Fluid Model for Relativistic, Magnetized Plasmas

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Many astrophysical plasmas and some laboratory plasmas are relativistic: either the thermal speed or the local bulk flow in some frame approaches the speed of light. Often, such plasmas are magnetized in the sense that the Larmor radius is smaller than any gradient scale length of interest. Conventionally, relativistic MHD is employed to treat relativistic, magnetized plasmas; however, MHD requires the collision time to be shorter than any other time scale in the system. Thus, MHD employs the thermodynamic equilibrium form of the stress tensor, neglecting pressure anisotropy and heat flow parallel to the magnetic field. Recent work has attempted to remedy these shortcomings. This paper re-examines the closure question and finds a more complete theory, which yields a more physical and self-consistent closure. Beginning with exact moments of the kinetic equation, we derive a closed set of Lorentz-covariant fluid equations for a magnetized plasma allowing for pressure and heat flow anisotropy. Basic predictions of the model, especially of the dispersion relation's dependence upon relativistic temperature, are examined.

I. INTRODUCTION

A plasma is relativistic if either the thermal speed—the rms speed of the individual particles—measured in the fluid rest-frame, or the local bulk flow measured in some relevant frame approach the speed of light. Such plasmas are ubiquitous in astrophysical phenomena (eg, galactic and extra-galactic jets [6], accretion discs of active galactic nuclei [12], and electron-positron-ion plasmas in the early universe [3], [16]; and in some laboratory fusion experiments.

Often (eg [1], [4], [11]), relativistic plasmas of interest are magnetized—meaning the dynamics are dominated by the magnetic field. The dynamics of such plasmas are typically described with magnetohydrodynamics (MHD), which captures the large-scale electromagnetic features of a magnetized plasma (eg, $\boldsymbol{E} \times \boldsymbol{B}$ drifts). A relativistic MHD closure has been presented by Anile [2]. Despite MHD's success at capturing some of the large scale physics, MHD plasmas are based on the use of a stress tensor whose origin is based on thermodynamic considerations (thermal equilibrium) rather than electrodynamics, in which electromagnetic forces dominate.

Chew, Goldberger, and Low [5] (CGL) present an early departure from the conventional MHD treatment of the stress tensor by allowing gyrotropic pressure: the CGL tensor differentiates between pressures parallel and perpendicular to the magnetic field. However, CGL neglects to include heat flow parallel to the magnetic field, which can be rapid in low collisionallity plasmas. Partly for this reason, the double adiabatic assumption used by CGL to achieve closure is not valid in many physical situations.

Hazeltine and Mahajan [7] (hereafter referred to as $\boldsymbol{I}$) attempted a more physical relativistic closure with gyrotropic pressure and parallel heat flow. However, close scrutiny of the Hazeltine-Mahajan model revealed fundamental deficiencies. The details of the deficiencies are covered in § II. The closure method employed in $\boldsymbol{I}$ uses the stress tensor as the constitutive relation for the fluid closure. The form of the stress tensor is derived from exact fluid equations together with orderings characterizing a magnetized plasma. Predictably, such an approach does not provide a closed system. Closure is achieved through a representative distribution function, consistent with relativity, magnetization, pressure anisotropy, and heat flow.

To achieve our closure, we take an approach parallel to $\boldsymbol{I}$; we use $\boldsymbol{I}$ as a guide in the search for a more physical and self-consistent relativistic, magnetized fluid closure.

II. CRITIQUE OF HAZELTINE AND MAHAJAN (2002)

We begin by discussing the covariant fluid closure of $\boldsymbol{I}$. We refer the reader to $\boldsymbol{I}$ and other related papers to observe the full treatment of the system rather than re-iterating both the relativistic closure and the non-relativistic limit here.

Study of the closure presented yielded several major shortcomings of the original model:

1. The first and most pertinent deficiency is apparent from the linearized, non-relativistic equations of motion. It is manifested by examining the electro-static response of the electron pressure anisotropy, $\Delta p_e = p^e_\parallel - p^e_\perp$. Parallel (perpendicular) here refers to being parallel (perpendicular) to the magnetic field. One finds that $\Delta p_e \sim m_i/m_e$,

leading to grossly exaggerated estimates of the electron anisotropy under the typical MHD assumption of vanishing electron inertia. Such anomalous scaling of the pressure anisotropy is not observed in conventional MHD or kinetic MHD.

The source of the anomalous scaling of the pressure anisotropy is the use of a single parallel heat flow, $Q_{\parallel}$, rather than separating the parallel heat flow into the parallel flow of parallel heat, $q_{\parallel}$, and the parallel flow of perpendicular heat, $q_{\perp}$. When a single heat flow is used, the evolution of parallel and perpendicular pressure are both coupled to parallel gradients of the heat flow, and the evolution of the single heat flow is driven by parallel pressure gradients. Using separate heat flows results in the expected evolution of the pressures and heat flows, namely $dp_{\parallel}/dt \sim \nabla_{\parallel} q_{\parallel}$, $dp_{\perp}/dt \sim \nabla_{\parallel} q_{\perp}$, $dq_{\parallel}/dt \sim \nabla_{\parallel} p_{\parallel}$, and $dq_{\perp}/dt \sim \nabla_{\parallel} p_{\perp}$.

Though separating the two forms of parallel heat is relatively common in the literature (see eg [5], [13], [15]), the distinction between the heat flows does not appear in the stress tensor, which forms the constitutive relation for a fluid plasma closure. Therefore, $\boldsymbol{I}$ attempts a closure involving the single heat flow, which corresponds to the sum: $Q_{\parallel} = q_{\parallel} + q_{\perp}$.

Including separate heat flows involves modifying the distribution used in $\boldsymbol{I}$ and using higher-order moment equations to obtain evolution equations for the two heat flows. However, the stress tensor is not changed.

Relatedly, $\boldsymbol{I}$ does a very poor job predicting the onset of the mirror instability. The source of this error is the unusual coupling of the pressures and heat flows noted above. The use of a relativistic bi-Maxwellian accurate to first order in the pressure anisotropy provides a better estimate of the mirror instability but still does not agree fully with kinetic MHD. However, a relativistic bi-Maxwellian retaining second-order pressure anisotropy terms captures the correct mirror instability. Note that, keeping accuracy to this order is reasonable since the fourth-rank moment (energy-weighted stress) will naturally have terms second-order in the anisotropy.

2. Examining the thermodynamics of $\boldsymbol{I}$ leads to a thermodynamic temperature of the following form:

$$T = \frac{\left(p + \frac{2}{3}\Delta p\right)^2}{n\left(p + \frac{4}{3}\Delta p\right)},$$

where p is the scalar pressure and Δp is the pressure anisotropy. This form makes thermodynamic calculations awkward and can lead to confusion with the more typical definition of the thermodynamic temperature, $T = p/n$.

Also in $\boldsymbol{I}$, the enthalpy density, h, is defined to be

$$h = u + p_{\|},$$

where u is the internal energy density. Typically, enthalpy is defined to be $h = u + p$. Again, there is nothing inherently incorrect with this definition, but it can also lead to confusion.

These shortcomings are addressed here by modifying the distribution in $\boldsymbol{I}$ to approximate a non-relativistic bi-Maxwellian expanded for small pressure anisotropy with only first-order terms retained, and by making a small modification to the $(0, 0)$ component of the stress tensor.

3. Approximate parallel and perpendicular projection operators were used in $\boldsymbol{I}$ as annihilators of the gyroscale portions of the exact moment equations to derive evolution equations for the parameters of the fluid system. Use of these operators leads to nearly redundant evolution equations which only agree in the non-relativistic limit. Thus, the redundancy leads to spurious instabilities in the moderate to ultra-relativistic temperature regimes of linear theory. This issue is solved by replacing the projection operators with more fundamental annihilators and discussed further in § IV B.

4. The form of the relativistic heat flux evolution equation provided in $\boldsymbol{I}$ omits relevant terms from the gyrophase dependent portion due to an ordering error. Also, the non-relativistic form of the closure presented in Hazeltine and Mahajan (2002b/c) contains algebraic errors which, when combined with the omission noted above, lead to an incorrect evolution equation for the parallel heat flux.

III. RELATIVISTIC PLASMA CONCEPTS

Here, we review some basic properties of relativistic electromagnetic theory, define what it means for a plasma to be magnetized, discuss some of the consequences of magnetization, and present the moments used in our theory. Because the majority of this material was covered in $\boldsymbol{I}$, the present treatment is brief.

We use the Einstein summation convention throughout, with Greek indices running from 0 to 3 and Roman indices from 1 to 3. Boldface type typically represents the 3-vector portion of a 4-vector, for instance an arbitrary 4-vector C^μ may be written as $C^\mu = (C^0, \mathbf{C})$. All speeds are normalized to the speed of light, so that $c = 1$. We use $\eta^{\mu\nu} = diag\{-1, 1, 1, 1\}$ as the signature for our Minkowski tensor.

A. Magnetized Plasma

We make use of the following Lorentz scalars formed from the Faraday tensor, F, and its dual, $\mathcal{F}$:

$$\frac{1}{2} F_{\mu\nu} F^{\mu\nu} = B^2 - E^2 \equiv W \tag{1}$$

$$\frac{1}{2} \mathcal{F}^{\mu\nu} F_{\nu\mu} = \boldsymbol{E} \cdot \boldsymbol{B} \equiv \lambda W. \tag{2}$$

The latter relation is of significant importance because λ, or equivalently $E_\parallel$, will be a small parameter of our theory.

Two conditions must be satisfied for our plasma to be considered magnetized:

1. The two electromagnetic field invariants must satisfy

$$W > 0, \tag{3}$$

$$\lambda \ll 1. \tag{4}$$

2. The thermal gyroradius must be small compared to any gradient scale length:

$$\delta \ll 1, \tag{5}$$

where δ is the ratio of the thermal gyroradius of any plasma species to any gradient scale length.

We assume the ordering $\lambda \sim \delta$ for convenience. We will implicitly use this definition of a magnetized plasma throughout the following analysis.

B. Quasi-Projectors

As is typical in a magnetized plasma, notions of parallel and perpendicular to the field play important roles. Thus, we need a covariant meaning for parallel and perpendicular. Such a meaning is provided by

$$e_\mu^\nu \equiv -F_{\mu\kappa}\frac{F^{\kappa\nu}}{W}, \tag{6}$$

$$b_\mu^\nu \equiv \eta_\mu^\nu - e_\mu^\nu. \tag{7}$$

e and b become approximate perpendicular and parallel projection operators in the magnetized limit. In a frame in which the transverse electric field vanishes (a subset of the instantaneous rest-frame (R)), the action of e and b on an arbitrary 4-vector $C_\mu = (C_0, \mathbf{C})$ is given by

$$b^{\mu\kappa}C_\kappa|_R = (C^0, \mathbf{C}_\parallel), \tag{8}$$

$$e^{\mu\kappa}C_\kappa|_R = (0, \mathbf{C}_\perp). \tag{9}$$

$\parallel$ and $\perp$ have the typical three-dimensional meaning: $\mathbf{C}_\parallel = \boldsymbol{BB}\cdot\mathbf{C}/B^2 = \boldsymbol{bb}\cdot\mathbf{C}$, $\mathbf{C}_\perp = \mathbf{C} - \mathbf{C}_\parallel$, where $\boldsymbol{b}$ is the standard abbreviation $\boldsymbol{b} \equiv \boldsymbol{B}/B$.

Gradients of the projection operators will be used implicitly later in our analysis. Thus, we present their forms. To do so, we begin by recalling the Maxwell stress tensor

$$\Theta^{\alpha\beta} = F_\kappa^\alpha F^{\kappa\beta} - \frac{1}{4}\eta^{\alpha\beta}F_{\kappa\lambda}F^{\kappa\lambda}$$

and observe

$$e^{\alpha\beta} = \frac{\eta^{\alpha\beta}}{2} + \frac{\Theta^{\alpha\beta}}{W},$$
$$b^{\alpha\beta} = \frac{\eta^{\alpha\beta}}{2} - \frac{\Theta^{\alpha\beta}}{W}.$$

Maxwell's equations (13) and (14) presented in the following section imply

$$\partial_\nu\Theta^{\mu\nu} = -F^{\mu\kappa}J_\kappa,$$

where $\partial_\nu = \frac{\partial}{\partial x^\nu}$. Thus, it is straightforward to show

$$\partial_\nu b^\nu_\mu = \frac{F_{\mu\nu}}{W} J^\nu + \left(\frac{1}{2}\eta^\nu_\mu - b^\nu_\mu\right) \partial_\nu \log W, \tag{10}$$

$$\partial_\nu e^\nu_\mu = -\frac{F_{\mu\nu}}{W} J^\nu + \left(\frac{1}{2}\eta^\nu_\mu - e^\nu_\mu\right) \partial_\nu \log W. \tag{11}$$

C. Closing Maxwell's Equations

Since plasmas are strongly coupled to the electromagnetic field, we must consider a closure involving Maxwell's equations. The coupling of the electromagnetic field to a plasma enters a fluid description through the second-moment equation, which constitutes the conservation of energy-momentum (Tsikanshvili et al. (1992)). In relativistic form, the second-moment equation takes the form

$$\partial_\nu \mathcal{T}^{\mu\nu} - F^{\mu\nu} J_\nu = 0, \tag{12}$$

where $\mathcal{T}$ represents the the total (summed over all species) energy-momentum tensor for the plasma and J_ν is the current density 4-vector. Thus, the second-moment equation is used as a constitutive relation for magnetized plasmas, providing closure to Maxwell's equations:

$$\partial_\nu F^{\mu\nu} = J^\mu, \tag{13}$$

$$\partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = 0. \tag{14}$$

It remains to compute the the current density in a magnetized plasma.

Equation (12), when composed with F^μ_κ, provides two components of the current density

$$e^{\mu\nu} J_\nu = -\frac{F^\mu_\kappa}{W} \partial_\nu \mathcal{T}^{\kappa\nu}. \tag{15}$$

There are two independent components because the perpendicular quasi-projector has a two-dimensional null space. Charge conservation

$$\partial_\nu J^\nu = 0 \tag{16}$$

and quasi-neutrality

$$J^\nu U_\nu = 0 \tag{17}$$

provide the two remaining components of the current density, where $U^\mu = (\gamma, \gamma\mathbf{V})$ is the local 4-velocity of the fluid, with $\gamma^2 = (1 - V^2)^{-1}$ the relativistic dilation factor. That equation (17) provides a good representation of quasi-neutrality will be presented in § III D.

We conclude that knowing the plasma stress tensor, and thus the current density, is sufficient to close Maxwell's equations.

D. Moments

Our analysis involves moments up to and including the fourth rank. We express each moment in terms of the distribution funtion $f(x, p)$, where p represents the four-momentum p^μ:

$$\Gamma^\alpha = \int \frac{d^3p}{p^0} f p^\alpha, \tag{18}$$

$$T^{\alpha\beta} = \int \frac{d^3p}{p^0} f p^\alpha p^\beta, \tag{19}$$

$$M^{\alpha\beta\gamma} = \int \frac{d^3p}{p^0} f p^\alpha p^\beta p^\gamma, \tag{20}$$

$$R^{\alpha\beta\gamma\delta} = \int \frac{d^3p}{p^0} f p^\alpha p^\beta p^\gamma p^\delta. \tag{21}$$

Here, $\frac{d^3p}{p^0}$ represents the invariant momentum-space volume, where

$$p^0 = \sqrt{m^2 + \mathbf{p}^2}. \tag{22}$$

Γ^α is the 4-vector fluid particle-flux density, $T^{\alpha\beta}$ is the stress-energy tensor, $M^{\alpha\beta\gamma}$ is typically referred to as the stress flow tensor, and $R^{\alpha\beta\gamma\delta}$ will be referred to as the energy-weighted stress tensor.

The exact moments of the collisionless kinetic equation associated with the four requisite moments for our analysis represent particle conservation, momentum evolution, stress-flow evolution, and energy-weighted stress evolution:

$$\partial_\alpha \Gamma^\alpha = 0, \tag{23}$$

$$\partial_\alpha T^{\alpha\beta} = eF^{\beta\alpha}\Gamma_\alpha, \tag{24}$$

$$\partial_\alpha M^{\alpha\beta\gamma} = e\left(F^{\beta\nu}T^\gamma_\nu + F^{\gamma\nu}T^\beta_\nu\right), \tag{25}$$

$$\partial_\alpha R^{\alpha\beta\gamma\delta} = e\left(F^{\beta\nu}M^{\gamma\delta}_\nu + F^{\gamma\nu}M^{\beta\delta}_\nu + F^{\delta\nu}M^{\beta\gamma}_\nu\right). \tag{26}$$

Note in the second and higher moment equations, the left hand side involves the macroscopic scale, while the right hand side deals with the short gyroscale. Thus, the small-gyroradius limit is obtained formally by allowing the charge to become arbitrarily large, $e \to \infty$.

No restrictions on the size of higher order moments is assumed. Our analysis does not require higher moments because we only need those corresponding to the scalar coefficients appearing in the energy-momentum tensor. This tensor provides the framework for the closure of the plasma-Maxwell system.

At this point, we restrict our analysis to a plasma with a single ion species in the interest of simplicity. We define the Lorentz scalar $\Gamma^0_R = \int d^3p f_R$ to be the rest-frame density, n_R, and define the fluid velocity of a species to be

$$U^\mu = \Gamma^\mu / n_R. \tag{27}$$

In order to satisfy quasi-neutrality, we require, to leading order, the electrons and ions have the same rest-frame densities, and reside in the same approximate rest-frame to avoid arbitrarily large current densities; we do not restrict plasma flow, however. Equation (17) then follows as the leading order expression of quasi-neutrality.

E. Gyro-Ordering

We must now determine evolution equations for the four components of the flux density. First, we note that all moments can be expanded in the form

$$\Gamma^\mu = \Gamma^\mu_{(0)} + \Gamma^\mu_{(1)},$$

where the parenthetical subscript refers to the order of the term with respect to the gyroradius (δ). Thus, equation (24) provides

$$F^{\mu\nu}_{(0)}\Gamma_{(0)\mu} = 0, \tag{28}$$

where we distinguish the lowest order Faraday tensor, $F^{\mu\nu}_{(0)} \equiv F^{\mu\nu}\left(E_{\|} = 0\right)$, from its first-order counterpart

$$F^{\mu\nu}_{(1)} \equiv F^{\mu\nu} - F^{\mu\nu}_{(0)} \propto E_{\|}.$$

Recalling the action of the Faraday tensor on a four-vector, equation (28) implies

$$\Gamma^0_{(0)}\boldsymbol{E} + \boldsymbol{\Gamma}_{(0)} \times \boldsymbol{B} = 0, \tag{29}$$

which reproduces the familiar MHD Ohm's law, $\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} = 0$. As such, equation (28) fixes the two perpendicular components of the flow. The particle conservation law, equation (23), fixes another of the components.

At this point, we drop the ordering subscripts and use Γ^μ and U^μ to refer to the zeroth-order fields from this point on. Similarly, we drop the ordering subscript from the Faraday tensor where it is nonessential. We can now write the flow in the form

$$\Gamma^\mu = \gamma n_R \left(1, \boldsymbol{V}_{\|} + \boldsymbol{V}_E\right), \tag{30}$$

where $\boldsymbol{V}_E = \boldsymbol{E} \times \boldsymbol{b}/B$, $\boldsymbol{V}_{\|} = \boldsymbol{b}\boldsymbol{b} \cdot \boldsymbol{V}$, and γ is evaluated at the lowest order flow velocity.

Before moving on, we note that equation (24) has become

$$\partial_\nu T^{\mu\nu}_{(0)} = eF^{\mu\nu}_{(0)}\Gamma_{(1)\nu} + eF^{\mu\nu}_{(1)}\Gamma_{(0)\nu} \tag{31}$$

taking gyro-ordering of the moments and the Faraday tensor into account.

The remainder of this paper is devoted to computing the stress tensor. Conventional MHD avoids this issue by assuming the stress tensor has the thermodynamic equilibrium form

$$\mathcal{T}^{\mu\nu} = p\eta^{\mu\nu} + hU^\mu U^\nu, \tag{32}$$

where p is the pressure and h the enthalpy density. This form only pertains to the highly collisional regime in which thermal relaxation occurs more rapidly than any other process of interest. Thus, our analysis can be viewed as taking place in a regime of much lower collisionallity. We ignore collisions altogether and compute the stress tensor subject to electromagnetic forces alone.

IV. COVARIANT EVOLUTION EQUATIONS

A. Magnetized Stress

We use the magnetized limit of equation (25) to find

$$F^{\alpha\nu}T^{\beta}_{\nu} + F^{\beta\nu}T^{\alpha}_{\nu} = 0. \tag{33}$$

We use indicial symmetry of the stress tensor, antisymmetry of the Faraday tensor, along with properties of the projection operators to conclude the stress tensor must have the following form:

$$\begin{aligned} T^{\alpha\beta} = p\eta^{\alpha\beta} + hU^{\alpha}U^{\beta} + \frac{1}{3}\Delta p\left(2k^{\alpha}k^{\beta} - e^{\alpha\beta}\right) \\ + Q_{\parallel}\left(k^{\alpha}U^{\beta} + U^{\alpha}k^{\beta}\right), \end{aligned} \tag{34}$$

where $p = (p_{\parallel} + 2p_{\perp})/3$, $\Delta p = p_{\parallel} - p_{\perp}$, $Q_{\parallel}$, and h are Lorentz scalars corresponding to pressure, pressure anisotropy, total parallel heat flow, and enthalpy density respectively. We differentiate between the parallel flow of parallel heat, $q_{\parallel}$, and the parallel flow of perpendicular heat, $q_{\perp}$, with $Q_{\parallel} = q_{\parallel} + q_{\perp}$ and $\Delta Q_{\parallel} = \frac{2}{5}q_{\parallel} - \frac{3}{5}q_{\perp}$. It is important to note that this distinction does not enter at this order in the moment equations. The total parallel heat flow is the only distinct component that appears in the stress tensor. This stress tensor differs from that in $\boldsymbol{I}$ primarily in notation. Here, the enthalpy presented corresponds to the standard thermodynamic definition, $h = u + p$, where $u = T^{00}_{R}$ is the energy density. In $\boldsymbol{I}$, $h = u + p_{\parallel}$.

k^{μ} must satisfy $e^{\alpha\beta}k_{\beta} = 0$ to satisfy force balance and $U_{\alpha}k^{\alpha} = 0$ to preserve the significance of $p_{\parallel}$ and $p_{\perp}$. These constraints on k^{μ} leave free only one component, which corresponds to the Lorentz boosted unit vector $\boldsymbol{b}$. Thus,

$$\begin{aligned} k^\alpha &= \frac{\mathcal{F}^{\alpha\beta}U_\beta}{\sqrt{W}} \\ &= \gamma\sqrt{\frac{W}{B^2}}\left(\frac{B^2}{W}V_\parallel, \boldsymbol{b} + \frac{B^2}{W}V_\parallel \boldsymbol{V}_E + \frac{E_\parallel}{W}\boldsymbol{E}\right). \end{aligned} \tag{35}$$

Two evolution equations are provided by equation (31), once we identify an annihilator of the $\Gamma_{(1)\nu}$ term. Appropriate choices in the magnetized limit are k and U, since $k_\nu\,(U_\nu)\,F^{\kappa\nu} \sim \delta$. We find

$$k_\kappa \partial_\nu T^{\kappa\nu} = e n_R \frac{B}{\sqrt{W}} E_\parallel \tag{36}$$

and

$$U_\kappa \partial_\nu T^{\kappa\nu} = 0. \tag{37}$$

These equations advance the parallel momentum and total scalar pressure/energy respectively.

B. Subtleties of Annihilator Choice

The evolution equations in $\boldsymbol{I}$ make use of the projection operators as annihilators of the gyroscale dependent portions of the moment equations, (18) - (21). This annihilator choice leads to subtle inconsistencies in the derived evolution equations, resulting from implicit, redundant use of evolution equations. Further, the inconsistencies cause spurious instabilities to develop in the linear theory for moderate to ultra-relativistic temperatures.

Consider first the parallel projection operator. We can write the operator in terms of U^μ and k^μ as $b^{\mu\nu} = k^\mu k^\nu - U^\mu U^\nu + \mathcal{O}\left(\lambda^2\right)$. Similarly, we can write $e^{\mu\nu} = \eta^{\mu\nu} + U^\mu U^\nu - k^\mu k^\nu + \mathcal{O}\left(\lambda^2\right)$. If we operate on the third rank moment equation, (20), with $U^\mu U^\nu$, the resulting equation would be an evolution equation for the total energy which agrees with that found at the second rank, (37), *only* in the non-relativistic limit. Therefore, operating with $b^{\mu\nu}$ would result in the implicit usage of a redundant energy equation that disagrees with the lower rank derived equation in all but the non-relativistic limit. The redundancy continues to higher order and with usage of the perpendicular operator. Thus, we avoid using the projection operators as annihilators in favor of more fundamental tensors in our system, U^μ and k^μ.

C. Magnetized Stress Flow

The expression for $M^{\alpha\beta\gamma}$ given in $\boldsymbol{I}$ does not allow for separate parallel and perpendicular heat flows. The three auxiliary parameters appearing in the stress flow tensor employed in $\boldsymbol{I}$ only permit dependence of the stress flow on $p_\parallel$, $p_\perp$, and $Q_\parallel$. We modify the model for the stress flow t include an additional auxiliary parameter (m_4 in what follows) to permit the freedom of having two parallel heat flows.

In the magnetized limit, the fourth-rank conservation law determines the form of the stress-flow tensor

$$F^{(\alpha\kappa}M_\kappa^{\beta\gamma)} = 0, \tag{38}$$

where the super(sub)-script parentheses indicate indicial symmetrization over non-contracted indices:

$$\eta^{(\alpha\beta}U^{\gamma)} \equiv \eta^{\alpha\beta}U^\gamma + \eta^{\alpha\gamma}U^\beta + \eta^{\beta\gamma}U^\alpha.$$

We are also constrained by the definition of the stress-flow (equation (20)) and particle flux (equation (18)). From the definitions, it can be seen that contracting two indices of the stress-flow reduces to the momentum flux

$$M_\alpha^{\alpha\gamma} = -m^2 n_R U^\gamma. \tag{39}$$

Given the above two constraints and assuming the only 4-vectors appearing in the stress-flow are U^μ and k^μ, the stress-flow must have the form

$$M^{\alpha\beta\gamma} = m^2 n_R U^\alpha U^\beta U^\gamma + \sum_k m_k M_k^{\alpha\beta\gamma}, \tag{40}$$

where

$$M_1^{\alpha\beta\gamma} = \eta^{(\alpha\beta}U^{\gamma)} + 6U^\alpha U^\beta U^\gamma, \tag{41}$$

$$M_2^{\alpha\beta\gamma} = b^{(\alpha\beta}U^{\gamma)} + 4U^\alpha U^\beta U^\gamma, \tag{42}$$

$$M_3^{\alpha\beta\gamma} = \eta^{(\alpha\beta}k^{\gamma)} + 6U^{(\alpha}U^\beta k^{\gamma)}, \tag{43}$$

$$M_4^{\alpha\beta\gamma} = b^{(\alpha\beta}k^{\gamma)} - \frac{2}{3}\eta^{(\alpha\beta}k^{\gamma)}, \tag{44}$$

and the m_k are scalars to be determined later. The M_k also satisfy $M^{\alpha\gamma}_{k\alpha} = 0$, so that the second constraint above is satisfied.

We construct an evolution equations for the magnetized stress-flow by finding annihilators for the right-hand side of equation (25). Two such equations are:

$$k_\alpha k_\beta \partial_\kappa M^{\kappa\alpha\beta} = 2e\frac{B}{\sqrt{W}} Q_\parallel E_\parallel, \tag{45}$$

$$(U_\alpha k_\beta + U_\beta k_\alpha)\, \partial_\kappa M^{\kappa\alpha\beta} = -2e\frac{B}{\sqrt{W}} E_\parallel \left(h + \frac{2}{3}\Delta p \right). \tag{46}$$

These equations can be considered to advance the parallel pressure and total parallel heat flow respectively.

We note that we cannot evolve the two parallel heat flows individually at this order. This is because evolving the separate heat flows requires a time-like (0-component) derivative of the elements of the stress flow containing each parallel heat flow. It will become clear after evaluating the m_k that such separation is not possible in this order.

The m_k appearing in the stress flow can be taken to be auxiliary parameters of our system. Thus, we will need to express them in terms of the dynamical variables appearing in our system. As such, it is convenient to examine the instantaneous rest frame components of the stress flow in terms of the m_k, which are listed in Appendix A.

D. Magnetized Energy-Weighted Stress

We construct the energy-weighted stress tensor in much the same way as the three previous tensors. We begin with the constraint provided by the fifth-rank conservation law in the magnetized limit

$$F^{(\alpha\kappa} R_\kappa^{\beta\gamma\delta)} = 0. \tag{47}$$

Our second constraint follows from the definitions of the energy-weighted stress (equation (21)) and the stress (equation (19)) when contracting two indices of the energy-weighted stress

$$R^{\alpha\gamma\delta}_{\alpha} = -m^2 T^{\gamma\delta} \tag{48}$$

Our third constraint follows from contracting all four indices of the energy-weighted stress

$$R^{\alpha\beta}_{\alpha\beta} = m^2\rho, \tag{49}$$

where $-\rho = T^{\alpha}_{\alpha} = -u + 3p$.

Unlike the third rank tensor whose indicial symmetrization is straightforward, the fourth rank tensor will have unique symmetrizations based on each tensor's construction, which are given in Appendix B.

The following expression gives the simplest fourth-rank tensor that satisfies the above constraints, without introducing new independent variables

$$\begin{aligned} R^{\alpha\beta\gamma\delta} = m^2 \big[& U^{(\alpha} U^{\beta} T^{\gamma\delta)} + 8pU^{\alpha}U^{\beta}U^{\gamma}U^{\delta} \\ & - Q_{\parallel} k^{(\alpha} U^{\beta} U^{\gamma} U^{\delta)} \big] + \sum_k r_k R_k^{\alpha\beta\gamma\delta}, \end{aligned} \tag{50}$$

where

$$\begin{aligned} R_1^{\alpha\beta\gamma\delta} = & \, \eta^{(\alpha\beta}\eta^{\gamma\delta)} + 6\eta^{(\alpha\beta}U^{\gamma}U^{\delta)} \\ & + 48U^{\alpha}U^{\beta}U^{\gamma}U^{\delta}, \end{aligned} \tag{51}$$

$$\begin{aligned} R_2^{\alpha\beta\gamma\delta} = & \, \eta^{(\alpha\beta}b^{\gamma\delta)} + 8b^{(\alpha\beta}U^{\gamma}U^{\delta)} + 2\eta^{(\alpha\beta}U^{\gamma}U^{\delta)} \\ & + 64U^{\alpha}U^{\beta}U^{\gamma}U^{\delta}, \end{aligned} \tag{52}$$

$$R_3^{\alpha\beta\gamma\delta} = \eta^{(\alpha\beta}U^{\gamma}k^{\delta)} - 8U^{(\alpha}k^{\beta}k^{\gamma}k^{\delta)}, \tag{53}$$

$$R_4^{\alpha\beta\gamma\delta} = b^{(\alpha\beta}U^{\gamma}k^{\delta)} - 6U^{(\alpha}k^{\beta}k^{\gamma}k^{\delta)}, \tag{54}$$

$$\begin{aligned} R_5^{\alpha\beta\gamma\delta} = & \, e^{(\alpha\beta}b^{\gamma\delta)} + 2e^{(\alpha\beta}U^{\gamma}U^{\delta)} \\ & + 2b^{(\alpha\beta}U^{\gamma}U^{\delta)} + 16U^{\alpha}U^{\beta}U^{\gamma}U^{\delta}. \end{aligned} \tag{55}$$

It can be seen that the $R^{\alpha\gamma\delta}_{k\alpha} = 0$ so that $R^{\alpha\gamma\delta}_{\alpha} = -m^2T^{\gamma\delta}$ and $R^{\alpha\beta}_{\alpha\beta} = m^2\rho$. The extra terms multiplying m^2 in equation (50) account for over counting certain elements of $T^{\alpha\beta}$ due to symmetry conditions on R.

Again, we construct evolution equations for the energy-weighted stress by identifying annihilators of the right-hand side of equation (26)

$$k_\alpha k_\beta k_\delta \partial_\gamma R^{\alpha\beta\gamma\delta} = 3e\frac{B}{\sqrt{W}}E_\parallel\left(m_1 + m_2\right). \tag{56}$$

This can be viewed as evolving the parallel component of the parallel heat flow.

As in the stress flow tensor, the r_k can be viewed as auxiliary parameters. Thus, we need to express them in terms of the rest-frame components of the energy-weighted stress. Such expressions are provided in Appendix A.

We now have evolution equations for n_R, p, $p_\parallel$, $Q_\parallel$, $q_\parallel$, and the three vector components of Γ^μ. We will take these to be our set of dynamical variables. We consider the enthalpy, h, to be an auxiliary parameter in much the same way we treat the m_k and r_k as auxiliary parameters. Thus, our fluid system is nearly closed; however, we still need to evaluate the auxiliary parameters in terms of the dynamical variables. For this, we need a distribution function.

V. DISTRIBUTION FUNCTION

A. Choosing a Distribution

Since we have auxiliary parameters not yet related to our dynamical variables, we require a distribution function to close our fluid system. Any lowest order distribution chosen must: be gyrotropic, solve the drift-kinetic equation, and reproduce the stress tensor, equation (34). Satisfying the first requirement is straightforward. The second is difficult to implement in a fluid treatment and typically abandons the fluid point of view in favor of kinetic MHD, making the drift-kinetic equation part of the closure [10], [14]. The third requirement restricts us to any of a class of distributions that reproduce the stress tensor.

Therefore, we choose a representative distribution from the equivalence class of distributions reproducing the stress tensor, capable of also representing the fluid equations of motion. The parameters in the distribution are proportional to the dynamical variables of the fluid system and evolve according to the fluid equations. We use such a parameterized distribution in place of the drift-kinetic equation to close our system.

B. Explicit Form

After examining previous literature [13], [15], it became clear in non-relativistic theory, a bi-Maxwellian (or two-temperature Maxwellian) is a good choice for capturing features of kinetic theory in a fluid approach. As such, our distribution can be considered the relativistic analog of the non-relativistic bi-Maxwellian. Our distribution has the form

$$\begin{aligned} f\,(x,p) = f_M \Big\{ & 1 + \hat{\Delta} + (\Delta + \Delta^*)\, p_\alpha e^{\alpha\beta} p_\beta \\ & + \left(\overline{\Delta} + \overline{\Delta}^*\right) p_\alpha b^{\alpha\beta} p_\beta \\ & + p_\alpha p_\beta p_\gamma p_\delta \left(\tilde{\Delta} k^\alpha k^\beta k^\gamma k^\delta + \tilde{\Delta}^* e^{\alpha\beta} e^{\gamma\delta}\right. \\ & \left. + \tilde{\Delta}^{**} k^\alpha k^\beta e^{\gamma\delta}\right) + Q_\alpha b^{\alpha\beta} p_\beta \Big[1 + \hat{Q} \\ & + p_\mu \left(e^{\mu\nu} + Q k^\mu k^\nu\right) p_\nu \Big] \Big\}\,, \end{aligned} \tag{57}$$

where f_M is a relativistic Maxwellian. The Δ scalars describe pressure anisotropy, while the Q scalars measure heat flow. Thus, our distribution can be parametrized by our dynamical variables: n_R, $p_\|$, $p_\perp$, $q_\|$, and $q_\perp$. The form of our distribution mirrors that found in $\boldsymbol{I}$ only in the first three terms and the last term multiplying the square brackets. Note that we do not simply write the distribution in the standard non-relativistic form with the directional temperature dependence in the exponent. If we were to make such an attempt, evaluating moments of the distribution would become intractable.

Recall that a relativistic Maxwellian has the following form

$$f_M(x,p) = N_M e^{U_\mu P^\mu / T},$$

where $P^\mu = p^\mu + eA^\mu$ is the canonical momentum, $U_\mu P^\mu$ defines the invariant energy, $T(x)$ the scalar temperature, and $N_M(x)$ the scalar normalization factor. In the rest frame, we have

$$f_{MR} = N_M e^{-P^0/T}.$$

Moments of the rest frame Maxwellian have the form

$$\int_0^\infty \frac{ds}{\sqrt{1+s^2}} s^{2n} e^{-\zeta\sqrt{1+s^2}} = \frac{1 \cdot 3 \cdots (2n-1) K_n(\zeta)}{\zeta^n},$$

where K_n is the n^{th} MacDonald function, $s = |\mathbf{p}|/m$, and $\zeta = m/T$. We can now compute the normalization factor

$$N_M = \frac{n_R e^{\Phi/T}}{4\pi m^2 T K_2(\zeta)},$$

where $\Phi = A^0$ is the electrostatic potential. Thus, the rest frame Maxwellian is

$$f_{MR} = \frac{n_R e^{-p^0/T}}{4\pi m^2 T K_2(\zeta)} \tag{58}$$

Returning to evaluating the parameters of our distribution, we compare our distribution to the non-relativistic bi-Maxwellian expanded for small pressure anisotropy to determine Δ^*/Δ, $\overline{\Delta^*}/\overline{\Delta}$, $\tilde{\Delta}^*/\tilde{\Delta}$, and $\tilde{\Delta}^{**}/\tilde{\Delta}$. Doing so yields

$$\begin{aligned} f(x,p) = f_M \Big\{ & 1 + \hat{\Delta} + \Delta\left(p_\alpha e^{\alpha\beta} p_\beta - 2 p_\alpha b^{\alpha\beta} p_\beta\right) \\ & + \overline{\Delta}\left(p_\alpha e^{\alpha\beta} p_\beta + 4 p_\alpha b^{\alpha\beta} p_\beta\right) \\ & + \tilde{\Delta} p_\alpha p_\beta p_\gamma p_\delta \left(4 k^\alpha k^\beta k^\gamma k^\delta + e^{\alpha\beta} e^{\gamma\delta}\right. \\ & \left. - \, 4 k^\alpha k^\beta e^{\gamma\delta}\right) + Q_\alpha b^{\alpha\beta} p_\beta \Big[1 + \hat{Q} \\ & + \, p_\mu \left(e^{\mu\nu} + Q k^\mu k^\nu\right) p_\nu \Big] \Big\} . \end{aligned}$$

For reference, expanding a bi-Maxwellian for small pressure anisotropy yields

$$\begin{aligned} f(x,v) &= \frac{N}{p_\perp p_\parallel^{1/2}} \exp\left[-\frac{mn}{2} \left(\frac{v_\perp^2}{p_\perp} + \frac{v_\parallel^2}{p_\parallel} \right) \right] \\ &= \frac{N}{p^{3/2}} e^{-\frac{mnv^2}{p}} \left[1 - \frac{mn\Delta p}{6p^2} \left(v_\perp^2 - 2 v_\parallel^2\right) \right. \\ & + \frac{\Delta p^2}{6p^2} - \frac{mn\Delta p^2}{18 p^3} \left(v_\perp^2 + 4 v_\parallel^2\right) \\ & \left. + \frac{1}{72} \left(\frac{mn\Delta p}{p^2} \right)^2 \left(4 v_\parallel^4 + v_\perp^4 - 4 v_\parallel^2 v_\perp^2\right) \right] , \end{aligned}$$

where N is the normalization factor, v is the particle velocity, $p_\parallel$ and $p_\perp$ refer the the parallel and perpendicular pressure, and p and Δp refer to the scalar pressure and pressure anisotropy.

In the instantaneous rest-frame with coordinates oriented such that $\boldsymbol{B} = (0, 0, B)$, our distribution reduces to

$$f_R\left(x,p\right) = f_{MR}\left\{1 + \hat{\Delta} + \frac{\Delta}{m^2}\left(p_\perp^2 - 2p_\parallel^2\right) + \frac{\overline{\Delta}}{m^2}\left(p_\perp^2 + 4p_\parallel^2\right) + \frac{\tilde{\Delta}}{m^4}\left(4p_\parallel^4 + p_\perp^4 - 4p_\perp^2 p_\parallel^2\right) + \frac{Q_3 p_3}{m}\left[1 + \hat{Q} + \frac{p_\perp^2}{m^2} + Q\frac{p_\parallel^2}{m^2}\right]\right\}, \tag{59}$$

where $p_\parallel$ and $p_\perp$ here refer to parallel and perpendicular components of momenta.

C. Scalar Moments

We choose $\hat{\Delta}$ and $\hat{Q}$ to ensure that the rest-frame density is Maxwellian and the rest-frame flow velocity vanishes. Δ and $\overline{\Delta}$ are chosen so that $p = nT = \frac{1}{3}\left(T_R^{33} + 2T_R^{11}\right)$ and $\Delta p = T_R^{33} - T_R^{11}$. $\tilde{\Delta}$ is chosen by matching the non-relativistic limit ($\zeta = m/T \to \infty$) of R_R^{1133} to its bi-Maxwellian counterpart, $\frac{m}{n}p_\parallel p_\perp$. Q_3 is chosen to satisfy $T_R^{03} = Q_\parallel$, and Q is chosen by matching the non-relativistic limits of the elements of the stress flow tensor involving heat flow to their bi-Maxwellian counterparts, ie $M_R^{003} = 2mQ_\parallel = M_R^{333} + 2M_R^{113} = 2mq_\parallel + 2mq_\perp$.

Thus, in the rest-frame, the distribution function becomes

$$f_R(x,p) = f_{MR}\left\{1 - \frac{1}{6}\frac{\Delta p}{p}\frac{\zeta K_2}{K_3}\left(\frac{p_\perp^2}{m^2} - \frac{2p_\parallel^2}{m^2}\right) + \frac{1}{6}\frac{\Delta p^2}{p^2}\frac{K_4}{K_2} - \frac{1}{18}\frac{\Delta p^2}{p^2}\frac{\zeta K_4}{K_3}\left(\frac{p_\perp^2}{m^2} + 4\frac{p_\parallel^2}{m^2}\right) + \frac{1}{72}\frac{\Delta p^2}{p^2}\zeta^2\left(4\frac{p_\parallel^4}{m^4} + \frac{p_\perp^4}{m^4} - 4\frac{p_\parallel^2 p_\perp^2}{m^4}\right) + \frac{n_R p_\parallel}{p^2}\frac{K_2}{K_3\mathcal{K}}\left[\frac{K_3}{\zeta K_2}Q_\parallel - \left(\frac{q_\parallel}{3}\frac{p_\parallel^2}{m^2} + \frac{q_\perp}{2}\frac{p_\perp^2}{m^2}\right)\right]\right\}. \tag{60}$$

where p and Δp refer to pressure and pressure anisotropy, while $p_\parallel$ and $p_\perp$ refer to parallel and perpendicular momenta. Explicitly, the scalar components of the distribution are

$$\zeta = mn_R/p, \tag{61}$$

$$\hat{\Delta} = \frac{1}{6}\frac{K_4}{K_2}\frac{\Delta p^2}{p^2}, \tag{62}$$

$$\Delta = -\frac{1}{6}\frac{K_2}{K_3}\zeta^2\frac{\Delta p}{mn_R}, \tag{63}$$

$$\overline{\Delta} = -\frac{1}{18}\frac{K_4}{K_3}\zeta\frac{\Delta p^2}{p^2}, \tag{64}$$

$$\tilde{\Delta} = \frac{\zeta^2}{72}\frac{\Delta p^2}{p^2}. \tag{65}$$

$$\hat{Q} = -\frac{K_3}{\zeta K_2}\frac{2Q_\parallel}{q_\perp} - 1 \tag{66}$$

$$Q_3 = -q_\perp\frac{\zeta^2}{2mn_R}\frac{K_2}{K_3\mathcal{K}}. \tag{67}$$

$$Q = \frac{2}{3}\frac{q_\parallel}{q_\perp}, \tag{68}$$

where $\mathcal{K} = \frac{K_3}{K_2} - \frac{K_4}{K_3}$.

VI. CLOSED FLUID EQUATIONS

A. Covariant Closure Summary

We have chosen n_R, $V_\parallel$, $p_\parallel$, $p_\perp$, $q_\parallel$, and $q_\perp$ as the dynamical variables of our collisionless, small gyroradius fluid system. The covariant evolution equations for the chosen dynamical variables of our system are:

$$\partial_\alpha \Gamma^\alpha = 0, \tag{69}$$

$$k_\kappa \partial_\nu T^{\kappa\nu} = en_R\frac{B}{\sqrt{W}}E_\parallel, \tag{70}$$

$$U_\kappa \partial_\nu T^{\kappa\nu} = 0, \tag{71}$$

$$k_\alpha k_\beta \partial_\kappa M^{\kappa\alpha\beta} = 2e\frac{B}{\sqrt{W}}Q_\parallel E_\parallel, \tag{72}$$

$$\left(U_\alpha k_\beta + U_\beta k_\alpha\right)\partial_\kappa M^{\kappa\alpha\beta} = -2e\frac{B}{\sqrt{W}}E_\parallel\left(h + \frac{2}{3}\Delta p\right) \tag{73}$$

$$k_\alpha k_\beta k_\delta \partial_\gamma R^{\alpha\beta\gamma\delta} = 3e\frac{B}{\sqrt{W}}E_\parallel\left(m_1 + m_2\right), \tag{74}$$

where the flux, Γ^α, stress, $T^{\alpha\beta}$, stress flow, $M^{\alpha\beta\gamma}$, and energy-weighted stress, $R^{\alpha\beta\gamma\delta}$, are given by equations (18)- (21) respectively, and the m_k are given in Appendix A. Therefore, equations (69)- (74) constitute a closed covariant set of fluid equations.

B. 3-Vector Form

It is often convenient to express fluid equations in 3-vector form, sacrificing explicit Lorentz covariance. As such, we present the 3-vector form of our closed fluid system here.

We begin by noting the following identities

$$U^\nu\partial_\nu = \gamma\frac{d}{dt} = \frac{d}{d\tau},$$

$$k^\nu\partial_\nu = \frac{d}{ds},$$

where d/dt is the conventional convective derivative and τ represents the proper time.

The explicit forms of (70) and (71) can be express as

$$\begin{aligned} &h\mathbf{k}\cdot\gamma\frac{d\mathbf{V}}{d\tau} + \frac{dp_\parallel}{ds} + \frac{dQ_\parallel}{d\tau} - Q_\parallel\frac{d\log n_R}{d\tau} + Q_\parallel\mathbf{k}\cdot\frac{d\mathbf{V}}{ds} \\ &-\frac{1}{6}\Delta p\frac{d\log W}{ds} + \frac{2}{3}\Delta p\partial_\mu k^\mu = en_R\frac{B}{\sqrt{W}}E_\parallel, \end{aligned} \tag{75}$$

$$\begin{aligned} &\frac{d}{d\tau}\left(p-h\right) + h\frac{d\log n_R}{d\tau} - \frac{dQ_\parallel}{ds} - \frac{2}{3}\Delta p\mathbf{k}\cdot\frac{d\boldsymbol{V}}{ds} \\ &-\frac{1}{6}\Delta p\frac{d\log W}{d\tau} - Q_\parallel\gamma\boldsymbol{k}\cdot\frac{d\boldsymbol{V}}{d\tau} - Q_\parallel\partial_\mu k^\mu = 0, \end{aligned} \tag{76}$$

where

$$\begin{aligned} \partial_\mu k^\mu = &\frac{1}{\sqrt{W}}\left[\frac{1}{\gamma}\mathbf{B}\cdot\frac{d}{dt}\left(\gamma\mathbf{V}\right) - E_\parallel\left(\mathbf{b}\times\nabla\right)\cdot\left(\gamma\mathbf{V}\right)\right] \\ &-\frac{1}{2}\frac{d\log W}{ds}. \end{aligned} \tag{77}$$

From the third moment equations, we have from equations (72) and (73)

$$\begin{aligned}
&\frac{d}{d\tau}\left(m_1+m_2\right)+\left(m_1+m_2\right)\left(2\boldsymbol{k}\cdot\frac{d\boldsymbol{V}}{ds}-\frac{d\log n_R}{d\tau}\right)\\
&+\left(m_3+\frac{1}{3}m_4\right)\partial_\mu k^\mu+12m_3\gamma\boldsymbol{k}\cdot\frac{d\boldsymbol{V}}{d\tau}\\
&-m_4\frac{d\log W}{ds}+3\frac{d}{ds}\left(m_3+\frac{1}{3}m_4\right)\\
&=2e\frac{B}{\sqrt{W}}Q_\parallel E_\parallel .
\end{aligned}\tag{78}$$

$$\begin{aligned}
&\frac{d}{d\tau}\left(5m_3-\frac{1}{3}m_4\right)+\frac{d}{ds}\left(m_1+m_2\right)\\
&+\left(5m_1+3m_2\right)\gamma\boldsymbol{k}\cdot\frac{d\boldsymbol{V}}{d\tau}-\frac{1}{2}m_2\frac{d\log W}{ds}\\
&+\left(7m_3+\frac{1}{3}m_4\right)\gamma\boldsymbol{k}\cdot\frac{d\boldsymbol{V}}{ds}-6m_3\frac{d\log n_R}{d\tau}\\
&+\frac{1}{2}m_4\frac{d\log W}{d\tau}+m^2n_R\gamma\boldsymbol{k}\cdot\frac{d\boldsymbol{V}}{d\tau}\\
&=e\frac{B}{\sqrt{W}}E_\parallel\left(h+\frac{2}{3}\Delta p\right).
\end{aligned}\tag{79}$$

Turning to the energy-weighted stress, we have from equation (74)

$$\begin{aligned}
&-\frac{d}{d\tau}\left(5r_3+3r_4\right)+\frac{d}{ds}\left(3r_1+6r_2\right)\\
&+18r_1\gamma\mathbf{k}\cdot\frac{d\mathbf{V}}{d\tau}+r_2\left(30\gamma\mathbf{k}\cdot\frac{d\mathbf{V}}{d\tau}-\frac{3}{2}\frac{d}{ds}\log W\right)\\
&-\left(5r_3+3r_4\right)\left(-\frac{d}{d\tau}\log n_R+3\gamma\mathbf{k}\cdot\frac{d\mathbf{V}}{ds}\right)\\
&+r_5\left(6\gamma\mathbf{k}\cdot\frac{d\mathbf{V}}{d\tau}+\frac{3}{2}\frac{d}{ds}\log W\right)\\
&+3m^2p_\parallel\gamma\mathbf{k}\cdot\frac{d\mathbf{V}}{d\tau}=3e\frac{B}{\sqrt{W}}E_\parallel\left(m_1+m_2\right),
\end{aligned}\tag{80}$$

where the r_k are given in Appendix A.

C. Non-Relativistic Limit

We now present the fully non-relativistic (NR), $\zeta=\frac{m}{T}\gg 1$ and $V_\parallel\sim V_\perp\sim\zeta^{-1/2}$, form of our closed system. Since labelling the rest-frame is somewhat inappropriate in this limit, we use $n\equiv n_R$. We also use the common notation $\nabla_\parallel\equiv\boldsymbol{b}\cdot\nabla$.

In the NR limit, equations (69), (75), (76), (78), (79), (80), become after some manipulation

$$\frac{dn}{dt} + n\nabla \cdot \boldsymbol{V} = 0, \tag{81}$$

$$mn\mathbf{b} \cdot \frac{d\mathbf{V}}{dt} + \nabla_{\|} p_{\|} + \left(p_{\perp} - p_{\|}\right) \nabla_{\|} \log B = enE_{\|}, \tag{82}$$

$$p_{\|} \frac{d}{dt} \log \left(\frac{p_{\|} B^2}{n^3}\right) + 2\nabla_{\|} q_{\|} + 2\left(q_{\perp} - q_{\|}\right) \nabla_{\|} \log B = 0, \tag{83}$$

$$p_{\perp} \frac{d}{dt} \log \left(\frac{p_{\perp}}{Bn}\right) + \nabla_{\|} q_{\perp} - 2q_{\perp} \nabla_{\|} \log B = 0, \tag{84}$$

$$q_{\|} \frac{d}{dt} \log \left(\frac{q_{\|} B^3}{n^4}\right) + \frac{3}{2} \frac{p_{\|}}{m} \nabla_{\|} \frac{p_{\|}}{n} = 0, \tag{85}$$

$$q_{\perp} \frac{d}{dt} \log \left(\frac{q_{\perp}}{n^2}\right) + \frac{p_{\|}}{m} \nabla_{\|} \frac{p_{\perp}}{n} - \frac{p_{\perp} \Delta p}{mn} \nabla_{\|} \log B = 0. \tag{86}$$

The NR limit of our closure coincides with a bi-Maxwellian MHD closure in which gyroviscous components of the stress tensor are retained as presented by Ramos [13]. As such, the system produces dispersion relations whose numerical coefficients coincide with those obtained through kinetic theory, and the system correctly predicts the onset of the mirror and firehose instabilities.

D. Linear Predictions

Having completed our closure, we now examine some basic predictions of the linearized relativistic system. Linearizing equations (15) and (69)- (74) about an isotropic equilibrium with no heat flow, equal electron and ion equilibrium temperatures, and non-relativistic flow speed yields a lengthy set of equations presented fully in Appendix C. We present the linearized version of equation (72) to compare with the non-relativistic limit of the same equation as an example:

$$
\begin{aligned}
&- v\zeta_s \left(f(\zeta_s) + 3\frac{K_3(\zeta_s)}{K_2(\zeta_s)} \right) \frac{\delta n}{n} \\
&+ \zeta_s \left(f(\zeta_s) + \frac{K_3(\zeta_s)}{K_2(\zeta_s)} \right) v\frac{\delta p_s}{p} + \frac{2}{3}\zeta_s \frac{K_4(\zeta_s)}{K_3(\zeta_s)} v\frac{\delta \Delta p_s}{p} \\
&- 3\left(1 - \frac{2}{5}\frac{K_4(\zeta_s)}{K_3(\zeta_s)\mathcal{K}(\zeta_s)} \right) \cos(\theta) \frac{\delta Q_{\parallel s}}{p} \\
&+ 2\frac{K_4(\zeta_s)}{K_3(\zeta_s)\mathcal{K}(\zeta_s)} \cos(\theta) \frac{\delta \Delta Q_{\parallel s}}{p} \\
&+ 2\zeta_s \frac{K_3(\zeta_s)}{K_2(\zeta_s)} \hat{\mathbf{k}}_\perp \cdot \delta\mathbf{v} = 0,
\end{aligned}
\tag{87}
$$

$$
\begin{aligned}
&- 3v\zeta_s \frac{\delta n}{n} + \zeta_s v \frac{\delta p_\parallel}{p} - \frac{6}{5}\cos(\theta)\frac{Q_{\parallel s}}{p} - 2\cos(\theta)\frac{\delta Q_{\parallel s}}{p} \\
&- 2\zeta_s \hat{\mathbf{k}}_\perp \cdot \delta\mathbf{v} = 0,
\end{aligned}
\tag{88}
$$

where $v = \omega/k$, $\hat{\mathbf{k}}_\perp = \mathbf{k}_\perp/k$, $\cos(\theta) = k_\parallel/k$, $v_A^2 = B^2/\mu_0(m_i + m_e)n$, subscript s denotes species, superscript T denotes a sum over the species, and

$$
f(\zeta_s) = \left[\zeta_s + \frac{K_3(\zeta_s)}{K_2(\zeta_s)} \left(1 - \zeta_s \frac{K_1(\zeta_s)}{K_2(\zeta_s)} \right) \right].
$$

Using the full set of linear equations, we plot the phase velocity squared versus ζ_i (ie, the inverse temperature) in figure (1) for $v_A^2 = 10^{-6}$, $\theta = 30^o$, and $m_i/m_e = 1833$. Also plotted in figure (1) as the dashed lines are the linearized version of the non-relativistic equations, (81)- (86). From lowest to highest phase speed for large ζ, we have the slow magnetosonic, two ion acoustic, shear Alfven, fast magnetosonic, and two electron acoustic modes.

For the plotted parameters, the electron modes are the first to show significant deviation for increasing temperature at roughly $100keV$. At this temperature, the non-relativistic theory begins to predict superluminal phase velocities for the electron acoustic modes. Also of note, the phase velocity of the shear and slow magnetosonic Alfven modes behave quite differently in the ultrarelativistic regime. In this regime, the correct dispersion relation is

$$
v^2 \sim \left(\frac{\zeta_i + \zeta_e}{8} \right) v_A^2,
$$

where the non-relativistic theory would simply state $v^2 \sim v_A^2$.

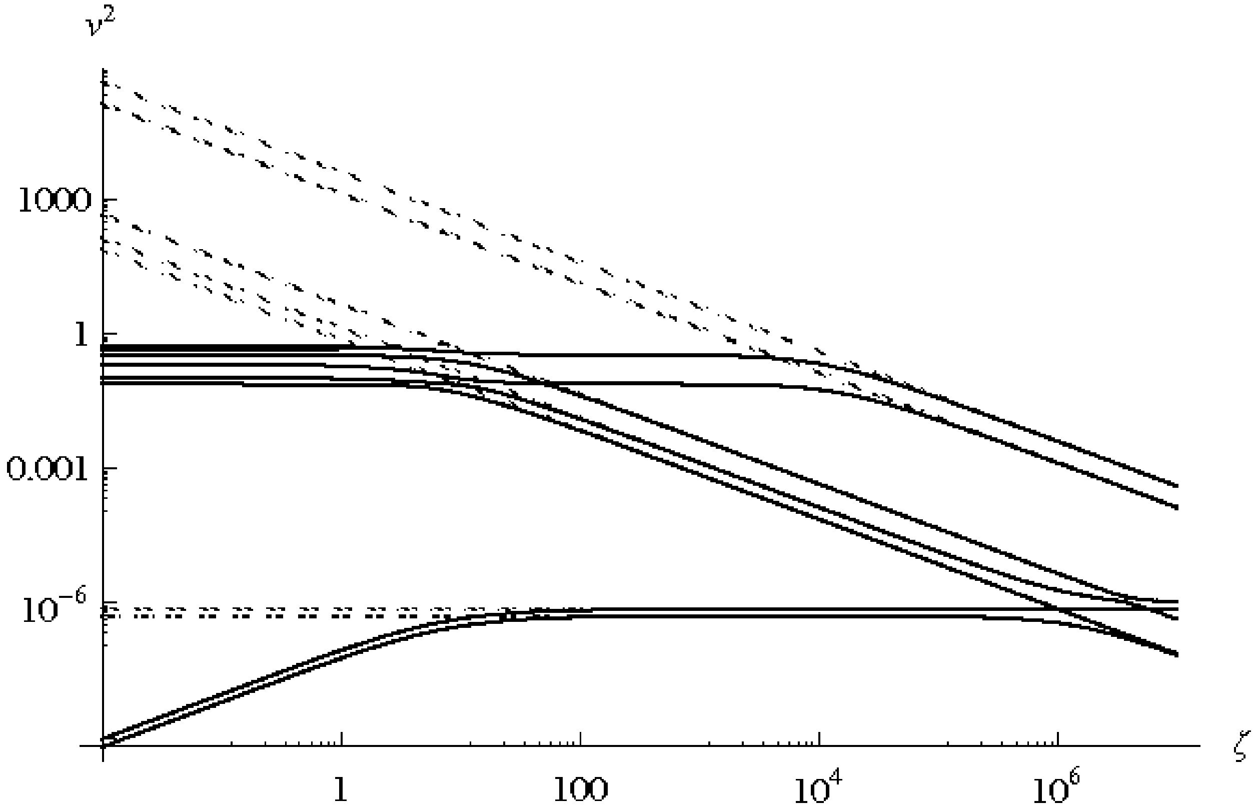

FIG. 1: Phase velocity squared versus $\zeta_i = m_i/T$ for the general linearized evolution equations (solid) and their non-relativistic limit (dashed) are plotted. $v_A^2 = 10^{-6}$, $\theta = 30^o$, and $m_i/m_e = 1833$

VII. SUMMARY

Maxwell's equations are closed in a magnetized plasma when the 4-vector current can be expressed in terms of the stress tensor,

$$\mathcal{T}^{\mu\nu} = \sum_{species} T^{\mu\nu},$$

where $T^{\mu\nu}$ is the stress tensor of the individual plasma species. This closure procedure is given by equation (15) and later equations.

Thus, a closed fluid description of plasma dynamics relies on equations that fix the evolution of the stress tensor of each plasma species. For this reason, the stress tensor is said to provide the constitutive relation for a plasma fluid closure. We obtain our description of the stress tensor, equation (34), via electromagnetic constraints rather than the simplier

MHD thermodynamic arguments

$$T^{\alpha\beta} = p\eta^{\alpha\beta} + hU^{\alpha}U^{\beta} + \frac{1}{3}\Delta p \left(2k^{\alpha}k^{\beta} - e^{\alpha\beta}\right) + Q_{\parallel}\left(k^{\alpha}U^{\beta} + U^{\alpha}k^{\beta}\right), \tag{89}$$

where $b^{\alpha\beta}$ and $e^{\alpha\beta}$ are approximate projection operators introduced in §III. The fluid 4-velocity, U^{μ}, and the heat flow, $Q_{\parallel}k^{\mu} = \left(q_{\parallel} + q_{\perp}\right) k^{\mu}$, are constrained by

$$F^{\mu}_{\nu}U^{\nu} = 0, \tag{90}$$

$$e^{\mu}_{\nu}k^{\nu} = 0, \tag{91}$$

$$U_{\nu}k^{\nu} = 0. \tag{92}$$

Equation (90) provides the first of our evolutionary constraints by reproducing the familiar $\boldsymbol{E}\times\boldsymbol{B}$ drift. We still need evolution equations for the two remaining free components of the flow, Γ^{μ}, which are the rest-frame density, n_R, and the parallel flow, $V_{\parallel}$. Also, from the stress tensor, we need to evolve $p = (p_{\parallel} + 2p_{\perp})/3$, $\Delta p = p_{\parallel} - p_{\perp}$, h, and the two components of the rest-frame heat flow, $q_{\parallel}$ and $q_{\perp}$.

Quasineutrality, equation (17), requires that n_R be the same for all species, while the other quantities in the stress tensor are free to vary from species to species. Thus, we choose the following six parameters n_R, $V_{\parallel}$, $p_{\parallel}$, $p_{\perp}$, $q_{\parallel}$, and $q_{\perp}$ as our dynamical variables. The evolution equations for the six dynamical variables of our system in various forms are given in §VI.

At this point in the closure, we have ten scalar auxiliary parameters which are not fixed. These are the enthalpy density, h, the four scalar parameters, m_k, of the stress flow, and the five scalar parameters, r_k, of the energy-weighted stress. We express these auxiliary parameters via a representative distribution, which is parameterized by our dynamical variables. Thus, the distribution evolves according to equations (69)- (74), and our auxiliary parameters can be expressed in terms of the dynamical variables, as presented in Appendix A.

Our closure provides a more accurate physical description of relativistic, magnetized fluid plasmas than previously presented by Hazeltine and Mahajan [7]. The system allows detailed study of various astrophysical and laboratory plasmas at a more realistic level than MHD. Also, in the non-relativistic limit, our closure reduces to a set of equations presented by

Ramos [13] obtained via a bi-Maxwellian closure in which gyroviscous terms of the stress tensor are retained. In forthcoming papers, we will explore the thermodynamic properties of an imperfect relativistic plasma through the inclusion of collisions.

Acknowledgments

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Appendices

A. AUXILIARY PARAMETERS

Here, we list the non-vanishing rest-frame moments of the third and fourth rank in terms of the auxiliary parameters m_k and r_k and express the auxiliary parameters of our system in terms of the dynamical variables. We orient our rest-frame such that $\boldsymbol{B} = (0, 0, B)$.

The non-vanishing components of the third rank moments are

$$M_R^{000} = m^2 n_R + 3m_1 + m_2,$$

$$M_R^{003} = 5m_3 - \frac{1}{3}m_4,$$

$$M_R^{011} = M_R^{022} = m_1,$$

$$M_R^{033} = m_1 + m_2,$$

$$M_R^{113} = M_R^{223} = m_3 - \frac{2}{3}m_4,$$

$$M_R^{333} = 3m_3 + m_4.$$

And for the fourth rank, we have

$$R_R^{0000} = m^2 (u + 3p) + 15r_1 + 10r_2 + 4r_5,$$

$$R_R^{0011} = R_R^{0022} = m^2 p_\perp + 5r_1 + r_2 + r_5,$$

$$R_R^{0033} = m^2 p_\parallel + 5r_1 + 8r_2 + 2r_5,$$

$$R_R^{0003} = m^2 Q_\parallel - 3r_3 - 3r_4,$$

$$R_R^{0113} = R_R^{0223} = r_3,$$

$$R_R^{0333} = -5r_3 - 3r_4,$$

$$R_R^{1111} = R_R^{2222} = 3R_R^{1122} = 3r_1,$$

$$R_R^{1133} = R_R^{2233} = r_1 + r_2 + r_5,$$

$$R_R^{3333} = 3r_1 + 6r_2.$$

The auxiliary parameters of our system are determined by evaluating the rest frame moments above via our distribution function, equation (60)

$$m_1 = m\left[p\frac{K_3}{K_2} - \frac{1}{3}\Delta p\frac{K_4}{K_3} + \frac{\Delta p^2}{p}\left(\frac{1}{6}\frac{K_4 K_3}{K_2^2} - \frac{4}{9}\frac{K_4^2}{K_3 K_2} + \frac{5}{18}\frac{K_5}{K_2}\right)\right],$$

$$m_2 = m\Delta p\left[\frac{K_4}{K_3} - \frac{1}{3}\frac{\Delta p}{p}\left(\frac{K_4^2}{K_3 K_2} - \frac{K_5}{K_2}\right)\right],$$

$$m_3 = \frac{mQ_\parallel}{\zeta}\left[1 - \frac{2}{3}\frac{K_4}{\mathcal{K}K_3}\frac{1+2Q}{2+3Q}\right],$$

$$m_4 = \frac{mQ_\parallel}{\zeta}\frac{K_4}{\mathcal{K}K_3}\frac{2\left(1-Q\right)}{2+3Q}.$$

$$r_1 = \frac{m}{n_R}\left[p^2\frac{K_3}{K_2} - \frac{2}{3}p\Delta p\frac{K_4}{K_3}\right.$$
$$\left. + \Delta p^2\left(\frac{1}{6}\frac{K_4K_3}{K_2^2} - \frac{5}{9}\frac{K_4^2}{K_3K_2} + \frac{1}{2}\frac{K_5}{K_2}\right)\right],$$

$$r_2 = \frac{1}{2}\frac{m}{n_R}\left[2p\Delta p\frac{K_4}{K_3} + \Delta p^2\left(-\frac{2}{3}\frac{K_4^2}{K_3K_2} + \frac{K_5}{K_2}\right)\right],$$

$$r_3 = \frac{m^2}{\zeta}\frac{K_4}{K_3\mathcal{K}}Q_\parallel\left(\mathcal{K}_2 - \frac{2}{2+3Q}\frac{K_5}{K_4}\right),$$

$$r_4 = -\frac{2}{3}\frac{m^2}{\zeta}\frac{K_4}{K_3\mathcal{K}}Q_\parallel\left(4\mathcal{K}_2 - \frac{5+3Q}{2+3Q}\frac{K_5}{K_4}\right),$$

$$r_5 = -\frac{1}{2}\frac{m}{n_R}\Delta p^2\frac{K_5}{K_2},$$

where

$$\mathcal{K}_2(\zeta_s) = \frac{K_3(\zeta_s)}{K_2(\zeta_s)} - \frac{K_5(\zeta_s)}{K_4(\zeta_s)}.$$

We can also now express the enthalpy density in terms of our dynamical variables by look at $T_R^{00} = h - p$

$$h = mn_R\frac{K_3}{K_2}.$$

B. FOURTH RANK SYMMETRIZATION

The construction of $R^{\alpha\beta\gamma\delta}$ will involve tensors of the three following forms, aside from fully asymmetric and fully symmetric

1. Symmetric times asymmetric, ie $\eta^{\alpha\beta}U^\gamma k^\delta$. This form will have 12 terms in the symmetrization

$$A_1^{(\alpha\beta\gamma\delta)} = A^{\alpha\beta\gamma\delta} + A^{\alpha\beta\delta\gamma} + A^{\alpha\delta\gamma\beta} + A^{\alpha\delta\beta\gamma}$$
$$+ A^{\alpha\gamma\beta\delta} + A^{\alpha\gamma\delta\beta} + A^{\beta\delta\alpha\gamma} + A^{\beta\delta\gamma\alpha}$$
$$+ A^{\beta\gamma\alpha\delta} + A^{\beta\gamma\delta\alpha} + A^{\delta\gamma\alpha\beta} + A^{\delta\gamma\beta\alpha}.$$

2. Symmetric times symmetric, ie $\eta^{\alpha\beta}U^\gamma U^\delta$. This form will have 6 terms

$$A_2^{(\alpha\beta\gamma\delta)} = A^{\alpha\beta\gamma\delta} + A^{\alpha\gamma\beta\delta} + A^{\alpha\delta\beta\gamma} + A^{\beta\gamma\alpha\delta} + A^{\beta\delta\alpha\gamma} + A^{\delta\gamma\alpha\beta}.$$

For the special case of a fourth rank composed of the two identical symmetric second rank tensors, ie $\eta^{\alpha\beta}\eta^{\gamma\delta}$, only the first three terms contribute to symmetrization.

3. Third rank symmetric times a four-vector, ie $k^\alpha U^\beta U^\gamma U^\delta$. This form has four terms

$$A_3^{(\alpha\beta\gamma\delta)} = A^{\alpha\beta\gamma\delta} + A^{\beta\alpha\gamma\delta} + A^{\gamma\alpha\beta\delta} + A^{\delta\alpha\beta\gamma}.$$

C. LINEARIZED EVOLUTION EQUATIONS

Here, we present the full set of linearized equations (15) and (69)- (74) about an isotropic equilibrium with no heat flow, equal electron and ion equilibrium temperatures, and non-relativistic flow speed:

$$v\frac{\delta n}{n} - \cos(\theta)\delta v_\parallel - \hat{\mathbf{k}}_\perp \cdot \delta\mathbf{v} = 0, \tag{93}$$

$$\zeta_s\frac{K_3(\zeta_s)}{K_2(\zeta_s)}v\delta v_\parallel - \cos(\theta)\frac{\delta p_s}{p} - \frac{2}{3}\cos(\theta)\frac{\delta\Delta p_s}{p} + v\frac{\delta Q_{\parallel s}}{p} = \frac{iq_s n}{kp}E_\parallel, \tag{94}$$

$$\zeta_s f(\zeta_s)v\frac{\delta n}{n} + (1-\zeta_s f(\zeta_s))\, v\frac{\delta p_s}{p} + \cos(\theta)\frac{\delta Q_{\parallel s}}{p} = 0, \tag{95}$$

$$
\begin{aligned}
&- v\zeta_s \left(f(\zeta_s) + 3\frac{K_3(\zeta_s)}{K_2(\zeta_s)} \right) \frac{\delta n}{n} \\
&+ \zeta_s \left(f(\zeta_s) + \frac{K_3(\zeta_s)}{K_2(\zeta_s)} \right) v\frac{\delta p_s}{p} + \frac{2}{3}\zeta_s \frac{K_4(\zeta_s)}{K_3(\zeta_s)} v\frac{\delta \Delta p_s}{p} \\
&- 3\left(1 - \frac{2}{5}\frac{K_4(\zeta_s)}{K_3(\zeta_s)\mathcal{K}(\zeta_s)} \right) \cos(\theta) \frac{\delta Q_{\| s}}{p} \\
&+ 2\frac{K_4(\zeta_s)}{K_3(\zeta_s)\mathcal{K}(\zeta_s)} \cos(\theta) \frac{\delta \Delta Q_{\| s}}{p} \\
&+ 2\zeta_s \frac{K_3(\zeta_s)}{K_2(\zeta_s)} \hat{\mathbf{k}}_\perp \cdot \delta \mathbf{v} = 0,
\end{aligned} \tag{96}
$$

$$
\begin{aligned}
&\zeta_s \left(\zeta_s + 5\frac{K_3(\zeta_s)}{K_2(\zeta_s)} \right) v\delta v_\| + \zeta_s f(\zeta_s) \cos(\theta) \frac{\delta n}{n} \\
&- \zeta_s \left(f(\zeta_s) + \frac{K_3(\zeta_s)}{K_2(\zeta_s)} \right) \cos(\theta) \frac{\delta p_s}{p} \\
&- \frac{2}{3}\zeta_s \frac{K_4(\zeta_s)}{K_3(\zeta_s)} \cos(\theta) \frac{\delta \Delta p_s}{p} \\
&+ \left(5 - 2\frac{K_4(\zeta_s)}{K_3(\zeta_s)\mathcal{K}(\zeta_s)} \right) v\frac{\delta Q_{\| s}}{p} \\
&= \frac{iq_s n}{kp}\zeta_s \frac{K_3(\zeta_s)}{K_2(\zeta_s)} E_\|,
\end{aligned} \tag{97}
$$

$$
\begin{aligned}
&\left(6\frac{K_3(\zeta_s)}{K_2(\zeta_s)} + \zeta_s \right) v\delta v_\| + \left(f(\zeta_s) + \frac{K_3(\zeta_s)}{K_2(\zeta_s)} \right) \cos(\theta) \frac{\delta n}{n} \\
&- \left(f(\zeta_s) + 2\frac{K_3(\zeta_s)}{K_2(\zeta_s)} \right) \cos(\theta) \frac{\delta p_s}{p} - \frac{4}{3}\frac{K_4(\zeta_s)}{K_3(\zeta_s)} \cos(\theta) \frac{\delta \Delta p_s}{p} \\
&+ \frac{v}{K_3(\zeta_s)\mathcal{K}(\zeta_s)} \left(K_4(\zeta_s)\mathcal{K}_2(\zeta_s) - \frac{2}{5}K_5(\zeta_s) \right) \frac{\delta Q_{\| s}}{p} \\
&- \frac{2}{3}\frac{vK_5(\zeta_s)}{K_3(\zeta_s)\mathcal{K}(\zeta_s)} \frac{\delta \Delta Q_{\| s}}{p} = \frac{iq_s n}{kp}\frac{K_3(\zeta_s)}{K_2(\zeta_s)} E_\|,
\end{aligned} \tag{98}
$$

$$
\begin{aligned}
&\left[\left(\zeta_i \frac{K_3(\zeta_i)}{K_2(\zeta_i)} + \zeta_e \frac{K_3(\zeta_e)}{K_2(\zeta_e)} \right) v^2 - (\zeta_i + \zeta_e) v_A^2 \cos(\theta)^2 \right] \delta \mathbf{v}_\perp \\
&- (\zeta_i + \zeta_e) v_A^2 \hat{\mathbf{k}}_\perp \left(\hat{\mathbf{k}}_\perp \cdot \delta \mathbf{v} \right) - v\hat{\mathbf{k}}_\perp \frac{\delta p^T}{p} \\
&+ \frac{1}{3}\frac{v}{n}\hat{\mathbf{k}}_\perp \frac{\delta \Delta p^T}{p} = 0.
\end{aligned} \tag{99}
$$

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