^2/\mu_o)$ occurring in Eq. (29) for $\eta(x,t)$ we find

$$\gamma_k + \eta_c k_\perp^2/\mu_o = \frac{\Lambda}{\tau_A}\left(\varepsilon^{3/5} - \varepsilon^{3/5}[1 - z\chi(z)]\right) = \gamma_k z\chi(z) \ . \tag{46}$$

It is now evident from Eqs. (36), (39), and (41) that there is no phase shift between ψ_k and $d^2\psi_k/dx^2$ for the eigenfunctions. That the same is true for the solution outside the resistive layer follows directly from Eq. (33). Thus, the quantity $\mathscr{C}(t)$ defined in Eq. (27) vanishes identically.

Substituting Eqs. (46) and $\mathscr{C}(t) = 0$ into Eq. (29), we obtain the expression for the anomalous resistivity

$$\eta_a(x,t) = 2\varepsilon^{-1/5} \frac{\mu_o x_o^2}{\tau_A} \sum_k \left(\frac{\psi_{ko}(t)}{x_o B_y^o}\right)^2 \Lambda_k^{1/2} \, (kx_o) \, \frac{\chi(z)}{z} , \tag{47}$$

where we have used the definition of τ_r. Given an equilibrium profile, specified by the equilibrium flux function $\langle\psi\rangle$, we calculate $\Delta'(k)$ to determine $\Lambda(k,t)$. Then $\eta_a(x,t)$ can be determined as a function of x for a given fluctuation spectrum $(\psi_{ko}/x_o B_y^o)$. The relevant equations to calculate $\eta_a(x,t)$ are Eqs. (33), (34), (36), (42), and (44).

The overall characteristics of $\eta_a(x,t)$ follow from the behavior of $\chi(z)$. Considering $\Delta'(k) \neq 0$ and using Eqs. (42) and (43), we obtain $\eta_a \sim 1/x^2$ for $|z| >> 1$ and $\eta_a \approx$ const for $|z| << 1$. Thus, the function $\eta_a(x,t)$ is peaked at x=0 and decays rapidly as $|x| \to \infty$ for fixed t. At marginal stability, i.e., $\Delta'(k) \to 0$ for fixed x and t, we have $z \sim \Lambda^{-1/4} \to \infty$ and thus, $\eta_a \sim \Lambda \to 0$. This means that only unstable modes contribute to the anomalous resistivity. Furthermore, it is clear from Eq. (47) that η_a is always positive.

Let us estimate the magnitude of η_a as compared to η_c for a simple equilibrium model. The full width of a magnetic island, in the thin island approximation, is given by

$$w_k = 4 \left(\frac{\psi_{ko}(t)}{\partial^2\langle\psi\rangle/\partial x^2}\right)^{1/2} , \tag{48}$$

where $\partial^2\langle\psi\rangle/\partial x^2$ is calculated at x=0. Using the approximation $\langle\psi\rangle = B_y^o x^2/2x_o$, valid inside the resistive layer, in Eq. (48), we obtain

$$\frac{\psi_{ko}}{x_o B_y^o} = \frac{1}{16} \left(\frac{w_k}{x_o}\right)^2 . \tag{49}$$

Substituting this relationship into Eq. (47), dividing both sides by η_c, and using the definitions of ε and τ_r, we get

$$\frac{\eta_a(x,t)}{\eta_c} = \frac{\varepsilon^{-6/5}}{2^7} \sum_k \left(\frac{w_k}{x_o}\right)^4 \Lambda_k^{1/2} (kx_o) \frac{\chi(z)}{z} . \tag{50}$$

The factor $\varepsilon^{-6/5}/2^7$ is very large for typical values of the inverse magnetic Reynolds number ε. However, the width w_k of the magnetic islands associated with individual unstable modes is much smaller than x_o for the quasilinear calculation to remain valid.

For a given fluctuation spectrum the magnitude of η_a/η_c depends on the form factor $S(k) = \sqrt{\Lambda(k)}\,(kx_o)\,\chi(z)/z$. Let us consider $\Delta'(k) = 15(1 - kx_o)/x_o$, as shown in Fig. 2. This is a good model for actual configurations. In Fig. 2 we show the corresponding $S(k)$ calculated at $x=0$, where η_a is maximum. The form factor $S(k)$ has a single maximum and vanishes at $kx_o = 0$ and $kx_o = 1$. The same type of dependence is obtained for other models of $\Delta'(k)$. The wavenumber dependence of $\Lambda(k)$ and $S(k)$ for the Harris equilibrium is shown in Fig. 3. Usually, the largest growth rates occur for $kx_o \to 0$; however, since $S(k)$ vanishes in this limit, the contribution of this part of the spectrum to the anomalous resistivity is not necessarily larger than the contribution from shorter wavelengths $kx_o \sim 1/2$. The maximum value of $S(k)$ is a number of order one. Thus, considering $\varepsilon = 10^{-3}$-10^{-4}, it is clear from Eq. (50) that η_a/η_c can be greater than unity for a spectrum of thin islands, $|w_k/x_o| \ll 1$.

V. MOTION OF THE PARTICLES

To obtain a more direct understanding of the anomalous transport mechanism described by the quasilinear equations we consider the motion of test particles e_α, m_α in the spectrum of tearing modes. The trajectory of the test particle is given by the guiding center equations of motion $d_t \underset{\sim}{r} = v_\parallel(\underset{\sim}{B}/B) + (\underset{\sim}{E}\times\underset{\sim}{B})/B^2$ and $d_t v_\parallel = (e_\alpha/m_\alpha)E_\parallel$. For the two-dimensional system considered, z is an ignorable coordinate and the equations of motion reduce to

$$\frac{dv_\parallel^\alpha}{dt} = \frac{e_\alpha}{m_\alpha} E_\parallel = \frac{e_\alpha}{m_\alpha} \left(\frac{\partial\psi}{\partial t} + \frac{1}{B_o} [\phi,\psi]\right)$$

$$\frac{dx^\alpha}{dt} = -\frac{1}{B_o}\frac{\partial\phi}{\partial y} - \frac{v_\parallel^\alpha}{B_o}\frac{\partial\psi}{\partial y}$$

$$\frac{dy^\alpha}{dt} = \frac{1}{B_o}\frac{\partial\phi}{\partial x} + \frac{v_\parallel^\alpha}{B_o}\frac{\partial\psi}{\partial x}, \qquad (51)$$

where the small polarization and curvature drifts are neglected.

From the analysis in Sec. III the formulas for the tearing mode fields are

$$\frac{\psi(x,y,t)}{B_y^o x_o} = \ell n\left[\cosh\left(\frac{x}{x_o}\right)\right] + \sum_k \frac{\psi_k(t)}{B_y^o x_o}\cos(ky + \beta_k) \qquad (52)$$

and

$$\frac{\phi(x,y,t)}{B_o} = \frac{x_o^2 \varepsilon^{1/5}}{\tau_A} \sum_k \frac{\psi_k(t)}{B_y^o x_o} \frac{\Lambda_k^{3/4}}{(kx_o)^{1/2}} \chi\left(\frac{x}{\Delta_k}\right) \sin(ky + \beta_k) \quad , \qquad (53)$$

using Eqs. (36)-(42) and defining $\Delta_k = \varepsilon^{2/5} x_o \Lambda_k^{1/4}/(kx_o)^{1/2}$.

Thermal electrons have $v_\parallel \psi >> \phi$ and their x,y motion from Eq. (51) reduces to

$$\frac{dx^e}{dt} = \frac{v_\parallel B_y^o}{B_o} \sum_k \frac{k\psi_k(t)}{B_y^o} \sin(ky + \beta_k)$$

$$\frac{dy^e}{dt} = \frac{v_\parallel B_y(x)}{B_o} \quad . \qquad (54)$$

For constant $v_\parallel$ the motion in Eq. (54) is extensively studied in the literature on the stochasticity of magnetic field lines and the associated anomalous diffusion of electrons. In the limit $\gamma_k << k_\parallel v_\parallel = kv_\parallel B_y/B_o$ the fields are essentially static with respect to the electron motion and the results of Rechester et al.[22] are directly applicable.

Thermal ions have $v_\parallel \psi << \phi$ and move with the $\underset{\sim}{E}\times\underset{\sim}{B}$ motion. From Eqs. (51) and (53) the $\underset{\sim}{E}\times\underset{\sim}{B}$ motion reduces to

$$v_x = \frac{dx^i}{dt} = - \frac{x_o \varepsilon^{1/5}}{\tau_A} \sum_k \frac{\psi_k(t)(kx_o)^{1/2} \Lambda_k^{3/4}}{B_y^o x_o} \chi(\frac{x}{\Delta_k}) \cos(ky + \beta_k)$$

$$v_y = \frac{dy^i}{dt} = \frac{x_o \varepsilon^{-1/5}}{\tau_A} \sum_k \frac{\psi_k(t) \Lambda_k^{1/2}}{B_y^o x_o} \chi'(\frac{x}{\Delta_k}) \sin(ky + \beta_k) \quad . \tag{55}$$

The motion in Eq. (55) describes convection about a superposition of randomly phased convective cells of scale $(\Delta x, \Delta y) = (\Delta_k, \pi/k)$. The convection patterns bring plasma and magnetic flux ψ into and out of the resistive layer. During this flow the net magnetic flux convected across the surface x = const is readily calculated from Eqs. (52) and (55) as

$$F_\psi^a(x) = \langle v_x \psi \rangle = - B_y^o x_o (\frac{x_o \varepsilon^{1/5}}{\tau_A}) \sum_k (\frac{\psi_k(t)}{B_y^o x_o})^2$$

$$\Lambda_k^{3/4}(kx_o)^{1/2} \chi(\frac{x}{\Delta_k}) \langle \cos^2(ky + \beta_k) \rangle \; , \tag{56}$$

which reduces to

$$F_\psi^a = - \eta_a \frac{\partial \langle \psi \rangle}{\partial x} \; ,$$

with $\eta_a(x,t)$ given by Eq. (47).

From the point of view of the electron motion, the anomalous resistivity may be viewed as the effect of following the meandering magnetic field lines, which changes the local $E_\parallel = \underset{\sim}{E} \cdot \underset{\sim}{B}/B$, accelerating the electrons from the ambient field $\langle \underset{\sim}{E} \rangle$. From Eq. (4) the mean parallel electric field is

$$\langle E_{\parallel} \rangle = \frac{\partial \langle \psi \rangle}{\partial t} + \frac{\partial}{\partial x} \langle v_x \psi \rangle = \frac{\partial \langle \psi \rangle}{\partial t} - \frac{\partial}{\partial x} \left(\eta_a \frac{\partial \langle \psi \rangle}{\partial x} \right) . \tag{57}$$

Thus, in the regime where $\eta_a >> \eta_c$ the quasilinear evolution of the flux $\langle \psi \rangle$ assures that the mean parallel electric field accelerating the electrons vanishes. The collective effect of the tearing mode is to produce an effective anomalous resistivity mechanism that prevents the acceleration of the electrons from the strong ambient electric field $\langle \underset{\sim}{E} \rangle = \hat{z}\ \partial \langle \psi \rangle / \partial t$.

VI. CONCLUSIONS

We investigate the quasilinear evolution of a spectrum of tearing modes produced by a local current sheet. The problem is formulated in terms of the two dimensional nonlinear Eqs. (7) and (8) derived from incompressible resistive magnetohydrodynamics. We neglect the effects of density and temperature gradients and finite ion gyroradius. The two quasilinear fluxes that govern the evolution of the background are $\langle v_x \psi \rangle$ and $\langle j_z B_x \rangle$. From the asymptotic boundary layer solution of the problem we calculate these fluxes showing that $\langle j_z B_x \rangle = 0$ for initial equilibria with vanishing velocity fields.

The quasilinear value for the transport of poloidal flux $F_{\psi}^{a} = \langle v_x \psi \rangle = - \eta_a\ \partial \langle \psi \rangle / \partial x$ is calculated in Eq. (29) and reduced in Eq. (47). From the structure of tearing mode wave functions we show that the anomalous resistivity $\eta_a(x,t)$ is positive definite. The positivity of $\eta_a(x,t)$ follows from the particular $\delta \underset{\sim}{v} - \delta \underset{\sim}{B}$ correlations established in a tearing mode and is not satisfied by other $\delta \underset{\sim}{v} - \delta \underset{\sim}{B}$ fluctuations. In a related work[23] we show the relationship between the positive anomalous resistivity of tearing modes with the negative anomalous resistivity of a different form of resistive magnetohydrodynamic turbulence recently analyzed by Biskamp and Welter.[13] In Ref. 23 extension of the present theory to the collisionless regime is discussed.

The quasilinear theory presented here predicts that $\eta_a > \eta_c$ for magnetic fluctuation levels that exceed $\langle(\delta\tilde{B}/B_y^o)^2\rangle^{1/2} \sim \varepsilon^{3/5}$, or expressed in terms of a mean turbulent magnetic island width $\tilde{w}$, the condition becomes $\tilde{w}/x_o \sim \varepsilon^{3/10}$, which is the same (to within $\varepsilon^{1/10}$) as the condition that the turbulent island width equals the tearing layer width $x_o \varepsilon^{2/5}$. At this level of turbulence the condition that the quasilinear time scale $\tau_{q\ell} = \eta/x_o^2$ be long compared with the linear time scale $\tau_A/\varepsilon^{3/5}$ is well satisfied.

At higher fluctuation levels $(\delta\tilde{B}/B_y^o > \varepsilon^{3/5})$ the quasilinear evolution of $\langle\psi\rangle$ is faster than the collisional evolution. To estimate the maximum anomaly η_a/η_c we take the amplitude limit given by the breakdown of the quasilinear ordering $\gamma\tau_{q\ell} >> 1$, separating equilibrium and fluctuation time scales. At this limit where the two time scales merge the turbulent island width is $\tilde{w}/x_o \sim \varepsilon^{1/5}$ and the anomalous resistivity exceeds the collisional resistivity by approximately $\eta_a/\eta_c \sim \varepsilon^{-2/5}$.

It appears straightforward to investigate further the evolution of the poloidal flux $\langle\psi\rangle$ and the fluctuation spectrum $\langle\psi_k^2\rangle$ by performing numerical simulations based on the quasilinear equations derived in this work.

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FIGURE CAPTIONS

1. Equilibrium configuration with field reversal.

2. Form factor $S(k) = \sqrt{\Lambda}\,(kx_o)(\chi/z)$ [See Eq. (47)] plotted against kx_o for $x_o\Delta'(k) = 15(1 - kx_o)$.

3. Normalized growth rate $\Lambda(k)$ and form factor $S(k)$ for $x_o\Delta'(k) = 2(1 - k^2x_o^2)/kx_o$, corresponding to the Harris current sheet model.

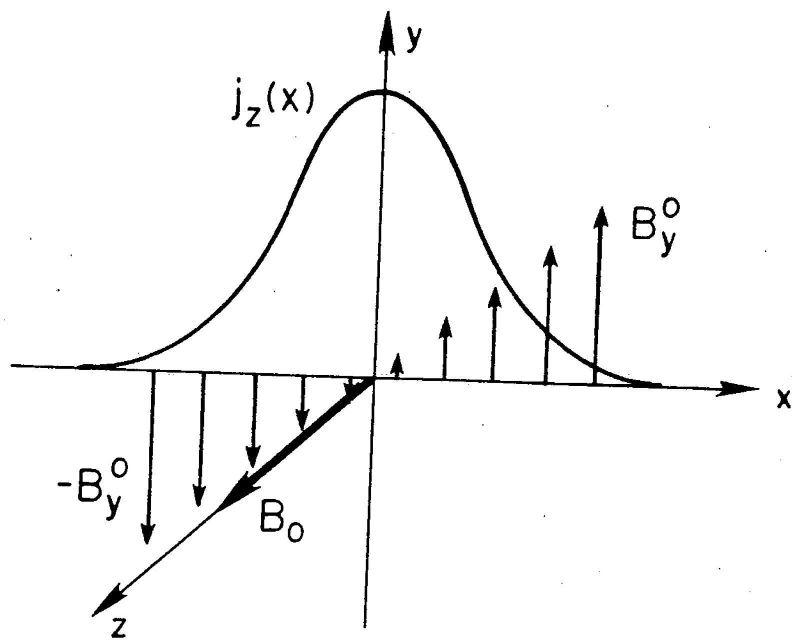

Fig. 1

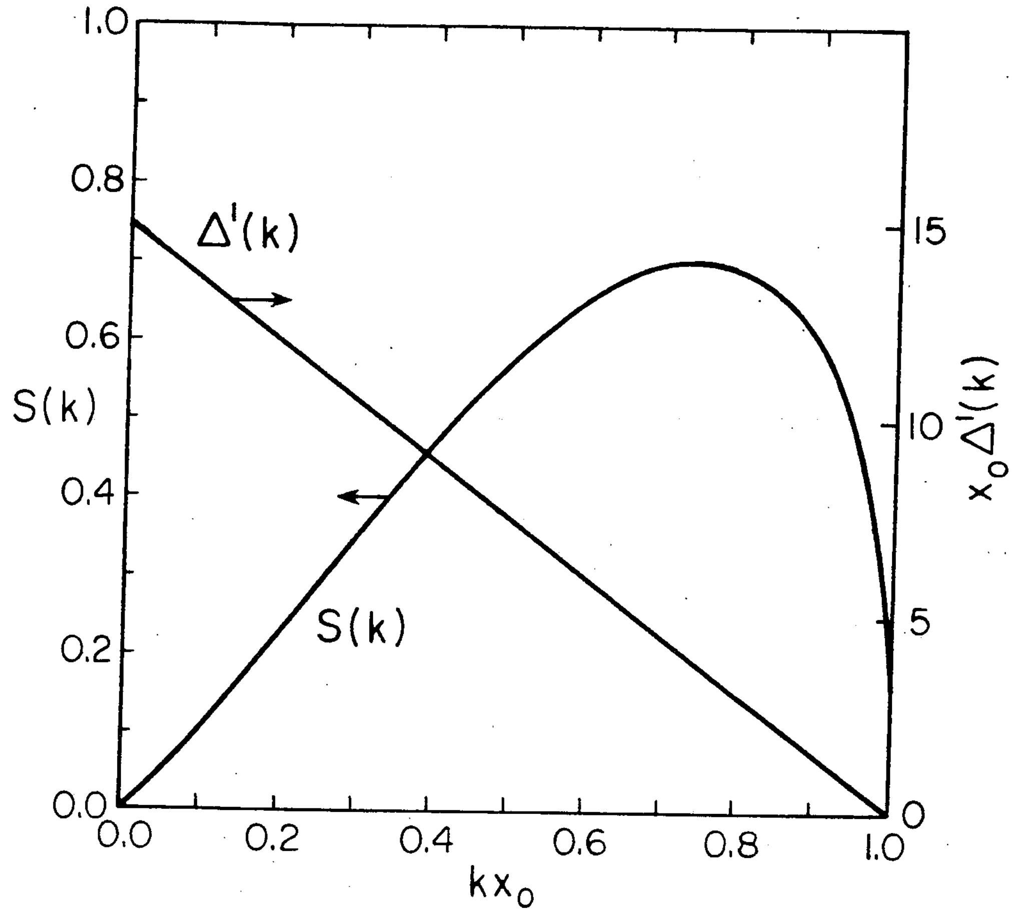
1.0
0.8
0.6
0.4
0.2
0.0
S(k)
$\Delta'(k)$
S(k)
0.0
0.2
0.4
0.6
0.8
1.0
kx_0
15
10
5
0
$x_0\Delta'(k)$

Fig. 2

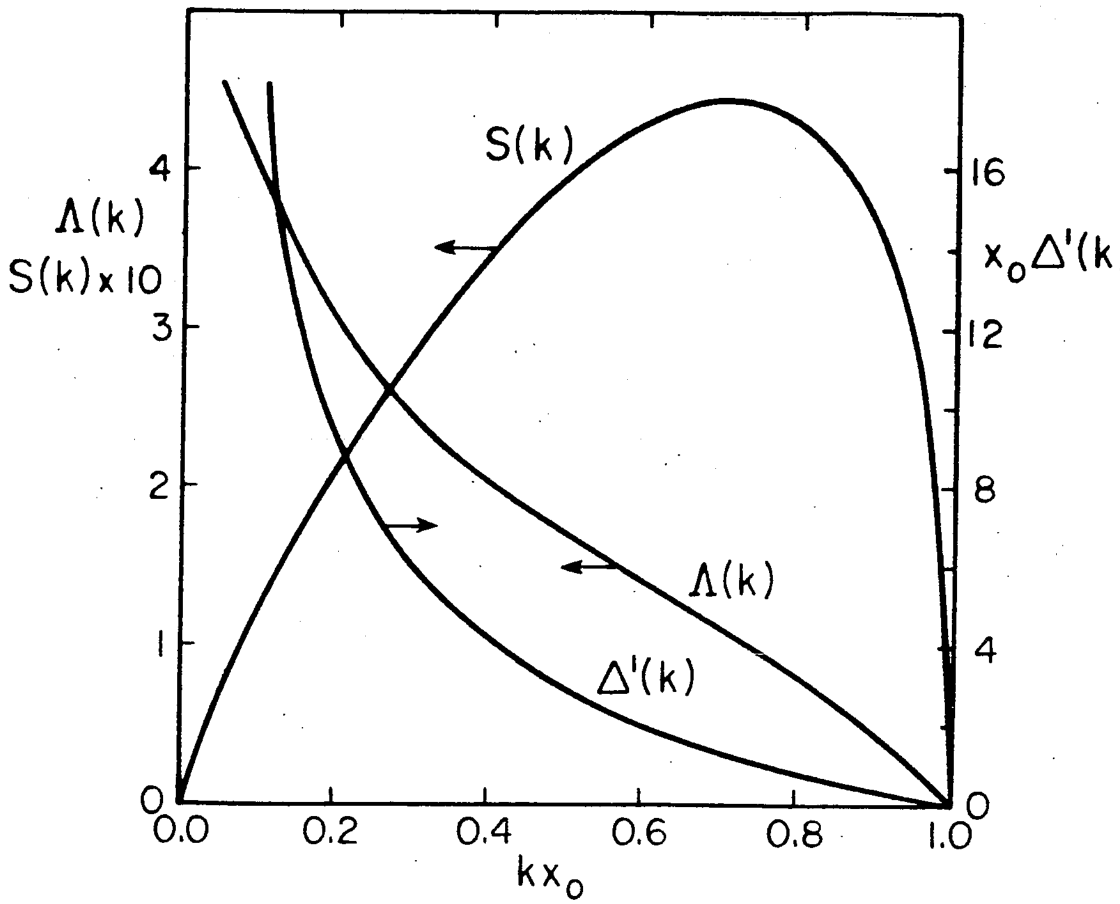

Fig. 3