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Abstract

The nonlinear trapped electron response to drift wave fluctuations is calculated using the coherent approximation to the DIA in action-angle variables appropriate for toroidal geometry. The bounce-averaged nonlinear response to low frequency electrostatic fluctuations is computed. Employing a spectrum of Pearlstein-Berk structure modes satisfying the symmetry of the ballooning representation, the nonlinear terms are evaluated explicitly. Nonlinear effects do not significantly modify the trapped electron response. Quasineutrality and the nonlinear ion response can then be used in a sheared slab to obtain the turbulence level at saturation level. The level of trapped electron diffusion is calculated.

I. INTRODUCTION

The relevance of turbulent nonlinear effects to the theoretical determination of drift-wave induced anomalous transport has been widely recognized.[1-3] Renormalized one point theories of drift-wave turbulence, including broadening of the circulating electron resonance[1] and Compton scattering contributions to the ion response[4], have been advanced. Saturation was achieved as a balance between circulating electron destabilization and shear damping. Subsequent work (also in a sheared slab) has demonstrated that ion Compton scattering effects have a strong stabilizing effect. Compton scattering causes saturation at a much lower level of fluctuations, so low as to cast the existence of nonlinear instability in doubt. Consequently, additional destabilizing physical mechanisms must be introduced to explain anomalous transport using drift wave turbulence models.

The present paper investigates collisionless trapped-electron toroidal drift resonance effects as such an additional source of destabilization. Many previous studies have examined linear trapped electron effects.[6] The work of Catto and Tsang[7] in particular is used here as a starting point. As in most previous slab hybrid theories, a Pearlstein-Berk type eigenmode structure is assumed. While electron temperature effects are neglected, it should be mentioned that a finite temperature gradient can have significant effects on both the trapped electron destabilization[7-8] and the eigenmode structure.[9] Because of the nature of trapped particle orbits a bounce-averaged response is computed, thus introducing coupling between poloidal harmonics with different poloidal mode numbers. The mode phases are chosen to satisfy the symmetry of the ballooning representation.[10] As in the linear work of Catto and Tsang, the dominant physical effect is found to be the radial localization of the trapped electron response resulting from the difference between field line and poloidal harmonic helicities.

A summary of the remainder of the paper follows. In Sec. II a renormalized drift-kinetic equation for trapped electrons is derived using a coherent response Direct Interaction Approximation[11-12], directly performing the gyro-phase and bounce averages. This procedure is considerably simplified by formulating the equation in action-angle variables appropriate to the toroidal geometry.[13] In Sec. III the nonlinear coefficients in the trapped electron kinetic equation are approximately evaluated, using the properties of the Pearlstein-Berk eigenfunction, the ballooning symmetry, and a simple model for the spectrum of modes. In Sec. IV the dominant nonlinear trapped electron effect is found to be poloidal diffusion. For typical levels of turbulence, this causes only small deviations from the linear response, allowing a simple, "patched", perturbative form to be written for the response at arbitrary energies. In Sec. V the dominant linear trapped electron response is thus combined with previously computed nonlinear ion response to form the eigenmode equation. The eigenmode equation and dispersion relation are solved variationally. The level of trapped electron diffusion is calculated. In Sec. VI the results of this paper and its relations to present and future work are discussed.

II. RENORMALIZED DRIFT-KINETIC EQUATION FOR TRAPPED ELECTRONS

In this section a renormalized drift-kinetic equation for the trapped electrons is derived. Action-angle variables appropriate to particle motion in a toroidal field[13-14] are employed; this simplifies and shortens the derivation considerably.

The three action coordinates, denoted by the vector $\underset{\sim}{J} \equiv (M,J,p)$ are given by:

$$M \equiv \frac{m_e}{2}\frac{v_\perp^2}{\Omega} = \frac{m_e^2 c}{e}\mu \ , \qquad (1)$$

$$J \equiv \frac{e}{c}\oint \frac{d\beta}{2\pi}\,\tilde{\alpha}\,(\beta;H_0,p,M) \ , \qquad (2)$$

and

$$p \equiv m_e \dot{\zeta} R^2 - \frac{e}{c}\psi \ . \qquad (3)$$

The action M is proportional to the magnetic moment $\mu \equiv v_\perp^2/2B$, where $v_\perp$ is the component of velocity perpendicular to the magnetic field, and B is the magnetic field strength. The other quantities in Eq. (1) are e , m_e , and Ω , the electron charge, mass, and gyro-frequency, and c , the speed of light. The action J is proportional to the toroidal magnetic flux enclosed by the particle orbit. In Eqs. (2) and (3) α , β , and ζ are toroidal flux coordinates, where α is the toroidal magnetic flux, and β and ζ are the usual poloidal and toroidal angles. The function $\tilde{\alpha}$ gives the radial position α as a function of β and the adiabatic invariants p, M , and H_0 ; H_0 is the unperturbed particle energy

$$\underset{\sim}{H_0} = \frac{m_e v^2}{2} + e\Phi_0 , \qquad (4)$$

where Φ_0 is the unperturbed potential. The action p is the toroidal angular momentum, where $\dot{\zeta}$ is the toroidal angular velocity, R is the major radial coordinate $R \approx R_0(1 + \varepsilon \cos\beta)$, R_0 is the major radius, ε is the inverse aspect ratio, and ψ is the poloidal magnetic flux.

Corresponding to the actions are the angles

$$\underset{\sim}{\Theta} \equiv (\Theta_g, \Theta, \phi) \qquad (5)$$

where Θ_g , corresponding to M , is the bounce-averaged gyro-angle, Θ , corresponding to J , is a poloidal angle which parameterizes the guiding center position along the banana orbit and ϕ , corresponding to p , is a toroidal angle which labels the point where the banana orbit intersects the circle determined by $\beta = 0$ and α = constant.

With these variables, the trapped electron Hamiltonian, including equilibrium and turbulent fields, may be written as

$$H(\underset{\sim}{J}, \underset{\sim}{\Theta}, t) = \underset{\sim}{H_0}(\underset{\sim}{J}) + \sum_{\underset{\sim}{\ell} \neq 0} \exp[i\underset{\sim}{\ell}\cdot\underset{\sim}{\Theta}_{H_{\underset{\sim}{\ell}}}(\underset{\sim}{J}, t)] . \qquad (6)$$

Here $\underset{\sim}{\ell} = (\ell, m, n)$ labels the Fourier harmonics in the three angles (Θ_g, Θ, ϕ). The quantity $\underset{\sim}{H_0}(\underset{\sim}{J})$ with $\underset{\sim}{\ell} = \underset{\sim}{0} = (0,0,0)$ is the angle-independent part of the Hamiltonian (Eq. (4), rewritten in terms of $\underset{\sim}{J}$), corresponding to unperturbed particle motion in the equilibrium fields, for which the actions are adiabatic invariants. The corresponding canonical equations are

$$\dot{\underset{\sim}{\Theta}} = \frac{\partial H_0}{\partial \underset{\sim}{J}} \equiv \underset{\sim}{\omega}_0 \tag{7}$$

and

$$\dot{\underset{\sim}{J}} = \frac{-\partial H_0}{\partial \underset{\sim}{\Theta}} = 0 \ . \tag{8}$$

Equation (7) leads to[14] the unperturbed orbit frequencies

$$\underset{\sim}{\omega}_0 = \left(\frac{\partial H_0}{\partial M}, \frac{\partial H_0}{\partial J}, \frac{\partial H_0}{\partial p} \right) = (\Omega, \omega_b, \omega_D) \ , \tag{9}$$

which are the bounce-averaged gyro-frequency, Ω , the bounce frequency, ω_b , and the toroidal drift frequency of the banana center, ω_D . Nonlinear effects, such as quasi-linear diffusion of the equilibrium distribution, will cause slow, diffusive changes in $\underset{\sim}{H_0}$, which are neglected in this section. This paper is restricted to the electrostatic case,

$$H_{\underset{\sim}{\ell}} = e\phi_{\underset{\sim}{\ell}} \ . \tag{10}$$

The corresponding perturbed orbital frequencies are denoted by

$$\underset{\sim}{\omega}_{\underset{\sim}{\ell}} = \frac{\partial H_{\underset{\sim}{\ell}}}{\partial \underset{\sim}{J}} \,. \tag{11}$$

(Note that for the $\underset{\sim}{\ell}$'s typical of drift wave structures, the $\underset{\sim}{\ell} \neq 0$ components of the equilibrium fields need not be included in $H_{\underset{\sim}{\ell}}$.) In action-angle variables, the Vlasov equation for the trapped electron distribution f is

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \underset{\sim}{\Theta}} \cdot (\dot{\underset{\sim}{\Theta}} f) + \frac{\partial}{\partial \underset{\sim}{J}} \cdot (\dot{\underset{\sim}{J}} f) = 0 \,, \tag{12}$$

with $\dot{\underset{\sim}{\Theta}}$ and $\dot{\underset{\sim}{J}}$ given by Eqs. (7) - (8). Fourier transforming Eq. (12), f is written, parallel to Eq. (6), as

$$f(\underset{\sim}{\Theta},\underset{\sim}{J},t) = f_0(\underset{\sim}{J}) + \sum_{\substack{\underset{\sim}{\ell}\neq 0 \\ \omega}} \exp(-i\omega t) \exp\left[i\underset{\sim}{\ell}\cdot\underset{\sim}{\Theta}\right] f_{\underset{\sim}{\ell}\omega}(\underset{\sim}{J}) \,, \tag{13}$$

where an additional sum over the system eigenfrequencies, ω, has been introduced; the ω subscript is often suppressed. Note that the eigenfrequencies ω are distinct from the orbital frequencies $\underset{\sim}{\omega}_{\underset{\sim}{\ell}}$. The $(\underset{\sim}{\ell},\omega)$ component of Eq. (12) is then (for $\underset{\sim}{\ell} \neq 0$, $\omega \neq 0$)

$$-i(\omega - \underset{\sim}{\ell}\cdot\underset{\sim}{\omega}_0) f_{\underset{\sim}{\ell}} + N_{\underset{\sim}{\ell}} = i\underset{\sim}{\ell} H_{\underset{\sim}{\ell}} \cdot \frac{\partial f_{\underset{\sim}{\ell}}}{\partial \underset{\sim}{J}} \,, \tag{14}$$

where $N_{\underset{\sim}{\ell}}$ is the nonlinear term

$$N_{\underset{\sim}{\ell}} = i \sum_{\substack{\underset{\sim}{\ell}',\omega' \\ (\underset{\sim}{\ell}'\neq \underset{\sim}{0},\underset{\sim}{\ell})}} \left(\underset{\sim}{\ell}' \cdot \underset{\sim}{\omega}_{\underset{\sim}{\ell}'} f_{\underset{\sim}{\ell}-\underset{\sim}{\ell}'} - \underset{\sim}{\ell}' \cdot \frac{\partial}{\partial \underset{\sim}{J}} H_{\underset{\sim}{\ell}'} f_{\underset{\sim}{\ell}-\underset{\sim}{\ell}'}\right) \,. \tag{15}$$

Here, and in the following, $(\partial/\partial \underset{\sim}{J})$ acts on all quantities to its right unless indicated otherwise.

Following Hazeltine[15], the transformed Vlasov equation is renormalized using a Direct Interaction Approximation[11-12] for the coherent response. This consists of keeping only the phase coherent part of the nonlinear term, which then has the form

$$N^c_{\underset{\sim}{\ell}} = d_{\underset{\sim}{\ell}} f_{\underset{\sim}{\ell}} + b_{\underset{\sim}{\ell}} H_{\underset{\sim}{\ell}} \ , \tag{16}$$

where the operators $d_{\underset{\sim}{\ell}}$ and $b_{\underset{\sim}{\ell}}$ are independent of phase, so that $N^c_{\underset{\sim}{\ell}}$ has the same phase as $H_{\underset{\sim}{\ell}}$. Considering $H_{\underset{\sim}{\ell}}$ as a test wave field, the coherent approximation thus gives the one-point nonlinear dielectric response. To compute $d_{\underset{\sim}{\ell}}$ and $b_{\underset{\sim}{\ell}}$ Eq. (14) is solved for $f_{\underset{\sim}{\ell}-\underset{\sim}{\ell}'}$, and the terms in this solution which, when substituted in Eq. (15), make "directly interacting" phase coherent contributions to $N_{\underset{\sim}{\ell}}$ are then retained. In this calculation, Eq. (16) is employed, and the Hamiltonian is treated as a first order quantity, so that, for simplicity, wave-wave interaction terms are excluded.[16] The resulting operators are

$$d_{\underset{\sim}{\ell}} f_{\underset{\sim}{\ell}} = \sum_{\underset{\sim}{\ell}'\omega'} (\underset{\sim}{\ell}\cdot\underset{\sim}{\omega}_{\underset{\sim}{\ell}'} - \underset{\sim}{\ell}'\cdot\partial_{\underset{\sim}{J}} H_{\underset{\sim}{\ell}'}) G_{\underset{\sim}{\ell}-\underset{\sim}{\ell}'} [f_{\underset{\sim}{\ell}} \underset{\sim}{\ell}\cdot\underset{\sim}{\omega}^*_{\underset{\sim}{\ell}'} + H^*_{\underset{\sim}{\ell}'} (\underset{\sim}{\ell}'\cdot\partial_{\underset{\sim}{J}} f_{\underset{\sim}{\ell}})] \tag{17}$$

and

$$b_{\underset{\sim}{\ell}} H_{\underset{\sim}{\ell}} = - \sum_{\underset{\sim}{\ell}'\omega'} (\underset{\sim}{\ell}\cdot\underset{\sim}{\omega}_{\underset{\sim}{\ell}'} - \underset{\sim}{\ell}'\cdot\partial_{\underset{\sim}{J}} H_{\underset{\sim}{\ell}'}) G_{\underset{\sim}{\ell}-\underset{\sim}{\ell}'} [f^*_{\underset{\sim}{\ell}'} \underset{\sim}{\ell}'\cdot\partial_{\underset{\sim}{J}} H_{\underset{\sim}{\ell}} + H_{\underset{\sim}{\ell}} (\underset{\sim}{\ell}\cdot\partial_{\underset{\sim}{J}} f^*_{\underset{\sim}{\ell}'})] \ , \tag{18}$$

where $G_{\underset{\sim}{\ell}}$ is the Green's function

$$G_{\underset{\sim}{\ell}} \equiv \left[-i(\omega-\underset{\sim}{\ell}\cdot\underset{\sim}{\omega}_0) + \underset{\sim}{d}_{\ell}\right]^{-1} , \tag{19}$$

and the minus one exponent denotes operator inversion.

To simplify the computation of the drift wave response, first note that, for typical tokamak parameters, the unperturbed orbit frequencies order as $\omega_D \ll \omega_b \ll \Omega$. Furthermore, the drift wave frequency $\omega \leqslant \omega_{*e}$ satisfies $\omega \ll \omega_b$. (Here ω_{*e} is the electron diamagnetic drift frequency). Thus, the gyro- and bounce-averaged drift wave response is appropriate. For such a response to be meaningful, the bounce time must be shorter than the wave-particle decorrelation time resulting from radial or poloidal diffusion. Prior to bounce averages, the guiding center decorrelation time is considered, while after the average it is the banana center diffusion that is of interest. The dominant guiding-center decorrelation mechanism has been shown[1] to be a combination of radial diffusion and free streaming along sheared magnetic field lines, which yields a decorrelation time

$$\tau_c = \left[\frac{1}{3}(k_\parallel' v_e)^2 D_r\right]^{-1/3} ,$$

where $k_\parallel' \equiv (m/rL_s)$ is the radial derivative of the parallel wavenumber in the vicinity of the rational surface, m is the poloidal mode number, r is the minor radius, L_s is the shear length, $v_e \equiv (2T_e/m_e)^{1/2}$ for electron temperature, T_e is the electron thermal velocity, and D_r is the radial diffusion coefficient. The bounce frequency is given by

$$\omega_b = \frac{\pi(\varepsilon\mu B)^{1/2}}{2qRK(\kappa)} , \tag{20}$$

where q is the safety factor, $q(\alpha) \equiv (d\psi/d\alpha)^{-1}$, κ is the trapping parameter

$$\kappa^2(\underset{\sim}{J}) \equiv \left[\frac{H_0 - e\Phi_0 - m\bar{B}\mu(1-\varepsilon)}{2\varepsilon m\mu\bar{B}}\right] , \tag{21}$$

$\bar{B}$ is the flux-surface averaged magnetic field, and K is the complete elliptic integral of the first kind. Note that κ is defined such that $0 < \kappa < 1$ for trapped particles, with the trapped-circulating boundary at $\kappa = 1$.

The condition for validity of the bounce-averaged response is then $\omega_b\tau_c > 1$. For drift waves satisfying the approximate dispersion relation

$$\omega(k_\perp) = \frac{k_\perp\rho_s v_s}{L_n} \left[1 + (k_\perp\rho_s)^2\right]^{-1} \tag{22}$$

(where $k_\perp = m/r$ is the perpendicular wavevector, v_s is the sound speed $v_s \equiv \left(T_e/T_i\right)^{1/2} v_i$, T_i and v_i are the ion temperature and thermal velocity, $\rho_s \equiv (v_s/\Omega_i)$ where Ω_i is the ion gyro-frequency, and L_n is the density scale length) this condition becomes

$$\omega_b\tau_c \approx \left(\frac{\pi}{\sqrt{2}qK(\kappa)}\right) \left(\frac{M_i}{m_e}\right)^{1/2} \varepsilon^{3/2}\left(\omega\tau_c\right) > 1 . \tag{23}$$

Here M_i is the ion mass , and $k_\perp \rho_s \approx 1$, $L_n \approx r$ and $v_\perp \approx v_e$ have also been assumed. For turbulence levels $\left(\omega\tau_c\right) \approx 1$ and other typical parameters ($q \approx 3$, $\varepsilon \approx 1/5$) Eq. (23) is satisfied for almost all values of κ (up to $\kappa = 0.97$ for the parameters given).

The gyro- and bounce-averaged response is easily effected by taking the $\underset{\sim}{\ell} = (0,0,n)$ component of Eqs. (14) and (16) - (19), since this is an average over the angles parameterizing the gyro and bounce motions. It is then useful to divide the trapped electron response into adiabatic and nonadiabatic pieces, so that

$$f_n = g_n - \frac{f_0}{T_e} H_n , \tag{24}$$

where $\underset{\sim}{\ell} = (0,0,n)$ is denoted by n alone, and g_n is the nonadiabatic response. With this form, it follows that

$$\begin{aligned} d_n(H_n,H_n)f_n + b_n(H_n,f_n)H_n &\approx d_n(H_n,H_n)g_n + b_n(H_n,g_n)H_n \\ &\equiv d_n g_n + \bar{b}_n H_n , \end{aligned} \tag{25}$$

where the functional dependence of d_n and b_n have been indicated explicitly. The nonlinear terms thus approximately annihilate the adiabatic response. This is easily demonstrated by an explicit computation of the nonlinear term, taking $f_0(\underset{\sim}{J}) = \exp\left[-H_0(\underset{\sim}{J})/T\right]F\left[\alpha_0(p)\right]$, a local Maxwellian with the density profile determined by the function F . The radial coordinate α is replaced by the average flux surface, or banana tip, coordinate α_0 which satisfies $p + (e/c)\psi(\alpha_0) = 0$; electron banana width corrections are neglected since they are small compared to any scale characteristic of the drift wave turbulence. It is then a straightforward matter to show[14] that corrections to Eq. (25) are of order (L_ϕ/L_n), where L_ϕ is the eigenmode scale length, and, for the model considered here [see Eq. (32)] $L\phi \sim x_T \ll L_n$.

Combining Eqs. (14), (16) - (18), (24) and (25) yields the renormalized drift-kinetic equation for the nonadiabatic trapped electron response

$$-i(\omega - n\omega_D)g_n + d_n g_n = \frac{-if_0}{T_e}\left[(\omega - \omega_{*n}) - ib_n\right]H_n \qquad (26)$$

where now b_n stands for $(T_e/f_0)\bar{b}_n$. The first term on the left-hand side of this equation displays the drift resonance, occurring when the banana center drift velocity v_D $[n\omega_D = (n/R)\, v_D \equiv k_T v_D$, where k_T is the toroidal wavenumber] equals the toroidal phase velocity of the wave (ω/k_T). The second term represents broadening of this resonance by diffusion due to nonlinear Landau resonance interactions with a bath of turbulent "background" waves. The first term on the right-hand side gives the basic drift wave response, with $\omega_{*n} = (nq/r)(\rho_e v_e/L_n)$, and the second term can be interpreted as a renormalization of the equilibrium distribution f_0.

It is important to note that the index n in Eq. (26) labels the n^{th} harmonic of the third orbital angle coordinate, rather than of the toroidal angle coordinate. However, the spatial harmonic is required to form the eigenmode equation from Poisson's equation. Furthermore, simple models of the eigenmodes are made most conveniently in terms of the spatial harmonics. Thus, the orbital harmonic source term $H_n(\alpha)$ must be related to the eigenmode spatial harmonics $\Phi_{mn}(\alpha)$, where m and n are harmonics of β and ζ, respectively. This relation, derived in Appendix A, is

$$H_n(\alpha) = e \sum_m \Phi_{nm}(\alpha) \langle \exp[i\beta(nq - m)] \rangle_b , \qquad (27)$$

where $\langle\ \rangle_b$ denotes the bounce average (Eq. A7). This result indicates, as verified later, that the response to $\Phi_{mn}(\alpha)$ is proportional to the bounce-averaged phase factor $\langle \exp[i\beta(nq - m)] \rangle_b$, which represents two important physical effects.[7] The factor $\exp[i\beta(nq - m)]$ represents the difference in helicities between a wavefront on the mode rational surface $r = r_{mn}$ where $q(r_{mn}) = m/n$, corresponding to the (m,n) mode, and a particle moving along a field line on a closely neighboring surface (where $r \neq r_{mn}$, but $|(r - r_{mn})/r_{mn}| \ll 1$). The quantity $\langle \exp[i\beta(nq - m)] \rangle_b$ is thus, first of all, a bounce-averaged measure of particle motion perpendicular to the wavefront. Since it decreases due to phase-mixing as $|r - r_{mn}|$ increases, the resonant interaction is localized near $r = r_{mn}$. The second effect is that, for a given difference in helicities between field line and wavefront, less deeply trapped particles with longer bounce lengths sample greater variations in the wave field. Combined, the helicity matching and bounce length constraints give the resonant region a characteristic width (Δ/κ) where $\Delta \equiv (1/nq')$, the distance between mode rational surfaces for fixed n, is the radial scale length for the phase $(m - nq)$. Note that the resonance is spatially widest, and thus strongest, for the particles most deeply trapped.

Returning now to Eqs. (17) and (18), these formulas can be put into a more useful form, independent of the specific form of radially localized eigenmodes $\Phi_{mn}(\alpha)$. These forms are

$$d_n(r) = \left(\frac{c}{B}\right)^2 \sum_{n'm'm''} G_{n-n'} \Big[\left(\frac{nq}{r}\right)^2 \{\frac{\partial}{\partial r} \langle \exp[i\beta(n'q - m')]\rangle_b \, \Phi_{n'm'}(r)\}$$

$$\{\frac{\partial}{\partial r} \langle \exp[-i\beta(n'q - m'')]\rangle_b \, \Phi^*_{n'm''}(r)\}$$

$$- \langle \exp[i\beta(n'q - m')]\rangle_b \langle \exp[-i\beta(n'q - m'')]\rangle_b$$

$$\times \left(\frac{n'q}{r}\right)^2 \Phi_{n'm'} \Phi^*_{n'm''} \frac{\partial^2}{\partial r^2}$$

$$+ \left(\frac{nq}{r}\right)\left(\frac{n'q}{r}\right) (\{\frac{\partial}{\partial r} \langle \exp[i\beta(n'q - m')]\rangle_b \, \Phi_{n'm'}\}$$

$$\Phi^*_{n'm''} \langle \exp[-i\beta(n'q - m'')]\rangle_b$$

$$- \{ \frac{\partial}{\partial r} \langle \exp[-i\beta(n'q - m'')]\rangle_b \, \Phi^*_{n'm''})$$

$$\Phi_{n'm'} \langle \exp[i\beta(n'q - m')]\rangle_b\} \frac{\partial}{\partial r} \Big] \tag{28}$$

and

$$b_n(r) = -\left(\frac{c}{B}\right)^2 \sum_{n'm'm''} G_{n-n'} \left(\frac{\omega' - \omega_{*n'}}{\omega' - n'\omega_D}\right) \Big[\left(\frac{nq}{r}\right)^2$$

$$\{\frac{\partial}{\partial r} \langle \exp[i\beta(n'q - m')]\rangle_b \, \Phi_{n'm'}(r)\}$$

$$\{ \frac{\partial}{\partial r} \langle \exp[-i\beta(n'q - m'')]\rangle_b \, \Phi^*_{n'm''}(r)\}$$

$$- \langle \exp[i\beta(n'q - m')]\rangle_b \langle \exp[-i\beta(n'q - m'')]\rangle_b$$

$$\times \left(\frac{n'q}{r} \right)^2 \Phi_{n'm'} \, \Phi^*_{n'm''} \frac{\partial^2}{\partial r^2}$$

$$+ \left(\frac{nq}{r} \right)\left(\frac{n'q}{r} \right) \left(\left\{ \frac{\partial}{\partial r} \langle \exp[i\beta(n'q - m')] \rangle_b \, \Phi_{n'm'} \right\} \right.$$

$$\Phi^*_{n'm''} \langle \exp[-i\beta(n'q - m'')] \rangle_b$$

$$- \left\{ \frac{\partial}{\partial r} \langle \exp[-i\beta(n'q - m'')] \rangle_b \, \Phi^*_{n'm''} \right)$$

$$\Phi_{n'm'} \langle \exp[i\beta(n'q - m')] \rangle_b \left. \right\} \frac{\partial}{\partial r} \Big] \, . \tag{29}$$

To derive these equations, Eq. (25) is evaluated for $\underset{\sim}{\ell} \to n$ and $\omega_{n'}$ is replaced by $(\partial H_{n'}/\partial p)$, where $\partial p = -(qc/e)(1/Br)(\partial/\partial r)$. The last approximation is true neglecting the implicit dependence of g_n on p , which occurs through its energy dependence. Such terms are of order $(\partial E/\partial p)(\partial g_n/\partial E) \approx (\omega_b g_n/T_e)$, where E is the particle energy; they are smaller than the terms retained by a factor of order $(L_\phi/L_s) \ll 1$. The basic assumption which simplifies d_n and b_n is then that $g_n \partial_r \Sigma_{n'} F_{n'}(r) \ll \Sigma_{n'} F_{n'}(r)(\partial g_n/\partial r)$ where $F_{n'}$ is any function occurring in the n' sums in Eqs. (17) and (18). Generally, $F_{n'}$ has a radial localization which varies with n' over the whole radial region where modes are excited. Thus, the summed quantity has radial scale length of order r , while g_n has the much smaller eigenmode scale length. Similarly, $G_{n-n'}$ varies slowly with radius, depending upon $\omega_D \alpha (1/r)$, and on $d_{n-n'}$ which, self-consistently, also has slow radial variation. The dominant terms in Eq. (17) are thus

$$\left[n^2 g_n \left|\frac{\partial H_{n'}}{\partial r}\right|^2 - (h')^2 |H_{n'}|^2 \frac{\partial^2 g_n}{\partial r^2} + nn' \left(\frac{\partial H_{n'}}{\partial r} H^*_{n'} - H_{n'} \frac{\partial H^*_{n'}}{\partial r} \right) \frac{\partial g_n}{\partial r} \right] .$$

This result, combined with Eq. (27), yields Eq. (28). Exactly parallel procedures are used to obtain the approximate expression for b_n . In addition, Eq. (26) must also be inverted. In doing so, d_n and b_n are taken as small, since, as verified a posteriori, d_n , $b_n \ll \omega$, and since it turns out to be unnecessary to include nonlinear corrections in computing the nonlinear coefficients themselves.[3] Thus $(\omega' - n'\omega_D)^{-1}$ is understood to mean

$$\lim_{\delta \to 0^+} (\omega' - n'\omega_D - i\delta)^{-1}$$

in Eq. (29).

Finally, note that the toroidal harmonic of the nonadiabatic trapped electron density response is simply related to g_n by

$$n_n = n_0 \int d^3\underset{\sim}{v} \exp(-inq\beta) g_n \quad , \tag{30}$$

where n_0 is the average density. To see this, note from Eq. (A2) that $g = \sum_{n'} \exp[in'(\zeta - q\beta)] g_{n'}$ so that by definition

$$n_n = n_0 \int d^3\underset{\sim}{v} \int_0^{2\pi} \left(\frac{d\zeta}{2\pi} \right) \exp(-in\zeta) \sum_{n'} \exp[in'(\zeta - q\beta)] g_{n'} .$$

Carrying out the ζ integral in this expression immediately yields Eq. (30).

III. COMPUTATION OF THE NONLINEAR COEFFICIENTS

To evaluate the nonlinear part of the trapped electron response, the expressions for d_n and b_n in Eqs. (28) and (29) must be simplified. While this involves many approximations, the nonlinear corrections to the linear response are, in any case, found to be quite small. For convenience, the terms in each nonlinear coefficient are considered separately, labelled according to the number of factors of $k_\Theta = (nq/r)$ and $(\partial/\partial r)$, corresponding to superscripts Θ and $(\partial/\partial r)$, respectively. Thus Eq. (28) becomes

$$d_n(r) = d_n^{\Theta\Theta}(r) + d_n^{rr}(r) + d_n^{\Theta r}(r) + d_n^{r\Theta}(r) .$$

The order of the last two terms is unimportant, since both turn out to be zero.

Consider first the simplest term,

$$d_n^{rr}(r) = -\left(\frac{c}{B}\right)^2 \sum_{h'm'm''} G_{n-n'} \langle \exp[i\beta(n'q - m')]\rangle_b \langle \exp[-i\beta(n'q - m'')]\rangle_b$$
$$\times \left(\frac{n'q}{r}\right)^2 \Phi_{n'm'} \Phi^*_{n'm''} \frac{\partial^2}{\partial r^2} .$$

Several approximations are made to simplify this expression. For convenience, r is repalced by $x \equiv r - r_{mn}$, the distance from a reference mode rational surface with typical mode numbers m and n. The response with local translational symmetry

$$\Phi_{m+p,n}(x + p\Delta) = \Phi_{mn}(x) \tag{31}$$

is studied. This symmetry corresponds to the case in which responses localized about different mode rational surfaces (the same n but different m's) are in phase. Departures from this symmetry tend to be stabilizing.[10] Formally, Eq. (31) is valid because the radial scale length of the coefficients of the eigenmode equation is large compared to Δ, while $p \ll m$ so that the coupling of responses with different m's be significant. Additionally, coupling is significant only for $|x| < x_T \ll r$, [where $x_T = (T_e/T_i)^{1/2}(L_s/L_n)^{1/2}\ \rho_i$] small compared to the scale length of q, so that $n'q(r) - m' \approx (x/\Delta)$ and $n'q - m'' \approx (m' - m'') + (x/\Delta)$. The bounce averages appearing in d_n^{rr} are evaluated in the small κ limit, $\langle\exp(i\beta x)\rangle_b \approx J_0(2\kappa x)$.[7] In fact, as has been verified numerically[17], this approximation is good for almost all κ, including those values for which $\omega_b\tau_c > 1$. Furthermore, as discussed previously, the small κ region makes the largest contribution to the spectral sums in d_n and b_n. With these approximations,

$$\langle\exp[i\beta(n'q - m')]\rangle_b \langle\exp[-i\beta(n'q - m'')]\rangle_b$$

$$\approx J_0\left(\frac{2\kappa x}{\Delta}\right) J_0\left\{\frac{2\kappa}{\Delta}[x + (m' - m'')]\Delta\right\}$$

and defining $p \equiv m'' - m'$, the results are

$$d_n^{rr}(r) \approx -\left(\frac{c}{B}\right)^2 \sum_{n'm'} G_{n-n'}\left(\frac{n'q}{r}\right)^2 \Phi_{n'm'}(x) J_0\left(\frac{2\kappa x}{\Delta}\right)$$

$$\times \sum_p J_0\left[\frac{2\kappa}{\Delta}(x - p\Delta)\right] \Phi^*_{n'm'}(x - p\Delta)\frac{\partial^2}{\partial r^2} \quad .$$

Since the summands are smooth over the intervals $\Delta m' = \Delta n' = 1$, and the mode scale-length $L_\phi >> \Delta$, the m' and n' sums may be replaced by continuous integrals according to the rules[4] $m' \rightarrow k'_\Theta r, n' \rightarrow [m'/q(r_{mn} + x)]$ and $\sum_{m'n'} \rightarrow \int dk'_\Theta \int dx \, (|k'_\Theta| R/L_s)$ where L_s is evaluated at $r=r_{mn}$.

Finally, in the present one-point theory an explicit model for the potential fluctuations must be adopted. As in Ref. (4), a Pearlstein-Berk eigenfunction model is adopted, so that

$$\Phi_{n'm'}(x) \propto \exp\left(\frac{-i\mu x^2}{2} \right) H\left(\frac{|x|}{W_{k'_\Theta}} \right)$$

where

$$\mu \approx \left(\frac{1}{x_T^2} \right) ,$$

$$H\left(\frac{|x|}{W_{k'_\Theta}} \right) \approx \Theta\left(\frac{1}{2} - \frac{|x|}{W_{k'_\Theta}} \right) ,$$

where Θ is the Heaviside step function, and $W_{k'_\Theta} = 2\alpha x'_i$, $\alpha \sim 1/2$ typically, is the spatial width of the mode in terms of the ion Landau damping point

$$x'_i = \frac{\omega'}{k'_\parallel v_i} \approx \left(\frac{L_s}{L_n} \right) \left(\frac{T_e}{T_i} \rho_i \right) .$$

Thus, the model spectrum

$$|\Phi_{nm}|^2 = \left(\frac{L_s}{rRW_{k_\Theta}}\right)\Phi_0^2 S(k_\Theta) H\left(\frac{x}{W_{k_\Theta}}\right) \tag{32}$$

is obtained, with normalization $\sum_{nm}|\Phi_{nm}|^2 = \Phi_0^2$. Here $S(k_\Theta)$ is the spectral density of modes, which is calculated from the nonlinear marginal stability condition in Sec. V. Thus,

$$\Phi_{m'n'}(x)\,\Phi^*_{m'n'}(x - p\Delta) \approx \left(\frac{L_s}{rRW_{k'_\Theta}}\right)\Phi_0^2 S(k'_\Theta)\, H\left(\frac{x}{W_{k'_\Theta}}\right) H\left(\frac{x - p\Delta}{W_{k'_\Theta}}\right)$$

$$\times \exp\left(\frac{i\mu p^2\Delta^2}{2}\right)\exp(-i\mu x p\Delta) ,$$

so that

$$d_n^{rr}(r) \approx -\left(\frac{c}{B}\right)^2 \Phi_0^2 \int_{-\infty}^{+\infty} dk'_\Theta \int_{-\infty}^{+\infty} dx \, \frac{(k'_\Theta)^3}{W_{k'_\Theta}} G_{n-n'} S(k'_\Theta)\, J_0\left(\frac{2\kappa x}{\Delta}\right) H\left(\frac{x}{W_{k'_\Theta}}\right)$$

$$\times \sum_{p=-\infty}^{+\infty} J_0\left[\frac{2\kappa}{\Delta}(x - p\Delta)\right] \exp\left(\frac{i\mu p^2\Delta^2}{2}\right)$$

$$\times \exp(-i\mu x p\Delta)\, H\left(\frac{x - p\Delta}{W_{k'_\Theta}}\right)\frac{\partial^2}{\partial r^2} \tag{33}$$

results.

As shown in Appendix B, the dominant contribution to the p sum in Eq. (33) is from the $p = 0$ term, essentially because of the rapid oscillation of the summand for $p \neq 0$. It is also permissible to perform the x integral in Eq. (33) for fixed $G_{n-n'}$, since the x variation of $G_{n-n'}$, via $n'\omega_D \propto [q(r_{mn}+x)]^{-1}$, is of scale length r, typically large compared to the (Δ/κ) scale on which $[J_0(2\kappa x/\Delta)]^2$ varies. Then, defining $u \equiv (x/W_{k'_\Theta})$, the final result for d_n^{rr} is

$$d_n^{rr}(r) \approx -\left(\frac{c\Phi_0}{B}\right)^2 \int dk'_\Theta G_{n-n'}|k'_\Theta|^3 S(k'_\Theta) \int_{-1}^{1} du \left[J_0\left(\frac{2\kappa u W_{k'_\Theta}}{\Delta}\right)\right]^2 \frac{\partial^2}{\partial r^2} \cdot \tag{34}$$

The other terms in the nonlinear coefficients are easily reduced to similar forms. The only additional information required to evaluate the (Θ,Θ) components is simply $[dJ_0(z)/dz] = -J_1(z)$. The (Θ,r) and (r,Θ) terms vanish because the integrands are proportional to $H(x/W_{k'_\Theta})[\partial/\partial x J_0(2\kappa x/\Delta)\Phi_{n'm'}(x)]\Phi^*_{n'm'}(x)J_0(2\kappa x/\Delta)$, which has odd parity in x. A straightforward computation then yields

$$d_n(r) = \left(\frac{c\Phi_0}{B}\right)^2 \int dk'_\Theta |k'_\Theta| G_{n-n'} S(k'_\Theta)$$

$$\times \left\{\left(\frac{nq}{r}\right)^2\left[\left(\frac{2\kappa}{\Delta}\right)^2 I_3 + \left(\mu W_{k'_\Theta}\right)^2 I_2\right] - (k'_\Theta)^2 I_1 \frac{\partial^2}{\partial r^2}\right\} \tag{35}$$

and

$$b_n(r) = -\left(\frac{c\Phi_0}{B}\right)^2 \int dk'_\Theta |k'_\Theta| G_{n-n'} S(k'_\Theta)\left(\frac{f_0}{T_e}\right)\left(\frac{\omega'-\omega_{*n'}}{\omega'-n'\omega_D}\right)$$

$$\times \quad \{ \left(\frac{nq}{r} \right)^2 \left[\left(\frac{2\kappa}{\Delta} \right)^2 I_3 + (\mu W_{k'_\Theta})^2 I_2 \right] - (k'_\Theta)^2 I_1 \frac{\partial^2}{\partial r^2} \} \, . \tag{36}$$

Here

$$I_1 \equiv \int_{-1}^{1} du \left[J_0 \left(\frac{2\kappa u W_{k'_\Theta}}{\Delta} \right) \right]^2 \tag{37}$$

$$I_2 \equiv \int_{-1}^{1} du \left[u J_0 \left(\frac{2\kappa u W_{k'_\Theta}}{\Delta} \right) \right]^2 \, , \tag{38}$$

and

$$I_3 \equiv \int_{-1}^{1} du \left[J_1 \left(\frac{2ku W_{k'_\Theta}}{\Delta} \right) \right]^2 \, . \tag{39}$$

These integrals are approximated according to the value of the parameter $s \equiv (\Delta/\kappa W_{k'_\Theta})$. Ordering $\Delta \sim \rho_s$, $s \sim (1/2\alpha\kappa)(L_n/L_s)(T_i/T_e)^{1/2}$. For typical numbers, $T_e \geqslant T_i$, $L_n/L_s \sim 1/20$ and $2\alpha \sim 1$, $S \leqslant (1/20)(1/\kappa)$, so that $s < 1$ for $\kappa > 1/20$. The $s < 1$ region is then consistent with approximations made later for the electron response [see Eq. (48)] and dominates the trapped electron contribution to the eigenmode equation [Eq. (C1)] which is weighted by an additional factor of κ . For these reasons, and for the sake of tractability, the relations $s < 1$, and $\kappa > 1/20$ are assumed in the remainder of this paper.

To simplify I_1 , I_2 , and I_3 , the Bessel functions are approximated by their asymptotic forms. Thus, taking $J_0(z) \approx 1$ and

$J_1(z) \approx z/2$ for $|z| < 1$ and $J_0(z) \approx (2/\pi z)^{1/2}\cos(z - \pi/4)$ and $J_1(z) \approx (2/\pi z)^{1/2}\cos(z - 3\pi/4)$ for $|z| > 1$,

$$I_1 \approx 2s + (s/\pi) \int_s^1 \left(\frac{du}{u} \right)\left[1 + \sin\left(\frac{4u}{s} \right) \right] ,$$

which implies

$$I_1 \approx 2s + \frac{s}{\pi} \ln\left(\frac{1}{s} \right) , \tag{40a}$$

where $\left[\mathrm{Si}(4/s) - \mathrm{Si}(4)\right] \ll \ln(1/s)$ has been taken.[18] Similarly,

$$I_2 \approx \frac{2}{3} s^3 + \frac{s}{2\pi} \left(1 - s^2\right) \approx \frac{s}{2\pi} , \tag{41}$$

and

$$I_3 \approx \frac{2s}{3} + \frac{s}{\pi} \ln\left(\frac{1}{s} \right) . \tag{42a}$$

Ordering $2\pi \gg \ln(1/s)$ when $s \gg (1/400)$

$$I_1 \approx 2s \tag{40b}$$

is valid. Since the ratio of coefficients of I_3 to those of I_2 in Eqs. (35) - (36) is of order $4\kappa^2\left(T_i/T_e\right)$, which is typically less than one, I_3 can be approximated similarly, as

$$I_3 \approx \frac{2s}{3} . \tag{42b}$$

With these approximations, the nonlinear coefficients become

$$d_n(r) = \left(\frac{c\Phi_0}{B}\right)^2 \int dk'_\Theta |k'_\Theta| G_{n-n'} S(k'_\Theta)$$

$$\times \left\{\left(\frac{nq}{r}\right)^2 \left[\left(\frac{2\kappa}{\Delta}\right)^2 \frac{2s}{3} + (\mu W_{k'_\Theta})^2 \frac{s}{2\pi}\right] - 2(k'_\Theta)^2 S \frac{\partial^2}{\partial r^2}\right\} \quad (43)$$

and

$$b_n(r) = -\left(\frac{c\Phi_0}{B}\right)^2 \int dk'_\Theta |k'_\Theta| G_{n-n'} S(k'_\Theta) \left(\frac{f_0}{T_e}\right)\left(\frac{\omega' - \omega_{*-n'}}{\omega' - n'\omega_D}\right)$$

$$\times \left\{\left(\frac{nq}{r}\right)^2 \left[\left(\frac{2\kappa}{\Delta}\right)^2 \frac{2s}{3} + (\mu W_{k'_\Theta})^2 \frac{s}{2\pi}\right] - 2(k'_\Theta)^2 S \frac{\partial^2}{\partial r^2}\right\} . \quad (44)$$

It is of interest to compare the results of this section to the corresponding coefficients for the circulating electrons.[1] The radial or (r,r) component of the circulating electron diffusion coefficient is approximately

$$d_n^{c,rr}(r) \approx -\pi\left(\frac{c\Phi_0}{B}\right)^2 \int dk'_\Theta |k'_\Theta|^3 S(k'_\Theta) \left(\frac{x_e}{W_{k'_\Theta}|\omega|}\right) \frac{\partial^2}{\partial r^2} ,$$

evaluated for typical resonant circulating electron velocity v_e , and using the model spectrum of Eq. (32). This is to be compared to the resonant trapped electron diffusion coefficient, taking

$$G_{n-n'} \approx \frac{1}{\omega' - n'\omega_D + d} \approx \frac{1}{\omega'}$$

(see Sec. IV) in Eq. (43), which yields $\left(d_n^{rr}/d_n^{c,rr}\right) \approx \left(\Delta/\pi\kappa x_e\right)$. This result is just the ratio of the spatial widths of the wave-particle resonances for the two processes, and, for typical $\kappa \sim 1/2$, is of order $\left(L_n/L_s\right)\left(T_i M_i/T_e m_e\right)^{1/2}$, typically greater than one. Computing the ratio of poloidal diffusion coefficients in a similar manner, $\left(d_n^{\Theta\Theta}/d_n^{c,\Theta\Theta}\right) \approx \left(\Delta/\kappa x_e\right)\left(W_{k_\Theta'}/x_e\right)^2 (1/4\pi) \gg 1$. This is a much larger ratio, since the trapped electron poloidal diffusion coefficient is dominated by the oscillations in the long ($\sim x_i$) eigenfunction tail. For the same reason, the dominant trapped electron diffusion is poloidal, with $\left(d_n^{\Theta\Theta}/d_n^{rr}\right) \approx \left(x_i/x_T\right)^2$.

IV. NONADIABATIC TRAPPED ELECTRON DENSITY PERTURBATION

An approximate solution for the nonadiabatic response g_n may now be found. Consider first the role of radial diffusion. Formally, g_n may be written as the orbit integral of Eq. (26)

$$g_n(x) = -\int_0^\infty d\tau \exp\left[i\left(\omega - n\omega_D + i d_n^{\Theta\Theta}\right)\tau\right]\left[\frac{if_0}{T_e}\left(\omega - \omega_{*n}\right) + b_n\right]$$

$$\left\langle \sum_p \Phi_{mn}\left[x - p\Delta + \delta x(\tau)\right] J_0\left\{\frac{2\kappa}{\Delta}\left[x - p\Delta + \delta x(\tau)\right]\right\}\right\rangle . \tag{45}$$

Here, Eqs. (27) and (31) have been used, and the slow radial variation of f_0, ω_{*n}, ω_{mn}, and b_n has been neglected. The radial orbit perturbation $\delta x(\tau)$ is the random part of the banana center radial position a time τ before the orbit integral endpoint at t, corresponding to position x. Thus, the orbit is $x_0(t - \tau) = x + \delta x(\tau)$

and $\delta x(\tau=0) = 0$. The ensemble average over such radial perturbations is taken to satisfy $\langle[\delta x(\tau)]^2\rangle = 2D^{rr}\tau$, where $d^{rr} = D^{rr}(\partial^2/\partial r^2)$.

Because of the factor $\exp(-d_n^{\Theta\Theta}\tau)$, $\delta x \leqslant x_c^{\Theta} \equiv (2D^{rr}\tau_c^{\Theta})^{1/2}$ may be assumed in performing the integral in Eq. (45). Since $\Phi_{mn}(x)$ develops fast oscillations in x for $|x| \geqslant x_T$, the main contribution to the p sum comes from $|p| < p_0 \equiv (\sqrt{2}\, x_T/\Delta)$, as in App. B. Note also that g_n is computed in order to find the charge density response $\rho_{nm}[\Phi_{nm}(x), x]$ which, for typical m , is localized to the region $|x| < \Delta << x_T$. Thus, $\Phi_{nm}(x - p\Delta + \delta x) \approx \Phi_{nm}(x + \delta x)$ in the p sum, since $|p\Delta| \lesssim x_T$. The fact that $x_c^{\Theta} << x_T$ has also been used, so that $p\Delta$ only shifts the argument of Φ_{nm} in a region where the eigenfunction varies slowly. The p sum then becomes

$$\sum_{|p|<p_0} J_0\left\{ \frac{2\kappa}{\Delta}\left[x - p\Delta + \delta x(\tau)\right] \right\} \quad ;$$

since $p_0 >> 1$ for typical numbers, J_0 undergoes rapid oscillations for $|p| > p_0$, and the sum can be approximated as

$$\sum_{p=-\infty}^{+\infty} J_0\left\{ \left(\frac{2\kappa}{\Delta}\right)\left[x - p\Delta + \delta x(\tau)\right] \right\} = \frac{1}{\kappa} \ .$$

(See Ref. 7.) Finally, note that $\langle\Phi_{nm}(x + \delta x)\rangle_{\delta x} \approx \Phi_{mn}(x)$ since, typically, $x_c^{\Theta} << x_T$. Thus, g_n is approximately independent of the radial diffusion of the electron orbits. Physically, poloidal diffusion decorrelates the particle from the wave before it can radially diffuse far enough to see significant variation in Φ_{nm} .

To proceed further, consider in more detail the wave-particle resonance condition $n\omega_D = \omega$. From previous formulas, this may be written as

$$\frac{nq}{r}\,\frac{\rho_e v_e}{R}\left(\frac{v}{v_e}\right)^2 \approx \left(\frac{k_\Theta \rho_e v_e}{R}\right)\left(\frac{E_R}{E_0}\right) = \omega_{mn} \tag{46a}$$

or

$$\frac{E_R}{E_0} = \left(\frac{R}{L_n}\right)\left\{\frac{1}{[1 + (k_\Theta \rho_s)^2]}\right\} . \tag{46b}$$

Here, E_R is the resonant particle energy, and E_0 is the thermal energy $(mv_e^2/2)$. Taking $L_n \sim r$, Eq. (46) implies $E_R \sim (1/\varepsilon)$ $\{E_0/[1 + (k_\Theta\rho_s)^2]\}$ which is typically a few times the thermal energy, in the tail of the trapped electron distribution. Then, provided resonance broadening is not so large as to make thermal electrons resonant, the broadening can be treated as a small perturbation of the linear response. This follows since the broadened resonance function has unit area, and includes a greater region of the distribution tail only, which is nearly constant in height.

Thus, to justify a perturbative treatment of the nonlinear response, typical turbulence levels $(e\Phi/T_e)$ are shown to be smaller than the critical level $(e\Phi/T_e)_c$ required to broaden the resonance to thermal energy. From Eqs. (22), (26) and (46), the resonance is broadened to thermal energy when $d_n^{\Theta\Theta} \approx \omega_{*e}(1 - \varepsilon)$. Substituting for the resonant value of $d_n^{\Theta\Theta}$ from Eq. (43) then yields

$$\left(\frac{e\phi}{T}\right)_c = \left(\frac{\rho_i}{L_n}\right)\left[\frac{2(1-\varepsilon)}{\bar{S}(k_\Theta)}\left(\frac{L_s}{L_n}\right)\kappa\right]^{1/2}\frac{(\rho_s\Delta k)}{\left[1+(k_\Theta\rho_s)^2\right]}\eta^{1/4} \quad . \tag{47}$$

Here $\eta \equiv (T_e/T_i)$ and $S(k_\Theta)$ has been written as $\bar{S}(k_\Theta)/(\Delta k_\Theta)^2$, where $\bar{S}(k_\Theta)$ is of order unity, and Δk_Θ is the k_Θ width of the spectrum; typically, $(\Delta k_\Theta)\rho_s \sim 1$. Since experimentally determined levels of $(e\Phi/T)$ are somewhat less than (ρ_i/L_n) , $(e\phi/T) < (e\phi/T)_c$ unless

$$\kappa < \left[(1-\varepsilon)2\eta^{1/2}\right]^{-1}\left(\frac{L_n}{L_s}\right)\left[1+(k_\Theta\rho_s)^2\right]^2 \quad , \tag{48}$$

which becomes $\kappa \lesssim 10^{-1}$ in practice. Consequently, resonance broadening is potentially significant only for a small region of κ space. This region is so small, and the contribution to ρ^T_{mn} small by another factor of κ as well (see next section), that the broadening is treated as everywhere small in the present computation.

Returning now to the computation of g_n , Eq. (28) becomes

$$g_n = \left(\frac{ef_0}{T_e}\right)\left[\frac{(\omega-\omega_{*n}) - ib_n}{(\omega - n\omega_D) + id_n}\right]\langle \exp(inq\beta)\Phi_n(\alpha,\beta)\rangle_b \quad .$$

As just shown, this response is nonresonant for $E \sim E_0$, resonant for $E \sim (1/\varepsilon)E_0$, and nonlinearly these responses are independent. Furthermore, the effects of both radial and poloidal diffusion are very small. Thus, g_n will be evaluated in the resonant (g_n^R) limit, and in the nonresonant (g_n^{NR}) limit, treating d_n and b_n as small perturbations. These two independent responses are then combined into a "patched" response approximately valid for all energies.

The resonant response is evaluated with $E = E_R$, $\omega = n\omega_D$, and superscript R on quantities evaluated at $E = E_R$. Thus,

$$g_n^R = \left(\frac{ef_0}{T_e}\right)\left(\frac{\omega - \omega_{*n} - ib_n^R}{id_n^R}\right) \langle \exp(inq\beta)\Phi_n(\alpha,\beta)\rangle_b \ .$$

The resonant Green's function is

$$G_{n-n'}^R = -i\left[\frac{P}{\omega' - n'\omega_D} + i\pi\delta(\omega' - n'\omega_D)\right] .$$

The principal value part, odd in k'_Θ, is annihilated by the k'_Θ integral. The delta function is

$$\delta(\omega' - n'\omega_D) =$$

$$\frac{\delta(k'_\Theta)}{\left|(\rho_e v_e/L_n) - (\rho_e v_e E_R/RE_0)\right|} + \frac{\left[\delta(k'_\Theta - \bar{k}) + \delta(k'_\Theta + \bar{k})\right]}{2|\bar{k}|\omega(\bar{k})(\rho_s)^2}\left|\frac{E_0}{\varepsilon E_R}\right| ,$$

where

$$\bar{k}(E) \equiv \frac{1}{\rho_s}\left(\frac{E_0}{\varepsilon E} - 1\right)^{1/2} . \qquad (49)$$

Note that Eq. (46) implies $\bar{k}^R = k_\Theta$. With these results, and using Eq. (43),

$$d_n^R \approx \pi\left(\frac{c\Phi_0}{B}\right)^2 \left(\frac{k_\Theta}{\rho_s}\right)^2 \frac{S(k_\Theta)}{\omega(k_\Theta)} \left[1 + (k_\Theta\rho_s)^2\right]$$

$$\times \left\{\left[\frac{2}{3}\left(\frac{2\kappa}{\Delta}\right)^2 + \frac{(\mu W_{k'_\Theta})^2}{2\pi}\right]s - 2s\,\frac{\partial^2}{\partial r^2}\right\} . \qquad (50)$$

The quantity g_n^R can be obtained from this result by retaining the dominant poloidal diffusion terms only.

The coefficient b_n^R is evaluated by rewriting Eq. (44) as

$$b_n^R(r) = -\left(\frac{c\Phi_0}{B}\right)^2 \partial_{v_D} \int_{-\infty}^{+\infty} dk'_\Theta \left(\frac{k'_\Theta}{|k'_\Theta|}\right) S(k'_\Theta)$$

$$\times \left(\frac{\bar{v}'_d - v_d}{v'_d - v_D}\right) \left\{\left(\frac{nq}{r}\right)^2 \left[\left(\frac{2\kappa}{\Delta}\right)^2 \frac{2s}{3} + (\mu W_{k'_\Theta})^2 \frac{s}{2\pi}\right] - 2(k'_\Theta)^2\, s\, \frac{\partial^2}{\partial r^2}\right\} .$$

Here $v_D \equiv (\rho_e v_e/R)(v_R/v_e)^2$, $v_d \equiv (\rho_e v_e/L_n)$ and $\bar{v}_d \equiv \{v_d/[1 + (k_\Theta\rho_s)^2]\}$. In this form, it is apparent that $b_n^R(r) = 0$, since $(k'_\Theta/|k'_\Theta|)$ is odd in k'_Θ , while the rest of the integrand is even in k'_Θ . Thus, resonant particles make no contribution to the renormalization of the background distribution.

The nonresonant response g_n^{NR} is computed for thermal energies much less than the resonant energy (the number of particles with larger energy is exponentially small). From Eq. (46), $(n\omega_D/\omega)$ evaluated at thermal energy is approximately $\varepsilon[1 + (k_\Theta\rho_s)^2] \ll 1$. The nonresonant nonlinear coefficients are consequently computed in the $\omega_D \to 0$ limit, denoted by superscript NR . The corresponding nonresonant response is

$$g_n^{NR} = \left(\frac{ef_0}{T_e}\right)\left[\frac{(\omega - \omega_{*n} - ib_n^{NR})}{\omega + id_n^{NR}}\right] \langle \exp(inq\beta)\,\Phi_n(\alpha,\beta)\rangle_b \quad .$$

Expanding this expression in the small quantities (b_n^{NR}/ω) and (d_n^{NR}/ω) yields, to first order,

$$g_n^{NR} \approx \left(\frac{ef_0}{T_e}\right)\left[\left(1 - \frac{\omega_{*n}}{\omega}\right)\left(1 - \frac{id_n^{NR}}{\omega}\right) - \left(\frac{ib_n^{NR}}{\omega}\right)\right]\langle \exp(inqB)\,\Phi_n(\alpha,\beta)\rangle_b \quad . \tag{51}$$

Eqs. (50) and (51) can be "patched" into a single formula valid for all energies, and simple enough to make the calculation of the trapped electron charge density tractable. This formula is

$$g_n \approx \left(\frac{ef_0}{T_e}\right)\left[\frac{\omega - \omega_{*n}}{\omega - n\omega_D + i(E/E_R)d_n^R} - i\left(\frac{d_n^{NR}}{\omega}\right)\left(1 - \frac{\omega_{*n}}{\omega}\right) - \frac{ib_n^{NR}}{\omega}\right]$$
$$\times \langle \exp(inq\beta)\,\Phi_n(\alpha,\beta)\rangle_b \quad . \tag{52}$$

In the nonresonant limit, taking $E \ll E_R$, the factor (E/E_R) allows d_n^R to be dropped from the denominator of the first term on the right-hand side of Eq. (52), and dropping $n\omega_D$ as well, Eq. (51) for g_n^{NR} is recovered. When $E = E_R$, $\omega = n\omega_D$, and the first term is of order (ω/d_n^R) , while the others are order (d_n^R/ω) , two orders smaller. The other terms can thus be neglected, and the formula for g_n^R is recovered. At intermediate energies, the factor (E/E_R) gives a smooth interpolation of the response. Note that Eq. (52) is in sharp contrast to the nonlinear circulating electron response[1], where the resonance is at thermal energies and broad enough to include the bulk of the electrons.

The physical meaning of the various terms in g_n is straightforward to interpret. Thus, the first term on the right-hand side of Eq. (52) represents the resonance between a particle of energy E and the test wave labelled by k_Θ and $\omega = \omega(k_\Theta)$. This resonance is broadened by diffusion due to background waves, labelled by k'_Θ in the integral for d_n^R , via the wave-particle resonance $\omega' = n'\omega_D$. The coefficients d_n^{NR} and b_n^{NR} involve contributions from resonances in the denominator $(\omega - \omega' - n'\omega_D)$ in the integral over background waves. The resonances are at the roots of a cubic equation in k'_Θ , approximately solved by expanding in ε . Two roots occur for $\omega \approx \omega' >> n'\omega_D$ and $k_\Theta\rho_s \approx k'_\Theta\rho_s \approx 1$. The third root occurs at $\omega \approx -n'\omega_D >> \omega'$, and $k_\Theta\rho_s \approx 1$, $k'_\Theta\rho_s \approx (1/\varepsilon)$. These roots represent nonlinear Landau damping. Thus, the first two roots occur when the test wave ω beats with a background wave ω' , and the beat wave is in resonance with the particle. The third root is the same process, except that the beating causes a very small shift in the test wave frequency. The third root is suppressed, since typical wave spectra are in the range $|k'_\Theta\rho_s| \lesssim 3$.

The nonadiabatic trapped electron charge density, ρ_{nm}^T , can now be computed. From Eq. (30)

$$\rho_{nm}^T = n_0 e \int_{-\pi}^{\pi} \frac{d\beta}{2\pi} \int d^3\underset{\sim}{v} \, \exp[i\beta(m - nq)] g_n(x) \ , \qquad (53)$$

where the β integral with $\exp(i\beta m)$ projects out the m^{th} component of ρ_n^T . To carry out the velocity integral, $g_n(x)$ must be calculated in further detail. Thus, d_n^{NR} and b_n^{NR} are evaluated using the identity

$$\left(\frac{\omega}{\omega - \omega' + n'\omega_D} \right)\left(\frac{1}{\omega' - n'\omega_D} \right) = \left(\frac{1}{\omega' - n'\omega_D} \right) + \left(\frac{1}{\omega - \omega' + n'\omega_D} \right) ,$$

resulting in

$$\rho_{nm}^T = \frac{n_0 e^2}{T_e} \int_{-\pi}^{\pi} \frac{d\beta}{2\pi} \int d^3\underset{\sim}{v} \exp[i\beta(m - nq)] f_0(\underset{\sim}{v})$$

$$\left(\frac{(\omega - \omega_{*n})}{[\omega - n\omega_D + i(E/E_R)d_n^R]} + \left(\frac{c\Phi_0}{B} \right)^2 \int_{-\infty}^{+\infty} dk'_\Theta \, |k'_\Theta| S(k'_\Theta) \right.$$

$$\times \frac{1}{\omega^2} \left\{ 2(k'_\Theta)^2 \, S \frac{\partial^2}{\partial x^2} - \left(\frac{nq}{r} \right)^2 \left[\frac{2}{3} \left(\frac{2\kappa^2}{\Delta} + \mu W_{k'_\Theta} \right)^2 \frac{1}{2\pi} \right] S \right\}$$

$$\left. \times \left[\frac{(\omega' - \omega_{*n'})}{(\omega' - n'\omega_D)} + \frac{\omega' - \omega - (\omega_{*n'} - \omega_{*n})}{\omega - \omega' + n'\omega_D} \right] \right) \langle \exp(inq\beta) \Phi_n \rangle_b \; .$$

The velocity integrals can be evaluated using the relation

$$d^3\underset{\sim}{v} = 2\pi \left(\frac{\varepsilon}{2} \right)^{1/2} v^2 dv d\kappa^2 [\kappa^2 - \sin^2\beta/2]^{-1/2}$$

valid for an axisymmetric system.[7] The corresponding limits of integration are 0 to ∞ for v and $(\kappa_0)^2$ to 1 for κ^2 , where $\kappa_0 \equiv \sin(\beta/2)$. An additional factor of two is included to count electrons with both positive and negative parallel velocities. Using a Maxwellian equilibrium $f_0(\underset{\sim}{v}) = \left(\pi^{3/2} v_e^3\right)^{-1} \exp\left(-E/E_0\right)$ then yields

$$\rho_{nm}^T = \left(\frac{n_0 e^2}{T_e} \right) \left(\frac{\varepsilon}{2} \right)^{1/2} \int_{-\pi}^{\pi} \frac{d\beta}{2\pi} \int_{\kappa_0^2}^{1} d\kappa^2 \left[\kappa^2 - \sin^2 \beta/2\right]^{-1/2}$$

$$\times \exp\left[i\beta(m - nq)\right]\left(L_0 + L_1 \frac{\partial^2}{\partial x^2}\right) \langle \exp(inq\beta)\,\Phi_n(\alpha,\beta)\rangle_b \,. \qquad (54)$$

Here

$$L_0 \equiv A_1(\omega - \omega_{*n}) - \left(\frac{c\Phi_0}{B} \right)^2 \int dk'_\Theta |k'_\Theta| S(k'_\Theta) \frac{1}{\omega^2}$$

$$\times \left(\frac{nq}{r} \right)^2 \left[\frac{2}{3} \left(\frac{2\kappa}{\Delta} \right)^2 + \frac{1}{2\pi} \left(\mu W_{k'_\Theta}\right)^2\right]\left\{(\omega' - \omega_{*n'})A_3\right.$$

$$\left. + \left[(\omega' - \omega) - (\omega_{*n'} - \omega_{*n})\right]A_2 \right\} s, \qquad (55)$$

and

$$L_1 \equiv \left(\frac{c\Phi_0}{B} \right)^2 \int dk'_\Theta |k'_\Theta|^3 S(k'_\Theta) \frac{2s}{\omega^2}$$

$$\left\{(\omega' - \omega_{*n'})A_3 + \left[(\omega' - \omega) - (\omega_{*n'} - \omega_{*n})\right]A_2\right\} . \qquad (56)$$

The functions A_1, A_2 and A_3 are given by $A_1 \equiv \left[-2(y_1)^2/\omega\right]\left[1 + y_1 Z(y_1)\right]$ where $y_1 \equiv \left\{\omega/\left[n\bar{\omega}_D - i\left(E_0/E_R\right)d_n^R\right]\right\}^{1/2}$, Z is the plasma dispersion function, and $n\bar{\omega}_D \equiv \left(n\omega_D E_0/E\right)$, $A_2 \equiv \left[2(y_2)^2/(\omega' - \omega)\right]\left[1 + y_2 Z(y_2)\right]$, where $y_2 \equiv \left(\omega' - \omega/n'\bar{\omega}_D\right)^{1/2}$, and $A_3 \equiv \left[-2(y_3)^2/\omega'\right]\left[1 + y_3 Z(y_3)\right]$ where $y_3 \equiv \left(\omega'/n'\bar{\omega}_D\right)^{1/2}$. Again, the physical meaning of the various terms in ρ^T_{nm} is clear, with A_1 corresponding to the diffusively broadened linear response, and A_2 and A_3 to the nonlinear Landau resonances.

Finally, ρ^T_{nm} must be expressed as an operator applied to $\Phi^{(x)}_{mn}$. Using Eqs. (27) and (31), the results of App. A, and defining $C_{mn} \equiv \langle \exp[i\beta(m - nq)]\rangle_b$ directly leads to

$$\rho^T_{nm}(\Phi_{nm}(x),x) = \left(\frac{\varepsilon}{2}\right)^{1/2}\left(\frac{n_0 e^2}{T_e}\right)\int_0^1 d\kappa^2 C_{mn}\left(L_0 + L_1 \frac{\partial^2}{\partial x^2}\right)$$
$$\times \sum_{p=-\infty}^{+\infty} \Phi_{nm}(x - p\Delta)C^*_{m+p,n} \; . \tag{57}$$

V. EIGENMODE EQUATION AND DISPERSION RELATION

The equation for the radial structure of the eigenmode is the quasi-neutral limit of Poisson's equation, which has the form

$$\rho^T_{nm}[x,\Phi_{nm}(x)] + \rho^c_{nm}[x,\Phi_{nm}(x)] - \frac{n_0 e^2}{T_e}\Phi_{nm}(x) + \rho^I_{nm}[x,\Phi_{nm}(x)] = 0 \; . \tag{58}$$

Here, ρ^T_{nm} is given by Eq. (57), the adiabatic electrons, $[-n_0 e^2 \Phi_{nm}(x)/T_e]$, and the ions $\rho^I_{nm}[x,\Phi_{nm}(x)]$.

For ρ^I_{nm} , the result found by Diamond and Rosenbluth[4], including the nonlinear effect of ion-Compton scattering, is taken. This response is appropriate to the present calculation which assumes eigenmodes of the Pearlstein-Berk form, and is given by

$$\rho^{I}_{nm}[x,\Phi_{nm}(x)] =$$

$$\left(\frac{n_0 e^2}{T_e}\right)\left[\rho_s^2 \frac{\partial^2}{\partial x^2} + \left(\frac{x}{x_s}\right)^2 + \frac{\omega_{*n}}{\omega} - k_\Theta^2\rho_s^2 + Q_{n\ell}(0)\right]\Phi_{nm}(x) .$$

Here $x_s^2 \equiv 2\left(L_s\omega/k_\Theta c_s\right)^2$, where c_s is the sound speed, and $Q_{n\ell}(0)$, the nonlinear part of the ion response, is given by Eqs. (34d) and (51) of Ref. 4.

The resulting eigenmode equation is

$$- \left(\frac{\varepsilon}{2}\right)^{1/2} \int_0^1 d\kappa^2 C_{mn}\left(L_0 + L_1 \frac{\partial^2}{\partial x^2}\right) \sum_p \Phi_{nm}(x - p\Delta)C^*_{m+p,n} + \Phi_{nm}(x)$$

$$- \left[\rho_s^2 \frac{\partial^2}{\partial x^2} + \left(\frac{x}{x_s}\right)^2 + \left(\frac{\omega_{*e}}{\omega}\right) - k_\Theta\rho_s)^2\right]\Phi_{nm}(x)$$

$$+ i\left\{\frac{[1+(k_\Theta\rho_s)^2]}{2\alpha\omega_{*n}}\right\}\left\{(k_\Theta\rho_s)^2\left(\frac{1}{\omega}\right)\left(\frac{e\Phi_0\Omega_i}{T_i}\right)^2\right\}$$

$$\times \int dk'_\Theta \left|\frac{k'_\Theta}{\omega'}\right| S(k'_\Theta)\left\{(\omega'' - \omega''_{*i})\left(\frac{\omega''}{\omega'}\right)^2 E_2\left[\left(\frac{\omega''}{\omega'\alpha}\right)^2\right]\right\}\Phi_{nm}(x)$$

$$= 0 \equiv \mathscr{L}(x)\Phi_{nm}(x) = 0 , \qquad (59)$$

where $\mathscr{L}$ is the eigenmode operator. Here $Q_{n\ell}(0)$ has been evaluated using the model spectrum of Eq. (32), Ω_i is the ion gyro-frequency, ω_{*i} is the ion drift frequency, neglected here for $T_i < T_e$, $\omega'' \equiv \omega + \omega'$, $\omega^{(\prime)} \equiv \omega(k_\Theta^{(\prime)})$ and E_2 is an exponential function.[19]

Except for the trapped electron term, the eigenmode equation is obviously self-adjoint. To see that the trapped electron term is also self-adjoint, note that for an arbitrary function $\psi(x)$

$$\int_{-\infty}^{+\infty} dx\ \psi(x)\rho_{nm}^{T}[x,\Phi_{nm}(x)] \ \propto\ \int_{-\infty}^{+\infty} dx\ \psi(x)J_0\left(\frac{2\kappa x}{\Delta}\right)\Phi_{nm}(x-p\Delta)J_0\left[\frac{2\kappa}{\Delta}(x-p\Delta)\right] ,$$

where the small κ approximation has been used. Changing variables from x to $x' \equiv (x - p\Delta)$ and the summation index from p to $p' \equiv -p$ then yields

$$\begin{aligned}\int dx\ \psi(x)\rho_{nm}^{T}[x,\Phi_{nm}(x)] \ &\propto\ \sum_p \int dx\ \psi(x-p\Delta) \\ &\qquad \times J_0\left[\frac{2\kappa(x-p\Delta)}{\Delta}\right]\Phi_{nm}(x)J_0\left(\frac{2\kappa x}{\Delta}\right) \\ &= \int dx\ \Phi_{nm}(x)\rho_{nm}^{T}[x,\psi(x)] \quad ,\end{aligned}$$

which proves self-adjointness. The eigenmode equation can thus be solved by a variational principle.[20] An appropriate quadratic form $V(a) = V(\phi,\phi)$ is then constructed such that for the trial function $\phi_{nm}(a,x)$, $(\partial V/\partial a) = 0$ determines $a = a_0$, and thus the eigenfunction $\Phi_{nm}(x) = \phi_{nm}(a_0,x)$. The dispersion relation becomes $V(a_0) = 0$, which determines the eigenvalue ω .

By inspection of Eq. (59) an appropriate quadratic form is

$$V(a) \equiv \frac{\int_{-\infty}^{+\infty} dx\ \phi(a,x)\ \mathscr{L}(x)\phi(a,x)}{\int_{-\infty}^{+\infty} dx\ [\phi(a,x)]^2}\left(\frac{T_e}{n_0 e^2}\right) . \tag{60}$$

Here the numerator is itself variational, and the denominator, which does not change this property, is added as a convenient normalization factor. Anticipating an eigenmode of Pearlstein-Berk form, the trial function $\phi(a,x) = \exp(-ax^2/2)$ is chosen. The integrals in Eq. (60) are then carried out, resulting in

$$V(a) = i\left\{\frac{[1 + (k_\Theta\rho_s)^2]}{2\alpha\omega_{*n}}\right\}\left[\frac{(k_\Theta\rho_s)^2}{\omega}\right]\left(\frac{e\Phi_0\Omega_i}{T_i}\right)^2$$

$$\times \int dk'_\Theta\left|\frac{k'_\Theta}{\omega'}\right| S(k'_\Theta)\left\{\omega''\left(\frac{\omega''}{\omega'}\right)^2 E_2\left[\left(\frac{\omega''}{\omega'\alpha}\right)^2\right]\right\} + \frac{a\rho_s^2}{2} - \frac{1}{2ax_s^2}$$

$$- S^T(a) - \left(\frac{\omega_{*n}}{\omega} - 1\right) + (k_\Theta\rho_s)^2 , \qquad (61)$$

where $S^T(a)$ is given by Eq. (C13) Taking $(\partial V/\partial a)$, the eigenmode equation is

$$\left(\frac{\rho_s^2}{2}\right) + \left(\frac{1}{2a^2x_s^2}\right) = \left[\frac{\partial S^T(a)}{\partial a}\right] . \qquad (62)$$

To zeroeth order in $(\varepsilon)^{1/2}$, Eq. (62) thus yields $a = (i/x_s\rho_s) \equiv i\mu$, which gives the familiar Pearlstein-Berk mode structure. For simplicity, the small trapped electron modification of the eigenmode is neglected here. Since the eigenvalue, which determines marginal stability, is insensitive to the eigenmode structure[20], this is a reasonable approximation.

Thus, setting $a = i\mu$ in $V(a)$ yields the dispersion relation

$$i \left\{\frac{[1 + (k_\Theta \rho_s)^2]}{2\alpha\omega_{*n}}\right\}\left[\frac{(k_\Theta \rho_s)^2}{\omega}\right] \left(\frac{e\Phi_0 \Omega_i}{T_i}\right)^2 \int dk'_\Theta \left|\frac{k'_\Theta}{\omega'}\right| S(k'_\Theta)$$

$$\left\{\omega'' \left(\frac{\omega''}{\omega'}\right)^2 E_2\left[\left(\frac{\omega''}{\omega'\alpha}\right)^2\right]\right\} + \frac{i\mu\rho_s^2}{2} + \frac{i\mu\rho_s^2}{2}$$

$$- S^T(a{=}i\mu) - \left(\frac{\omega_{*n}}{\omega} - 1\right) + (k_\Theta \rho_s)^2 = 0 \; , \qquad (63)$$

where $x_s \rho_s = (1/\mu)$, valid in the Pearlstein-Berk limit, has been used. The first two terms in Eq. (63) correspond to the nonlinear ion response and the third to linear shear damping. The fourth term contains both linear and nonlinear trapped electron effects. The last two terms correspond to the basic drift wave oscillation.

Setting ω real and taking the imaginary part of the dispersion relation yields the marginal stability condition

$$\left(\frac{1}{2\alpha}\right)\left(\frac{k_\Theta \rho_s}{\omega}\right)^2 \left(\frac{e\Phi_0 \Omega_i}{T_i}\right)^2 \int dk'_\Theta \left|\frac{k'_\Theta}{\omega'}\right| S(k'_\Theta) \left\{\omega''\left(\frac{\omega''}{\omega}\right)^2 E_2\left[\left(\frac{\omega''}{\omega'\alpha}\right)^2\right]\right\}$$

$$+ \mu\rho_s^2 - \mathrm{Im}\, S^T(i\mu) = 0 \; . \qquad (64)$$

To compute the consequences of this condition the ion and trapped electron terms in Eq. (64) must be simplified.

The ion term is treated as in Ref. 4. Thus, noting that E_2 is sharply peaked at $\omega'' = 0$, the rest of the k'_Θ integrand is expanded about the point $\omega' = -\omega$. With the change of variable $x \equiv (-\omega''/\omega\alpha)$, the ion term in Eq. (64) is easily found to be

$$\frac{(k_\Theta\rho_s)^2}{2}\left(\frac{e\Phi_0}{T_i}\,\Omega_i\right)^2\frac{\alpha^4|k_\Theta|}{|d\omega/dk_\Theta|}\left(\frac{\partial S}{\partial\tilde{\omega}}\bigg|_{\tilde{\omega}=-\omega}\right)A,$$

where $A \equiv \int_{-\infty}^{+\infty} dx x^4 E_2(x^2)$. To the extent that ω_c , being weakly dependent on D_r , may be treated as a parameter rather than as a moment of $S(k_\Theta)$, this result allows the dispersion relation to be treated as a differential rather than integral equation for $S(k_\Theta)$.

Only the linear part of the trapped electron term is retained, since, as shown in Sec. IV, the nonlinear terms are small corrections to the linear response. For the same reason, d_n^R is neglected in the argument of A_1 , and from Eq. (C13)

$$-\mathrm{Im}S^T(i\mu) \approx \mathrm{Im}\left(\frac{2\varepsilon\mu}{\pi}\right)^{1/2}(i)^{1/2}\ln\left[\frac{1}{\Delta(i\mu)^{1/2}}\right]$$

$$\times\left(\frac{\omega-\omega_{*n}}{\omega}\right)\left(\frac{\omega}{n\bar{\omega}_D}\right)\Delta\{1+\left(\frac{\omega}{n\bar{\omega}_D}\right)^{1/2} Z[\left(\frac{\omega}{n\bar{\omega}_D}\right)^{1/2}]\} .$$

Using the simplified ion and trapped electron terms, replacing $(dS/d\omega)$ by $(dS/dk_\Theta)(d\omega/dk_\Theta)^{-1}$, and noting $(\partial S/\partial\tilde{\omega})|_{\tilde{\omega}=-\omega} = -(\partial S/\partial\omega)$ for S even in ω , Eq. (64) may be rewritten as an expression for (dS/dk_Θ) at marginal stability

$$\frac{dS}{dk_\Theta} = \frac{2}{A}\left(\frac{1}{\alpha}\right)^4\frac{1}{|k_\Theta|}\left(\frac{T_i}{e\Phi_0\Omega_i}\right)^2\left(\frac{\rho_s c_s}{L_n}\right)^2\frac{[1-(k_\Theta\rho_s)^2]^2}{[1+(k_\Theta\rho_s)^2]^4}$$

$$\left[\frac{\mu\rho_s^2}{(k_\Theta\rho_s)^2} - \mathrm{Im}\left(\frac{2\epsilon\mu}{\pi}\right)^{1/2}\Delta(i)^{1/2}\ell n\left(\frac{1}{\Delta(i\mu)^{1/2}}\right)\right.$$

$$\times\left(\frac{\omega}{n\bar{\omega}_D}\right)\left\{1 + \left(\frac{\omega}{n\bar{\omega}_D}\right)^{1/2} Z\left[\left(\frac{\omega}{n\bar{\omega}_D}\right)^{1/2}\right]\right\} . \qquad (65)$$

With this result, the radial diffusion coefficient trapped electrons can be computed. It follows straightforwardly that:

$$S(b) = \frac{T_i^2}{T_e^2\alpha^4}\,\frac{\rho_s^4}{L_n^2}\, I \qquad (66)$$

where

$$I = \int_{\bar{b}}^{b_{max}} db\, \frac{|1-b|^2}{(1+b^2)^4}\left\{\mathrm{Im}\left[\left(\frac{2\epsilon\mu}{\pi}\right)^{1/2}\Delta(i)'k\,\ell n[1/\Delta(i\mu)^{1/2}]\right.\right.$$

$$\left.\left.\left(\frac{\omega}{n\bar{\omega}_D}\right)\left(1+\left(\frac{\omega}{n\bar{\omega}_D}\right)^{1/2} Z\left[\left(\frac{\omega}{n\bar{\omega}_D}\right)^{1/2}\right]\right)\right] - \mu\,\rho_s^2/b^2\right\}$$

and $b = k_\theta\rho_s$. Finally, the trapped electron diffusion coefficient is:

$$D(E/E_o) \cong \left(\frac{\rho_s^2 c_s}{L_n}\right)\left(\frac{E_o}{\epsilon E}\right)\left(\frac{E_o}{\epsilon E} - 1\right)^{1/2}\left(\frac{T_i^2}{T_e^2\alpha^4}\right) I \quad .$$

VI. DISCUSSION

The role of trapped electron dynamics in a simple model of drift wave turbulence in toroidal geometry has been studied. The nonlinear modifications of the wave-particle resonance have been calculated by renormalizing the gyro-phase and bounce averaged trapped electron response. Poloidal diffusion was found to be the dominant nonlinear effect, but, for turbulence levels and other parameters typical of current experiments, this was found to make only a small perturbation in the linear response. The saturated spectrum and diffusion coefficient were thus calculated including only linear trapped electron effects.

While use of parameters different than those taken in Sec. V. may somewhat enhance trapped electron destabilization, changes in the physical model seem to be required for large enhancement. The present work has been limited to the case where there is no electron temperature gradient, since that is the limit in which nonlinear ion and circulating electron responses have previously been studied. It is well known, however[7], that such gradients can greatly enhance the linear trapped electron destabilization. This effect is especially strong for values of $(k_\theta \rho_s)$ less than one. Since, however, temperature gradients seriously alter the mode structure, the present calculations would have to be reconsidered.

Within this context, trapped electron clumps provide a new source of drift-frequency turbulence.[24] Clump calculations involve a two-point kinetic theory where the spectrum may be calculated directly, without an assumed model as in the present one-point theory. Since clumps create non-mode type fluctuations, such a theory has the additional advantage of offering a possible explanation of the observed sizable frequency width of the turbulence at fixed values of the wavelength. The clump theory also exploits the form of the nonlinear one-point response calculated here. Thus, the one point response, including the linear part only of the trapped electron response, is used to construct the two-point correlations from incoherent clump sources. Further, because of enhanced transport effects resulting from correlated diffusion, realistic radial electron energy fluxes can be obtained.

Appendix A

In this appendix the relation between the orbital harmonic of the Hamiltonian, H_n , and the spatial harmonic of the potential, Φ_n , is derived. By definition,

$$H_n^{(\alpha)} = e\int_0^{2\pi} \frac{d\Theta}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \exp(-in\phi)\Phi(\Theta,\phi,\alpha) \ , \tag{A1}$$

where the $m = 0$ component is taken, and Φ has been gyro-averaged and is evaluated at the guiding center position. The relation desired is found by transforming variables from (Θ,ϕ) to (β,ζ) . Following Ref. 14 this is done in the limit of large aspect ratio, keeping terms to order $\varepsilon^{1/2}$, and neglecting corrections of order $\delta \equiv (\rho_p/L_n) \ll 1$, where ρ_p is the poloidal gyro-radius. The Jacobian of the transformation is thus computed from the relations [see Eqs. (41) and (47) of Ref. 14]

$$\phi = \zeta - q\beta \tag{A2}$$

and

$$\Theta = \frac{\pi}{2}\frac{F(\xi,\kappa)}{K(\kappa)} \ , \tag{A3}$$

where

$$\kappa \sin\xi \equiv \sin(\beta/2) \ , \tag{A4}$$

and $F(\xi,\kappa)$ is the elliptic integral of the first kind.[25] From Eqs. (21) and (A4), $(\partial\Theta/\partial\zeta) = 0$, so the Jacobian is

$$\mathscr{J} \equiv \left|\frac{\partial(\Theta,\phi)}{\partial(\beta,\zeta)}\right| = \left|\frac{\partial\Theta}{\partial\beta}\frac{\partial\phi}{\partial\xi}\right| = \left[\frac{\pi}{4K(\kappa)}\right]\left(\frac{1}{\kappa\sigma\cos\xi}\right), \tag{A5}$$

where σ is the sign of the parallel velocity, $\sigma \equiv \mathrm{sgn}(\underset{\sim}{v}\cdot\underset{\sim}{B}/|\underset{\sim}{B}|)$. To account for the bounce motion, the β integral is of the form

$$\sum_\sigma \sigma \int_{-\beta_c}^{\beta_c} d\beta \equiv \oint d\beta \ .$$

Here the bounce-point angle β_c satisfies $\sin(\beta_c/2) = \kappa$, and the sum over $\sigma = (-1,1)$ corresponds to the reversal of motion in β at the bounce points. For β fixed, Eq. (A2) implies the ζ integral is $\int_0^{2\pi} d\zeta$. Thus, transforming (Θ,ϕ) to (β,ζ) , H_n becomes

$$H_n = e \sum_\sigma \int_{-\beta_c}^{\beta_c} \frac{d\beta}{2\pi} \int_0^{2\pi} \frac{d\zeta}{2\pi} \left[\frac{\pi}{4K(\kappa)}\right] \frac{\exp[-in(\zeta - q\beta)]\Phi(\alpha,\beta,\zeta)}{[\kappa^2 - \sin^2(\beta/2)]^{1/2}} , \tag{A6}$$

where Eqs. (A2), (A4), and (A5) have been used.

Equation (A6) must be expressed in terms of the bounce average to obtain Eq. (27). The bounce average of any function $f(\beta)$ is defined as

$$\langle f(\beta)\rangle_b \equiv \left[\frac{\oint(d\beta/\dot{\beta})f(\beta)}{\oint(d\beta/\dot{\beta}}\right] . \tag{A7}$$

The average is evaluated explicitly, using

$$\dot{\beta} = \left(\frac{2\sigma}{qR} \right) (\varepsilon\mu\bar{B})^{1/2} \left[\kappa^2 - \sin^2(\beta/2)\right]^{1/2} ,$$

valid to lowest order in ε and δ. Then, changing variables from β to β' such that $\kappa \sin\beta' = \sin(\beta/2)$,

$$\langle f(\beta)\rangle_b = \sum_\sigma \int_{-\beta_c}^{\beta_c} \frac{d\beta}{\left[\kappa^2 - \sin^2(\beta/2)\right]^{1/2}} \frac{f(\beta)}{8K(\kappa)} . \tag{A8}$$

Noting that $\int_0^{2\pi}(d\zeta/2\pi)\exp(-in\zeta)\Phi(\alpha,\beta,\zeta) = \Phi_n(\alpha,\beta)$,

$$H_n = \exp\langle\exp(inq\beta)\Phi_n(\alpha,\beta)\rangle_b \tag{A9}$$

immediately follows from Eqs. (A6) and (A8). Expanding Φ_n in poloidal harmonics

$$\Phi_n = \sum_m \Phi_{nm} \exp(-i\beta m) \tag{A10}$$

and substituting in Eq. (A9) then yields Eq. (27).

Appendix B

An estimate of the sum of the $p \neq 0$ terms in the expression for d_n^{rr} [Eq. (33)] is made in this appendix. This sum is shown to be small compared to the $p = 0$ term retained in Sec. III. For other coefficients, such as $d_n^{\Theta\Theta}$, very similar estimates justify the neglect of the $p \neq 0$ terms.

Writing $d_n^{rr} = -D_n^{rr}(\partial^2/\partial r^2)$, Eq. (33) implies

$$D_n^{rr} = \left(\frac{c\Phi_0}{B}\right)^2 \int_{-\infty}^{+\infty} dk'_\Theta \int_{-\infty}^{+\infty} dx \frac{|k'_\Theta|^3}{W_{k'_\Theta}} G_{n-n'} S(k'_\Theta)\, J_0\left(\frac{2\kappa x}{\Delta}\right) H\left(\frac{x}{W_{k'_\Theta}}\right)$$

$$\times \sum_{p=-\infty}^{+\infty} J_0\left[\frac{2\kappa}{\Delta}(x - p\Delta)\right] \exp\left(\frac{i\mu p^2\Delta^2}{2}\right) \exp(-i\mu x p\Delta)\, H\left(\frac{x - p\Delta}{W_{k'_\Theta}}\right) . \tag{B1}$$

The $p \neq 0$ contribution to D_n^{rr} is denoted as ΔD and, for convenience, is rewritten as

$$\Delta D = \left(\frac{c\Phi_0}{B}\right)^2 \int_{-\infty}^{+\infty} dk'_\Theta\, |k'_\Theta|^3\, G_{n-n'} S(k'_\Theta) \sum_{p=-\infty}^{+\infty}{}' I_p , \tag{B2}$$

where

$$I_p \equiv \int_{-\infty}^{+\infty} \frac{dx}{W_{k'_\Theta}} f(x,p) H\left(\frac{x}{W_{k'_\Theta}}\right) H\left(\frac{x - p\Delta}{W_{k'_\Theta}}\right) , \tag{B3}$$

$$f(x,p) \equiv J_0\left(\frac{2\kappa x}{\Delta}\right) J_0\left[\frac{2\kappa}{\Delta}(x - p\Delta)\right] \exp\left(\frac{i\mu p^2\Delta^2}{2}\right) \exp(-i\mu x p\Delta) , \tag{B4}$$

and the prime on the summation indicates $p = 0$ is excluded.

To compute ΔD, first consider the effect of the H functions on I_p, for H approximated by a step function, as in Sec. III. For $p\Delta > 0$, the combined restrictions $|x| < W$ (W will be written for $W_{k'_\Theta}$ wherever possible in this appendix) and $|x - p\Delta| < W$ imply $-W + p\Delta < x < W$, while for $p\Delta < 0$ the result is $-W < x < W + p\Delta$. Furthermore, $|p\Delta| < 2W$ is required, since the first step function restricts $|x| < W$, so that for $|p\Delta| > 2W$, $|x - p\Delta| > W$ would result, violating the restriction of the second one. Thus, Eq. (B3) implies

$$\sum_p{}' I_p = \sum_{p=1}^{[2W/\Delta]} \int_{-W+p\Delta}^{W} dx \;\; f(x,p) + \sum_{p=-[2W/\Delta]}^{-1} \int_{-W}^{W+p\Delta} dx \;\; f(x,p) \;, \tag{B5}$$

where $[2W/\Delta]$ is the largest integer less than $(2W/\Delta)$. From Eq. (B4) $(-x,-p) = (x,p)$ so that (B5) becomes

$$\sum_p{}' I_p = 2 \sum_{p=1}^{[2W/\Delta]} I_p \;, \tag{B6}$$

where now

$$I_p = \int_{-W+p\Delta}^{W} \frac{dx}{W} J_0\left(\frac{2\kappa x}{\Delta}\right) J_0\left[\frac{2\kappa}{\Delta}(x - p\Delta)\right] \exp\left(\frac{i\mu p^2\Delta^2}{2}\right) \exp(-i\mu x p\Delta) \;. \tag{B7}$$

The p sum is further restricted to $|p| < (\sqrt{2}x_T/\Delta) \equiv p_0$. For larger values of $|p|$, contributions to ΔD are small due to oscillations in p from the term $(i\mu p^2\Delta^2/2)$, which dominates the exponent. Then, neglecting a small correction of order $[(1/2)(L_n/L_s)(T_i/T_e)]^{1/2}$, and defining $u \equiv (x/W)$, Eqs. (B6) and (B7) imply

$$\sum_p \acute{I}_p \approx 2 \sum_{p=1}^{p_0} \exp\left[i\left(\frac{\mu}{2}\right)p^2\Delta^2 \int_{-1}^{1} du\, J_0\left(\frac{2\kappa W u}{\Delta}\right) J_0\left[\frac{2\kappa W}{\Delta}\left(u - \frac{p\Delta}{W}\right)\right]\exp(-i\mu W u p \Delta)\right. . \tag{B8}$$

The u integral in Eq. (B8) is evaluated by approximating the integrand in the $(2\kappa Wu/\Delta) \gg 1$ limit. This limit is equivalent to $u \gg (\Delta/2\kappa W) \approx (1/\kappa)(L_n/L_s)(T_i/T_e)$; for typical parameters, this excludes only the small region $|u| \lesssim (1/160)$, where the integrand is order unity. Thus the Bessel functions are approximated by[26]

$$J_0\left(\frac{2\kappa W u}{\Delta}\right)J_0\left[\frac{2\kappa W}{\Delta}\left(u - \frac{p\Delta}{W}\right)\right] \sim$$

$$\frac{2}{\pi}\left(\frac{\Delta}{2\kappa W|u|}\right)^{1/2}\left(\frac{\Delta}{2\kappa W}\,\frac{1}{|u - (p\Delta/W)|}\right)^{1/2} \times \cos\left(\frac{2\kappa W u}{\Delta} - \frac{\pi}{4}\right)\cos\left(\frac{2\kappa W}{\Delta}u - 2\kappa p - \frac{\pi}{4}\right) . \tag{B9}$$

Here $(p\Delta/W) < (\sqrt{2}\, x_T/W) \sim (2L_n/L_s)^{1/2} \ll 1$, so $(u - p\Delta/W)^{1/2} \approx u^{1/2}$, and the cosines are expanded as

$$\cos\left(\frac{2\kappa W u}{\Delta} - \frac{\pi}{4}\right)\cos\left(\frac{2\kappa W}{\Delta}u - 2\kappa p - \frac{\pi}{4}\right)$$

$$= \frac{1}{2}\left[\cos\left(\frac{4\kappa W u}{\Delta} - 2\kappa p - \frac{\pi}{2}\right) + \cos(2\kappa p)\right] . \tag{B10}$$

The first term on the right-hand side of Eq. (B10) contributes integrals of the form

$$\int_{-1}^{1} du\, \frac{\exp(-i\mu W u p \Delta)}{u} \begin{Bmatrix} \sin\left(\frac{4\kappa W u}{\Delta}\right) \\ \cos\left(\frac{4\kappa W u}{\Delta}\right) \end{Bmatrix} ,$$

while the second term makes contributions of the form

$$\int_{-1}^{1} du\left[\frac{\exp(-\mu W u p \Delta)}{u}\right] \quad ;$$

these integrals are multiplied by other factors of equal magnitude. Since $4\kappa(x_T/\Delta)^2 \gg p_0$ when $\kappa \gg (1/2\sqrt{2})(L_n/L_s)^{1/2}(T_i/T_e)^{1/2} \sim 1/20$, consistent with the approximations of Secs. III and IV, the sine and cosine of $(4\kappa Wu/\Delta)$ vary rapidly compared to the sine and cosine of $(\mu W u p \Delta)$. Furthermore, $(\Delta/2\kappa W) \ll 1$, so that the u integrals with the sine and cosine $(4\kappa Wu/\Delta)$ factors may be neglected. Then $\sum_p' I_p$ may be evaluated by retaining only the $\cos(2\kappa p)$ term in Eq. (B10), and evaluating the u integral with Eq. (B9) and the self-consistent restriction $|u| > (\Delta/2\kappa W)$. The result is

$$\sum_p{}' I_p \approx$$

$$\left(\frac{2\Delta}{\pi\kappa W}\right) \sum_{p=1}^{p_0} \cos(2\kappa p)\cos\left(\frac{\mu}{2}\, p^2\Delta^2\right) \times\left\{\mathrm{Ci}\left[2\alpha\left(\frac{T_e}{T_i}\right)^{1/2} p\right] - \mathrm{Ci}\left(\frac{\Delta^2 p\mu}{2\kappa}\right)\right\}. \quad \text{(B11)}$$

To estimate ΔD, Eq. (B11) is evaluated taking $\cos[(\mu/2)p^2\Delta^2] \sim 1$, since $p < p_0$, and neglecting $\mathrm{Ci}[2\alpha(T_e/T_i)^{1/2}p]$, since, for $2\alpha(T_e/T_i)^{1/2} \approx (T_e/T_i)^{1/2} \gg \Delta^2\mu/2\kappa$, it oscillates about zero as p varies.[18] Then, for typical values $p_0 \approx 6$, $\kappa \approx 1/2$, $(\Delta/x_i) \approx 1/20$,

$$\sum_{p=1}^{p_0} \cos(p)\mathrm{Ci}\left(\frac{p}{20}\right) \approx 0.6 \quad ,$$

and

$$\Delta D \approx \left(\frac{c\Phi_0}{B}\right)^2 \int_{-\infty}^{+\infty} dk'_\Theta |k'_\Theta|^3 G_{n-n'} S(k'_\Theta) \left[\frac{(1.2)\Delta}{\pi\kappa W}\right] \quad .$$

This is to be compared to the $p = 0$ contribution to D_n^{rr} . From Eq. (43) it thus follows that

$$\left(\frac{\Delta D}{D_n^{rr}}\right) \sim \left(\frac{1.2\Delta}{\pi\kappa W}\right)\left[\frac{1}{2(\Delta/\kappa W)}\right] \sim 0.2 \ , \qquad \text{(B12)}$$

which is small enough to justify the neglect of the $p \neq 0$ terms in the evaluation of d_n^r . Exactly parallel estimates show contributions of the same order from the $p \neq 0$ terms in the other nonlinear coefficients.

Appendix C

From Eqs. (59) and (60), $S^T(a)$, which is evaluated in this appendix, is, in the small κ approximation,

$$S^T(a) = \left(\frac{a\varepsilon}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} dx \exp\left(\frac{-ax^2}{2}\right) \int_0^1 d\kappa^2 J_0\left(\frac{2\kappa x}{\Delta}\right)$$

$$\left(L_0 + L_1 \frac{\partial^2}{\partial x^2}\right) \sum_p \exp\left[-\frac{a(x-p\Delta)^2}{2}\right] J_0\left[\frac{2\kappa}{\Delta}(x-p\Delta)\right] . \qquad (C1)$$

With the aid of Eqs. (55) and (56) each term in $S^T(a)$ corresponding to the terms in L_0 and L_1 will now be evaluated.

As in App. B, the factor $\exp(-ap^2\Delta^2/2)$ in Eq. (C1) restricts the p sum to values $|p| < \left[(2/a)^{1/2}(1/\Delta)\right] \equiv p_a$. Under this restriction, $\exp(-ap^2\Delta^2/2)$ may be roughly approximated by unity. The contributions to $S^T(a)$ from the L_0 term are then proportional to

$$\sum_{|p|<p_a} \int_{-\infty}^{+\infty} dx \exp\left(-ax^2\right) \int_0^1 d\kappa^2 J_0\left(\frac{2\kappa x}{\Delta}\right) \exp(ap\Delta x)\ J_0\left[\frac{2\kappa}{\Delta}(x-p\Delta)\right] \left\{{1 \atop 1/k}\right\} . \qquad (C2)$$

Because of the factor $\exp\left(-ax^2\right)$, the x integral is approximately restricted to $|x| < \left(1/a^{1/2}\right)$, which allows the rough approximation $\exp\left(-ax^2\right) \approx 1$. Noting that the integrand in Eq. (C2) is symmetric under $(x,p) \to (-x,-p)$, the $p = 0$ contribution to Eq. (C2) is approximately

$$4 \int_0^1 \kappa \, d\kappa \int_0^{(1/\sqrt{a})} dx \, \left[J_0\left(\frac{2\kappa x}{\Delta}\right)\right]^2 \left\{ {1 \atop 1/k} \right\} .$$

Consider then the integral

$$R_1 \equiv 2 \int_0^1 \kappa \, d\kappa \int_0^{(1/\sqrt{a})} dx \, \left[J_0\left(\frac{2\kappa x}{\Delta}\right)\right]^2 . \tag{C3}$$

The κ integral is tabulated[23], yielding

$$R_1 = \int_0^{1/\sqrt{a}} dx \, \left\{ \left[J_0\left(\frac{2x}{\Delta}\right)\right]^2 - J_{-1}\left(\frac{2x}{\Delta}\right) J_1\left(\frac{2x}{\Delta}\right) \right\} .$$

To compute the x integral, the Bessel functions are evaluated for $(x/\Delta) \gg 1$. Since the integrand is of order 1 for $(x/\Delta) \ll 1$, while $\Delta \ll (2/\sqrt{a})$, self-consistent with the result $a^{-1/2} = x_T$, this approximation is good to zeroeth order in (Δ/x_T), restricting $x > \Delta$ in the x integral. Asymptotically, the integrand is approximately $(\Delta/\pi x)$ (see Ref. 26), so that

$$R_1 \approx \frac{\Delta}{\pi} \ln \left(\frac{1}{\Delta\sqrt{a}} \right) . \tag{C4}$$

Consider next the integral

$$R_2 \equiv 2 \int_0^1 d\kappa \int_0^{1/\sqrt{a}} dx \, \left[J_0\left(\frac{2\kappa x}{\Delta}\right)\right]^2 \tag{C5}$$

required to compute the other $p = 0$ contribution to (C2). In this case, the κ integral cannot be done exactly. Then, reversing the order of

κ and x integration, the Bessel function is approximated for $(2\kappa x/\Delta) >> 1$. The limits of integration thus become $(\Delta/2x) < \kappa < 1$, while $(\Delta/2) < x < (1/\sqrt{a})$, since $\kappa < 1$; because $(1/\sqrt{a}) \sim x_T >> \Delta/2$ an error of order $\Delta\sqrt{a} << 1$ results. Then $[J_0(2\kappa x/\Delta)]^2 \sim (\Delta/2\pi\kappa x)[1 + \sin(4\kappa x/\Delta)]$, and the sine term is neglected because of its fast oscillations in the x integral. These approximations lead straightforwardly to

$$R_2 \approx \left(\frac{\Delta}{2\pi}\right)\left[\ln\left(\frac{2}{\Delta\sqrt{a}}\right)\right]^2 . \tag{C6}$$

The $p \neq 0$ contributions to Eq. (C2) again involve two integrals, the first of which is

$$R_3 \equiv \frac{1}{2}\int_0^1 d\kappa^2 \sum_{|p|<p_a}{}' \int_{-1/\sqrt{a}}^{1/\sqrt{a}} dx\, J_0\left(\frac{2\kappa x}{\Delta}\right) J_0\left[\frac{2\kappa}{\Delta}(x - p\Delta)\right]\exp(-ax^2 + axp\Delta) . \tag{C7}$$

As in the evaluation of R_1 and R_2, R_3 is evaluated taking $\exp(-ax^2 + axp\Delta) \approx 1$, and the Bessell functions are evaluated for large argument, with error of order $(\Delta\sqrt{a})$. The Bessel functions are approximately

$$J_0\left(\frac{2\kappa x}{\Delta}\right)J_0\left[\frac{2\kappa}{\Delta}(x - p\Delta)\right] \sim \left(\frac{\Delta}{2\pi\kappa x}\right)\left[\sin\left(\frac{4\kappa x}{\Delta} - 2\kappa p\right) + \cos(2\kappa p)\right] ,$$

and, as in Eqs. (B10) and (B11), $\cos(2\kappa p)$ is the dominant term. With these approximations,

$$R_3 \approx \frac{\Delta}{\pi}\left(\frac{\pi}{2} - 1\right)\ell n\left(\frac{1}{\Delta\sqrt{a}}\right) \quad , \tag{C8}$$

where

$$\sum_{p=1}^{p_s} \frac{\sin 2p}{p} \approx \left(\frac{\pi}{2} - 1\right)$$

is taken since p_a is typically large enough that use of the infinite sum[23] is adequate; contributions from $|p| > p_a$ are oscillatory and thus small.

Similar approximation of the other $p \neq 0$ contributions to Eq. (C2) yields

$$R_4 \equiv \frac{1}{2}\int_0^1 \frac{d\kappa^2}{\kappa} \sum_{|p|<p_a}{}' \int_{-1/\sqrt{a}}^{1/\sqrt{a}} dx\, J_0\left(\frac{2\kappa x}{\Delta}\right) J_0\left[\frac{2\kappa}{\Delta}(x - p\Delta)\right] \exp\left(-ax^2 + axp\Delta\right)$$

$$\simeq \frac{2\Delta}{\pi} \sum_{p=1}^{p_a} \int_{[\Delta\sqrt{a}/2]}^{1} \frac{d\kappa}{\kappa} \int_{\Delta/2\kappa}^{1/\sqrt{a}} \frac{dx}{x} \cos(2\kappa p) \ . \tag{C9}$$

After performing the x integration, the κ integral is done by parts, and

$$R_4 \approx \frac{2\Delta}{\pi} \sum_{p=1}^{p_a} \left\{\frac{\cos(2p)}{2}\left[\ell n\left(\frac{2}{\Delta\sqrt{a}}\right)\right]^2 + \int_{\Delta\sqrt{a}/2}^{1} p \sin(2\kappa p)\left[\ell n\left(\frac{2\kappa}{\Delta\sqrt{a}}\right)\right]^2 d\kappa\right\} \ . \tag{C10}$$

Because of the oscillation of $\sin(2\kappa p)$ in the κ integral, the surface term is dominant, and, using

$$\sum_{p=1}^{p_a} \cos(2p) = \sin(p_a + 1)\sin\left(\frac{p_a}{\sin 1} \right) ,$$

(see Ref. 23)

$$R_4 \approx \frac{\Delta}{\pi} \left[\ell n\left(\frac{2}{\Delta\sqrt{a}} \right)\right]^2 \left[\frac{\sin(p_a + 1)\sin(p_a)}{\sin(1)}\right] \tag{C11}$$

results.

The contributions to $S^T(a)$ from the L_1 term are evaluated starting from

$$\frac{\partial^2}{\partial x^2} \left\{\exp\left[- \frac{a}{2} (x - p\Delta)^2\right]J_0\left[\frac{2\kappa}{\Delta} (x - p\Delta)\right] \right\}$$

$$= \exp\left[- \frac{a}{2} (x - p\Delta)^2\right]\left(a\left[a(x - p\Delta)^2-1\right]J_0\left[\frac{2\kappa}{\Delta} (x - p\Delta)\right]\right.$$

$$+ a(x - p\Delta)\left(\frac{4\kappa}{\Delta} \right)J_1\left[\frac{2\kappa}{\Delta} (x - p\Delta)\right]$$

$$\left. + 2\left(\frac{\kappa}{\Delta} \right)^2\left\{J_2\left[\frac{2\kappa}{\Delta} (x - p\Delta)\right] - J_0\left[\frac{2\kappa}{\Delta} (x - p\Delta)\right] \right\} \right) . \tag{C12}$$

Then, proceeding as with the L_0 terms, and neglecting corrections of order $(L_n/L_s)^{1/2}$, finally yields

$$S^T(a) = \left(\frac{\varepsilon}{2\pi}\right)^{1/2} \Delta a^{1/2} \ln\left(\frac{1}{\Delta a^{1/2}}\right) A_1 (\omega - \omega_{*n})$$

$$- \left(\frac{\varepsilon a}{8\pi^5}\right)^{1/2} \left(\frac{\Delta\mu}{\omega}\right)^2 \left[\ln\left(\frac{2}{\Delta a^{1/2}}\right)\right]^2 \times \left[1 + \frac{2\sin(p_a+1)\sin(p_a)}{\sin(1)}\right]$$

$$\times \left(\frac{c\Phi_0}{B}\right)^2 \int dk'_\Theta |k'_\Theta| S(k'_\Theta) W_{k'_\Theta} \left(\frac{nq}{r}\right)^2$$

$$\times \{(\omega' - \omega_{*n'})A_3 + [(\omega' - \omega) - (\omega_{*n'} - \omega_{*n})]A_2\}$$

$$- \left(\frac{2a\varepsilon}{\pi^3}\right)^{1/2} \ln\left(\frac{1}{\Delta\sqrt{a}}\right)\left[1 + \frac{4\sin(p_a+1)\sin(p_a)}{\sin(1)}\right]$$

$$\times \frac{1}{\omega^2} \left(\frac{c\Phi_0}{B}\right)^2 \int dk'_\Theta |k'_\Theta|^3 \frac{S(k'_\Theta)}{W_{k'_\Theta}} \{(\omega' - \omega_{*n'})A_3$$

$$+ [(\omega - \omega') - (\omega_{*n'} - \omega_{*n})]A_2\} \,. \qquad \text{(C13)}$$

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