

# Threshold of Global Stochasticity and Universality in Hamiltonian Systems

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The robustness of noble tori is interpreted as a hierarchy of the fixed points of the renormalization group for KAM tori. The threshold of global stochasticity for a large class of area-preserving maps and 1.5 or 2 degree-of-freedom Hamiltonian systems is estimated by a simple method which relies upon a new version of an approximate renormalization scheme for  $H(v,x,t) = v^2/2 - M\cos x - P\cos k(x-t)$ , that is consistent with that hierarchy.

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Much work has been devoted to the computation of the threshold of global stochasticity (TGS for short) for Hamiltonian systems and area-preserving maps.<sup>1-4</sup> On the other hand, motivated by the current trend in dynamical systems, universal properties of these systems have been studied for their own sake (see Refs. 5 and 6, and references therein). These studies concentrate on KAM (after Kolmogorov, Arnold, and Moser) tori<sup>2</sup> of dimension two contained within an energy surface of dimension three. The increase of the perturbation to a given integrable case eventually leads to the breakup of all KAM tori in a given domain of the phase space; the disappearance of the last torus triggers global stochasticity (or diffusion-like motion) in that domain.<sup>2</sup>

A KAM torus  $\mathcal{T}(w)$  is characterized by an irrational winding number  $w$ . Let  $a_j$ ,  $j = 0, 1, \dots$ , be the coefficients of the continued fraction expansion of  $w$ ,  $w = a_0 + 1/[a_1 + 1/(a_2 + 1/\dots)]$ . The renormalization approach to KAM tori<sup>2,3,5,6</sup> reveals that there is a universality class for each periodic sequence of  $a_j$ 's. One should therefore ask what universal features can be physically relevant. Fortunately, numerical calculations show the robustness of noble tori (they correspond to  $a_j = 1$ , for  $j$  large enough and belong to the universality class of the golden mean  $g = (\sqrt{5}+1)/2$ ): locally (in  $w$ ), a noble torus is the last to disappear.<sup>6</sup>

The results of this Letter are the following: (i) the robustness of noble tori is geometrically interpreted as due to a hierarchy of the fixed points of the renormalization group for KAM tori. This implies that only the noble stable manifold  $\mathcal{S}_1$  associated to  $\mathcal{T}(g)$  is useful for computing a TGS; (ii) our interpretation is born out by an improved

version of an approximate renormalization scheme<sup>2,3</sup> derived for the Hamiltonian

$$H(v, x, t) = v^2/2 - M \cos x - P \cos(x-t), \quad (1)$$

that allows the computation of a simple formula for  $\mathcal{H}_1$  and good estimates of the TGS for any value of  $k$  and  $M/P$ ; (iii) similar estimates can be given for Hamiltonians of the form

$$\mathcal{H}(I, \theta, t) = H_0(I) + \varepsilon \sum_{n \in J} V_n(I) \cos(q_{n1} \theta + q_{n2} t + \psi_n), \quad (2)$$

$$\mathcal{H}^1(\vec{I}, \vec{\theta}) = H_0(\vec{I}) + \varepsilon \sum_{n \in J} V_n(\vec{I}) \cos(\vec{q}_n \vec{\theta} + \psi_n),$$

where  $\psi_n$  is an arbitrary phase,  $J$  is a denumerable set with at least two elements, all vectors are two-dimensional, and where the  $\vec{q}_n$ 's define a series of discrete directions; (iv) our improved renormalization scheme is consistent with the use of a technique introduced by Chirikov<sup>1</sup>, that allows one to take into account any number of primary resonances; for the standard map<sup>1,2</sup> this yields the critical parameter with a 2% error only; (v) the improvement of our scheme is made possible by an approximation suggested by a non-trivial symmetry of Hamiltonian (1) where  $M$  and  $P$  depend linearly on  $v$  and (vi) we establish for the first time from theoretical grounds why the noble unstable eigenvalue is close<sup>5,6</sup> to  $g$ .

We now define the hierarchy of the fixed points. Figure 1 is a schematic two-dimensional description of the space of Hamiltonians in the vicinity of a given integrable one located at the origin. Two

fixed points  $F_i$  of the renormalization group and their stable manifolds  $\mathcal{S}_i$ ,  $i = 1, 2$  are plotted. They correspond to winding numbers  $w_i$  with  $a_j = i$  for all  $j$ 's; thus  $F_1$  is the noble fixed point since  $w_1 = g$ .  $\mathcal{S}_1$  is of codimension one and separates the basins of attraction of  $(0,0)$  and  $(\infty, \infty)$ , and corresponds to the breakup of the KAM torus  $\mathcal{T}(w_1)$ . We say that a point is above (resp. below)  $\mathcal{S}_1$  if itself and the origin are on the same side (resp. opposite sides) of  $\mathcal{S}_1$ . We now interpret the robustness of noble tori in the following way:  $F_1$  is above all non-noble stable manifolds, and all non-noble fixed points are below  $\mathcal{S}_1$ . Let us show how this conjecture implies the robustness of noble tori. Figure 1 displays a one-parameter family of Hamiltonians which correspond to the line  $\Delta$  that intersects  $\mathcal{S}_2$  at A. Since A is above  $\mathcal{S}_1$ ,  $\mathcal{T}(w_2)$  is stabler than  $\mathcal{T}(w_1)$  for the  $\Delta$  family. Renormalize from A for  $\mathcal{T}(w_2)$ ; then, as A belongs to  $\mathcal{S}_2$ , its iterates converge toward  $F_2$ , and lie below  $\mathcal{S}_1$  after, say,  $n$  iterations. Each iterate has a new  $w$  with  $a_j$ 's that are the  $a_{j+1}$ 's of the previous one.<sup>3,5,6</sup> Choose a number  $m \geq n$ , and let  $w_m$  be the winding number with  $a_j = 2$  for  $0 \leq j < m$  and  $a_j = 1$  for  $j \geq m$ . Then, renormalize from A for  $\mathcal{T}(w_m)$ . The first  $m$  steps are the same as for  $\mathcal{T}(w_2)$ , but after that, the renormalization mapping is ruled by  $F_1$ . Since the  $m$ -th iterate lies below  $\mathcal{S}_1$ , the forthcoming ones converge to the origin. Therefore  $\mathcal{T}(w_m)$  is undercritical at A, when  $\mathcal{T}(w_2)$  is critical:  $\mathcal{T}(w_m)$  is a noble torus more robust than  $\mathcal{T}(w_2)$ , which is arbitrarily close to it by choosing  $m$  large enough. This rationale can be extended to any non-noble torus. A similar one proves that " $F_1$  is above all non-noble stable manifolds" is a necessary condition for the robustness of noble tori. The threshold of breakdown of a noble torus of the  $\Delta$  family is obtained by

requiring that some iterate of large enough order belongs to  $\mathcal{P}_1$ , which is therefore the unique stable manifold useful for computing a TGS.

This paragraph describes an approximate procedure for analytically computing  $\mathcal{P}_1$  for the space of Hamiltonians (1). Reference 3 already gives a renormalization scheme for getting  $\mathcal{P}_1$ , but it is not consistent with the robustness of noble tori. Furthermore the approximations involved in the scheme completely neglect the distortion of tori due to the primary resonance with amplitude P (P resonance for short). We therefore look for a modified version of this scheme which takes into account that distortion too. Define the winding number  $w$  such that the torus  $\mathcal{T}(w)$  corresponds to a mean velocity  $\langle v(t) \rangle_t = k/(k+w)$ . The Kolmogorov transformation<sup>1,2</sup> with generating function  $F(I, x, t) = Ix + (M/I)\sin x + (P/kJ)\sin k(x-t)$ , where  $J = I-1$ , that kills both resonances M and P (the transformation used in Ref. 3 only kills resonance M) of H at lowest order, yields a new Hamiltonian

$$H'(I, \theta, t) = H_0 + \sum_{m=-\infty}^{+\infty} \sum_{n=0}^{\infty} K_{mn}(I) \cos \phi_{mn}(\theta, t), \quad (3)$$

where  $\phi_{mn}(\theta, t) = m\theta + nk(\theta-t)$ , and where  $H_0$  and the  $K_{mn}$ 's are explicitly computable.<sup>7</sup> Define  $I_r$ , for any  $r$ , by  $dH_0(I_r)/dI = k/(k+r)$ , and  $\ell = \text{integer part of } w$ . When dealing with  $\mathcal{T}(w)$ , a renormalized Hamiltonian of the type (1) is obtained from  $H'$  in analogy with Ref. 3: (i) by retaining only the two  $K_{mn}$  resonances with  $n = 1$  and  $m = \ell, \ell + 1$ ; (ii) by expanding  $H_0$  to second order about  $I_w$ ; (iii) by approximating the  $K_{m1}(I)$ 's by their value at  $I = I_m$  which is the location of the  $K_{m1}$  resonance (and not at  $I = I_w$ , the location of  $\mathcal{T}(w)$ , as in Ref. 3; this is motivated later) and (iv) by rescaling

$\Delta v = u_\ell - u_{\rho+1}$  to unity, where  $u_m = k/(k+m)$  is the mean velocity of the  $K_{m1}$  resonance. This modified renormalization scheme improves the estimates of the TGS for H, and simultaneously happens to verify our conjectured hierarchy of the fixed points. The amplitudes of the primary resonances of the renormalized Hamiltonian are  $M' = F(\ell+\lambda)$ ,  $P' = F(\ell+1-\lambda)$ , where

$$F(m) = \sigma U_m / (\Delta v)^2, \quad (4)$$

with  $\sigma = d^2 H_0(I_w) / dI^2$ ,  $U_m = |K_{m1}(I_m)|$ , and  $\lambda = 0$  (resp. 1) for  $w-\ell < 1/2$  (resp.  $> 1/2$ ). Our improved scheme can only deal with tori  $\mathcal{T}(w)$  such that  $w > 1$ , but this is no limitation since the torus  $\mathcal{T}(w)$  of the system with parameters  $(k, M, P)$  is the torus  $\mathcal{T}(1/w)$  of the equivalent system<sup>2,3</sup> with parameters  $(1/k, P, M)$ . At lowest order in  $M$  and  $P$ ,  $M' \propto PM^{\ell+\lambda}$  and  $P' \propto PM^{\ell+1-\lambda}$ . With the next order corrections  $\mathcal{P}_1$  is defined by

$$MP^{1/g} [1 + c(k)P^2] = R(k), \quad (5)$$

where  $R(k)$  and  $c(k)$  are plotted in Fig. 2 (their involved analytical expressions will be given elsewhere). Figure 1 shows a slice of  $\mathcal{P}_1$  at constant  $k$ . Since for  $\mathcal{T}(g)$ ,  $M' \propto M^2P$  and  $P' \propto MP$ , the noble unstable eigenvalue is readily shown to be  $\delta = g^2 + \beta$ , where  $0 < \beta \approx 0.13$  to be compared with the exact value<sup>5,6</sup>  $\beta \approx 0.032$  (One step of our scheme corresponds to two steps of the scheme of Refs. 5 and 6; our  $\delta$  is therefore the square of theirs).

We now show how to use the knowledge of  $\mathcal{S}_1$  for computing the TGS for  $H$ . Consistently with our conjectured hierarchy, a part of  $\mathcal{S}_1$  is above all non-noble stable manifolds (see Fig. 1). More precisely for  $k = 1$  and  $1/a < M/P < a$  with  $a \approx 25$  and  $M = P$  and  $1/k_m < k < k_m$ , with  $k_m \approx 2.2$ . For the parameters of the corresponding domain, the last torus  $\mathcal{I}(w_*)$  corresponds to  $w_* = g$  or  $1/g$ . The TGS is thus strictly given by Eq. (5) computed with either  $(k, M, P)$  or  $(1/k, P, M)$ . Simple geometrical arguments allow one to only compute the higher threshold. Indeed  $\mathcal{I}(g)$  is the more robust when the overlap of resonance  $K_{11}(w=0)$  with resonance  $P(w=0)$  is faster than the one with resonance  $M(w=\infty)$ ; this is obviously the case for  $k = 1$ ,  $M/P < 1$ , and for  $M = P, k > 1$ . If  $k$  or  $M/P$  are too far from 1, the previous estimate yields only a lower bound to the TGS, since the last torus is another noble torus. For the practical purpose of estimating the TGS, we look for  $w_*$  as  $w_* = g + \mu$  or  $1/(g + \mu)$ , where  $\mu$  is a positive integer. One iteration of the renormalization scheme yields  $w' = g$ . Therefore Eq. (4) is first used for computing  $M'$  and  $P'$  with  $\ell = \mu + 1$  and  $\lambda = 1$  from  $(k, M, P)$  or  $(1/k, P, M)$ . Then the TGS is computed by setting these values in Eq. (5) where  $k$  is to be replaced by<sup>3</sup>  $k' = (k + \ell)/(k + \ell + 1)$ . The agreement between the numerical and the present theoretical estimates of the TGS as given by the parameter  $s = 2\sqrt{M} + 2\sqrt{P}$  is improved with respect to Ref. 3 and is better than 4% for  $1/25 \leq M/P \leq 25$ ,  $k = 1$ , and for  $M = P$ ,  $1/4 \leq k \leq 4$ ; outside the interval  $[1/k_m, k_m]$ ,  $w_*$  corresponds to  $\mu = 1$ .

Approximations similar to those involved in deriving the improved renormalization scheme for  $H$ , make it possible to estimate the TGS in the domain between two primary resonances with nearby directions  $\vec{q}^2$ ,  $n = m, p$  of Hamiltonians (2) by their local reduction to  $H$ . We compute

$n = m, p$  of Hamiltonians (2) by their local reduction to  $H$ . We compute again the threshold of noble tori  $\mathcal{T}(w)$  with  $w = g + \mu$  or  $1/(g + \mu)$  where  $\mu = 0, 1, \dots$ . For  $\mathcal{H}(I, \theta, t)$ , define  $I_r$  for  $r = 0, w, \infty$  by  $dH_0(I_r)/dI = k\Delta v/(k+r) - q_{m2}/q_{m1}$ , where  $k = |q_{p1}/q_{m1}|$  and  $\Delta v = q_{m2}/q_{m1} - q_{p2}/q_{p1}$ . For a torus  $\mathcal{T}(w)$ , the reduction of  $\mathcal{H}$  to  $H$  yields  $M = F(m)$ ,  $P = F(p)$ , where  $F$  is given by Eq. (4) with  $U_m = \varepsilon |V_m(I_\infty)|$ ,  $U_p = \varepsilon |V_p(I_0)|$ , and  $\sigma = |d^2 H_0(I_w)/dI^2|$ . For  $\mathcal{H}^1(\vec{I}, \vec{\theta})$ , the TGS is computed in terms of  $\varepsilon$  for a given energy  $E$ , or vice-versa.  $\mathcal{T}(w)$  corresponds for  $\varepsilon = 0$  to  $\vec{I}_w$  such that  $\vec{\omega} = dH_0(\vec{I}_w)/d\vec{I}$  verifies  $w = |\vec{\omega} \cdot \vec{q}_p / \vec{\omega} \cdot \vec{q}_m|$ . When taking into account the fact that locally the unperturbed energy line  $E = H_0(\vec{I})$  is a parabola<sup>3</sup>, the reduction of  $\mathcal{H}^1$  to  $H$  yields  $k = |\vec{r} \cdot \vec{q}_p / \vec{r} \cdot \vec{q}_m|$  with  $\vec{r} = (\omega_2, -\omega_1)$ ,  $M = F(m)$ ,  $P = F(p)$ , with  $U_n = \varepsilon |V_n(\vec{I}_n)|$ ,  $\sigma = |\vec{r} \cdot d^2 H_0(\vec{I}_w)/d\vec{I}^2 \cdot \vec{r}|$ ,  $\Delta v = \gamma_p/\alpha_p - \gamma_m/\alpha_m$ , where  $\gamma_n = -\vec{\omega} \cdot \vec{q}_n$ ,  $\alpha_n = \vec{r} \cdot \vec{q}_n / \vec{\omega}^2$ , and  $\vec{I}_n$  is defined by  $\vec{q}_n \cdot dH_0(\vec{I}_n)/d\vec{I} = 0$ ,  $n = m, p$ . Setting the parameters  $(k, M, P)$  computed from  $\mathcal{H}$  or  $\mathcal{H}^1$  for  $\mu = 0$  into Eq. (5) yields a first estimate of the TGS in terms of  $\varepsilon$  or  $E$ , that can be possibly improved as for  $H$  by considering  $\mu \geq 1$  and by iterating once the renormalization scheme before using Eq. (5). The present technique, when applied to the Hamiltonian given by Eq. (32) of Ref. 8, yields the estimate of the TGS:  $E = 0.198$  to be compared with our direct numerical estimate  $E = 0.195$ .

We now show how to take into account more than two primary resonances. For instance, consider the standard map that corresponds, as many other maps, to a time dependent Hamiltonian whose primary resonances all have an equal amplitude.<sup>1,2</sup> All these resonances can be killed by a Kolmogorov transformation.<sup>1</sup> When dealing with the



transformed Hamiltonian as we did for Eq. (3), one gets modified renormalized values of  $M'$  and  $P'$ . Setting these values in Eq. (5) with  $k' = 2/3$  yields the estimate  $\kappa_1 = 0.991$  of the TGS, to be compared with the exact value<sup>2</sup>  $\kappa_* \approx 0.9716$  and with the estimate  $\kappa_0 = 1.107$  obtained from the method of the preceding paragraph. This procedure amounts to  $L = 1$  iteration of the exact renormalization group. A general argument of hyperbolicity predicts an improvement  $i_L = (\kappa_0 - \kappa_*)/(\kappa_L - \kappa_*) \approx \delta^L$  for  $L$  large. Here  $i_1 \approx 7$  larger than  $\delta \approx 2.65$ . When using the resonance-overlap criterion instead of Eq. (5), Chirikov gets a similar improvement from the use of the same Kolmogorov transformation.<sup>1</sup>

We now motivate the computation of the amplitudes of the resonances at the location of the resonances rather than at the location of the KAM torus as in Ref. 3. Consider the Hamiltonian  $H_\lambda$  obtained from Eq. (1) by substituting  $M(v) = M_0 + M'_0 v$  and  $P(v) = P_0 + P'_0(1-v)$  for  $M$  and  $P$ . Write down the canonical equation for  $H_0$  and eliminate  $v$  in between. This yields a second order differential equation for  $x(t)$  that depends on  $M'_0$  and  $P'_0$ , only through  $M_0'^2$ ,  $P_0'^2$  and  $M'_0 P'_0$ . This non-trivial symmetry implies that the TGS has an extremum for  $M'_0 = P'_0 = 0$ . This extremum is a minimum for  $M'_0 = \alpha M_0$  and  $P'_0 = \alpha P_0$ , when the approximation of Ref. 3 or the resonance-overlap criterion of Ref. 8 leads to an estimate of the TGS that is a monotonically decreasing function of  $\alpha$ . The existence of the extremum suggests to approximate  $H_\lambda$  by a new  $H_\lambda$  with  $M'_0 = P'_0 = 0$ , that is by  $H$  with  $M = M(0)$  and  $P = P(1)$ , provided  $M'_0$  and  $P'_0$  are not too large. This approximation, also necessary for applying the resonance-overlap criterion to  $H_\lambda$ , was also checked to be good for the case  $M(v) = M_0(1 + \alpha v^2)$ ,  $P(v) = P_0[1 + \alpha(1-v)^2]$ , and independently recovered

as appropriate in a recent work by Reichl and Zheng.<sup>9</sup> It is crucial for improving our scheme, since the approximation of Ref. 3 gives a bad scheme when applied to Eq. (3).

The present Letter and a previous one<sup>4</sup> yield accurate analytical methods coherent with Hamiltonian universality for computing the TGS in most area preserving maps and 1.5 or 2 degree-of-freedom Hamiltonian systems of practical interest.

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Figure Captions

Fig. 1 - Schematic of the space of Hamiltonians.

Fig. 2 - Plot of  $R(k)$  (solid) and  $c(k)$  (dashed).

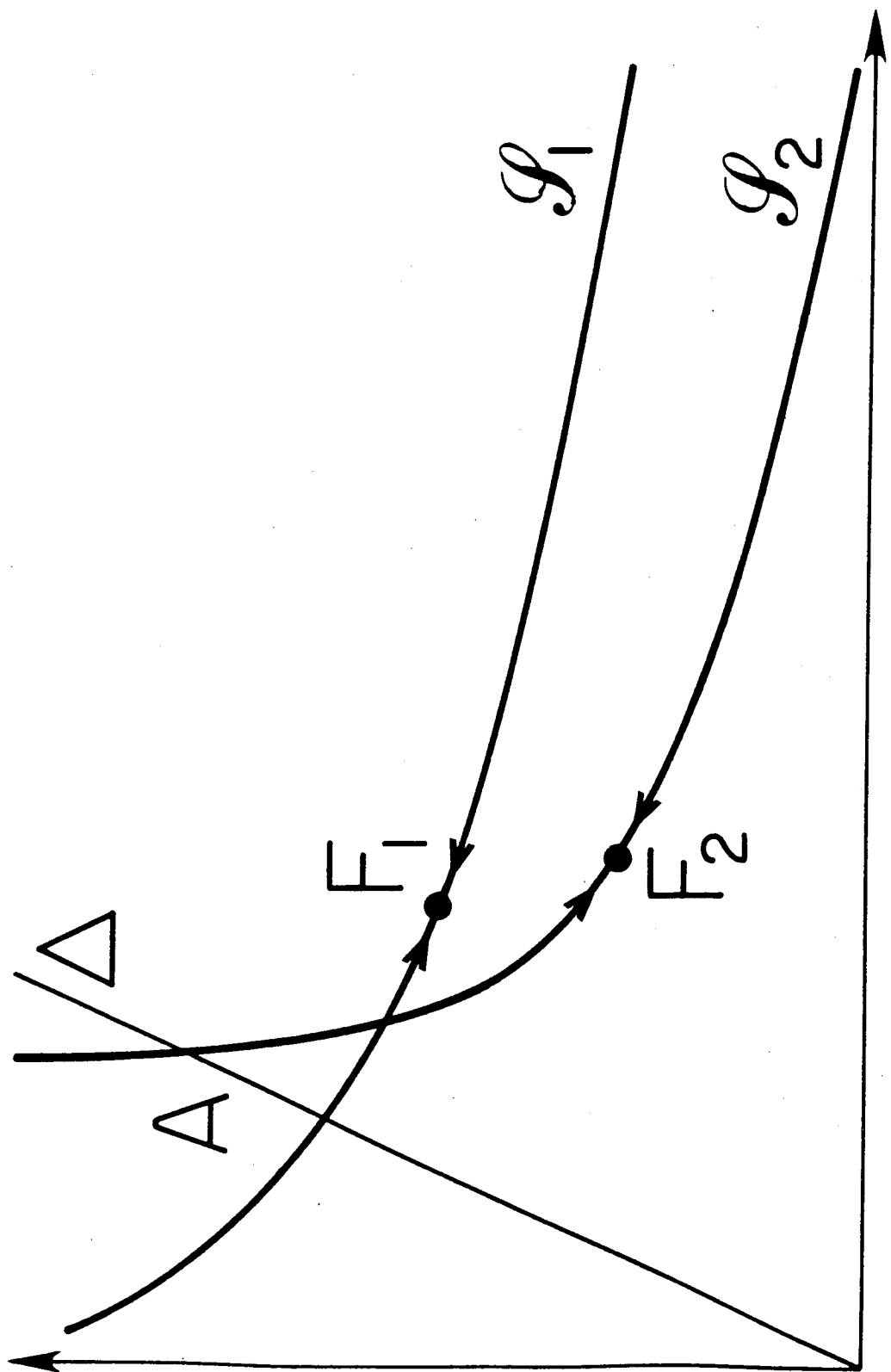


FIG. 1

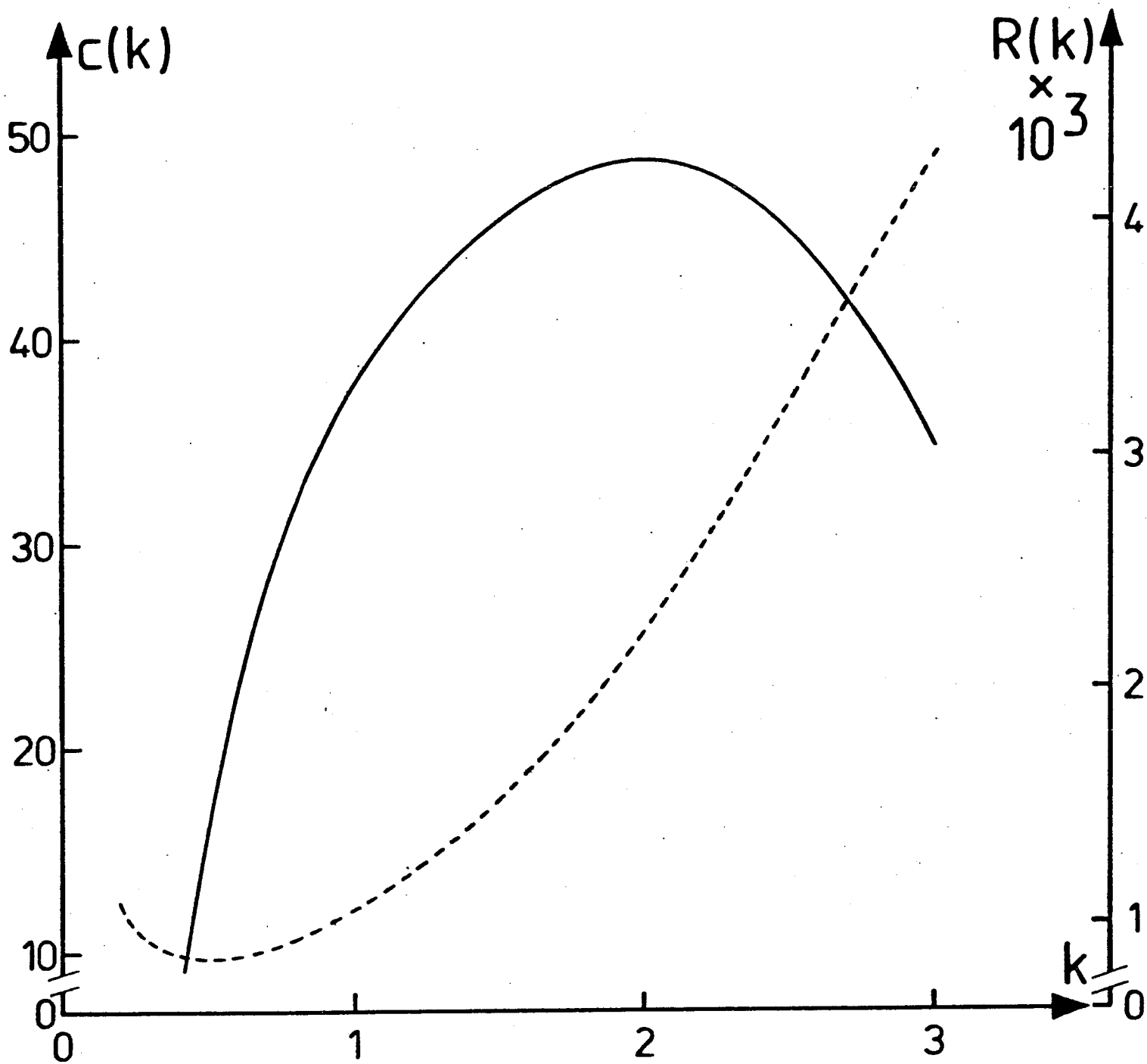


FIG. 2