Theory of Dissipative Density Gradient Driven Turbulence in the Tokamak Edge

P. W. Terry and P. H. Diamond

Institute for Fusion Studies
University of Texas at Austin
Austin, Texas 78712

Abstract

A theory for tokamak edge turbulence based on resistive, density gradient driven fluctuations is presented. From a fluid model for isothermal electrons in toroidal geometry, equations are obtained and solved analytically, retaining both coherent and incoherent contributions. The analytic results indicate that the spectrum is characterized by broad linewidths $(\Delta\omega_k/\omega_{*e}\sim 1)$ with energy predominantly in the small wavenumbers $(k_\perp \rho_s \sim 0.1)$. For larger wavenumbers and frequency the spectrum decays as $k^{-17/6}$ and ω^{-2}. The fluctuation level is large (~30%) and scales as ρ_s/L_n. Particle diffusion is Bohm-like in magnitude but does not follow Bohm scaling, going instead as $n^{2/3}T_e^{1/6}$. The density fluctuations exhibit non-adiabatic character due to the incoherent mode coupling. An expression for the departure from adiabaticity is given.

I. INTRODUCTION

Recently, turbulence in the edge region of tokamaks has received increased attention. Measurements of particle loss in this region and comparison with total particle loss suggest the importance of edge turbulence to global particle confinement.[1] Investigation of edge turbulence has also been stimulated by a need to understand edge phenomena in order to effectively design limiters and divertors. Discovery of the H mode with its improved confinement time and the correlation of the transition to H mode operation with edge related processes such as divertor operation underscore the need to understand edge turbulence and its effect on confinement. Because the edge region is accessible to probes, a significant amount of information is being obtained on the nature of edge turbulence.[1-4] Measurements show edge turbulence to be characterized by very large fluctuations having $\delta n/n \sim 10\%-50\%$ in contrast to the center where levels of 2% are typical. Both wavenumber and frequency spectra are broad. Strong non-adiabatic behavior of density fluctuations is observed ($\delta n/n \neq e\varphi/T$). Typical poloidal wavenumbers and frequencies are $k_\vartheta \rho_s \sim 1/10$ and $\omega \sim \omega_{*e}$, respectively. Both wavenumber and frequency spectra are broad, and $\Delta\omega_k$, the frequency linewidth at fixed k_ϑ, is comparable to the frequency ω.

Recent measurements of edge plasma parameters indicate that $\eta_e < 1$ ($\eta_e = L_n/L_T = \frac{n}{T}\frac{dT}{dn}$) and that the electron mean free path λ_{mfp} is short ($\lambda_{mfp} < Rq$). Hence, a fluid model of density gradient driven plasma turbulence is suggested as a simple paradigm for the dynamics of edge fluctuations. However, the large density fluctuation level and broad frequency linewidth suggest that a nonlinear density gradient relaxation process, rather than a linear instability, drives the edge turbulence. In this paper, a nonlinear theory of resistive, density gradient driven turbulence is presented. As in hydrodynamic shear flow, the density fluctuation spectrum is determined by the competition between production by gradient relaxation and perpendicular and parallel shear stress, exerted in this case on density fluctuation elements by $\hat{E}\times B_0$ convection and (parallel) collisional diffusion, respectively. Since the theory regards the spectrum as the fundamental quantity, both coherent and incoherent density fluctuations[4,5] are accounted for. Hence, density fluctuation elements are spatially localized structures which resemble fluid eddys rather than density perturbations associated with linear waves. The scales of density correlation determine the size of density fluctuation elements. Furthermore, the localized density elements extract expansion free energy from the average density gradient at a rate which exceeds that of quasilinear relaxation. Finally, the incoherent density fluctuations act as a nonlinear noise source for the collective resonances of the system, which in turn act as a sink. At steady state, nonlinear noise due to density gradient relaxation is balanced by emission of fluctuation energy into (nonlinearly) over-damped collective resonances. Such energy deposition ultimately results in ion heating.

A simple fluid model of resistive density gradient driven turbulence is investigated analytically. The model is cast in toroidal geomentry in order to include ion precessional motion, which annuls shear damping. The principal results of this investigation are:

i.) The effect of nonlinear noise on gradient relaxation rates has been calculated. Substantial growth rate enhancement occurs.

ii.) Analytical expressions for the frequency and wave number spectrum are obtained. These expressions indicate that:

a) energy is predominately in the small wavenumbers, $k_\perp \rho_s \lesssim 0.1$,

b) for larger $k_\perp$ and ω, the spectrum decays as $k^{-17/6}$ and ω^{-2}. These are in excellent agreement with experiment.[2,3]

c) $\Delta\omega_k$, the frequency linewidth at fixed k, is broad, and $\Delta\omega_k \sim \omega$. An expression for $\Delta\omega_k$ is obtained.

iii.) The fluctuation level $e\varphi/T_e$ scales as $e\varphi/T_e \sim \rho_s/L_n$, and is typically of the order of 30%.

iv.) The magnitude of the predicted particle diffusion is of the order of Bohm diffusion. However, the diffusion coefficient scales as $D \sim n^{2/3} T_e^{1/6}$, quite unlike Bohm scaling.

v.) The predicted incoherent mode coupling and density fluctuations offer a possible explanation of the strong non-adiabatic character of density fluctuations observed in experiments. These predictions are in qualitative (1-3) agreement with recent experimental results.

Furthermore, this investigation is a first attempt at the application of ideas and methods of renormalized two point theory and nonlinear relaxation to fluid plasma systems. The basic ideas utilized were originally proposed in the context of phase space density turbulence.[5] In that case, the phase space density field f is conserved along particle trajectories. In this case, the basic model is dissipative, and the effect of collisions on small scale density correlation (parallel viscosity) must be considered. Specifically, at very small scales when parallel collisional diffusion times are sufficiently faster than nonlinear density element relaxation times (eddy turnover time associated with $\hat{E} \times B_0$ convections), small scale density correlation decays with the coherent dissipative relaxation time and incoherent fluctuation is negligible. In order for frequency broadening due to incoherent fluctuations to occur at a particular scale, the effects of the E×B nonlinearity must be stronger than those of viscous diffusion. A Reynold's number parameterizes this relative strength. Quantitatively, it is shown that substantial incoherent frequency broadening occurs for moderate Reynold's numbers, i.e., for Reynold's numbers from a few tenths to order unity, up to large Reynold's numbers, i.e., greater than unity. The denisty correlation function exhibits Reynold's number similarity over scales where appreciable frequency broadening occurs. Observe that this is a necessary, but not sufficient, condition for the existance of an inertial range of density fluctuations. Finally, the theory quantitatively links the frequency broadening process with non-adiabatic density fluctuation behavior.

Recently, edge turbulence theories based on rippling modes and dissipative drift waves[6,7] have been proposed. In the dissipative drift wave theory of Hasegawa and Wakatani, numerical solution of two dimensional evolution equations for density and vorticity exhibits broad frequency line width, large fluctuation levels, and Bohm-like particle diffusion[7] (magnitude and scaling). Such a model is attractive because of its simplicity and because it exhibits many features observed in studies of strong plasma turbulence, including broad frequency spectra.[7-9] However, it omits the important energy sink associated with parallel ion heating and relies on (relatively small) perpendicular ion viscosity for saturation and treats the destabilization mechanism in an ad-hoc fashion. For these reasons, the general significance of the results for frequency width, fluctuation level, and diffusion are unclear. Furthermore, several advantages are realized from the analytic treatment of the incoherent nonlinearity. Aside from obtaining analytic formulas, the basic physics involved in the line broadening process is illuminated and its impact on other observable spectrum properties is determined. As it turns out, the line broadening is related to a nonlinear instability that can strongly affect the saturation amplitude and transport.[9,10] The analytical theory elucidates the effect of this process.

The paper is organized as follows: in Sec. II we present the basic equations and review the relevant linear theory. Sec. III develops the analytic theory of coherent and incoherent mode coupling. Some introductory comments preceed two major subsections, one on the physics of the driving mechanism of incoherent fluctuations and the other on the evolution of small scale correlation, which determines the solution of the two-point density equation. In Sec. IV we derive the detailed predictions of the theory for edge turbulence in tokamaks. The linewidth, wavenumber spectrum and particle diffusion are discussed. Section V includes a summary and conclusions.

II. BASIC EQUATIONS AND LINEAR BEHAVIOR

A. Electron and Ion Dynamics

At the plasma edge the electron temperature is sufficiently low so that the electron-ion collision frequency exceeds the inverse transit time $\nu_{ei} > \omega_{T_e} = v_{te}/Rq$. We consider the electrons to be isothermal and hence neglect temperature gradients and fluctuations. Since $\nu_{ei} >> \omega$, electron inertia is also neglected. Thus, the electron fluid equations are[11]:

$$\frac{\partial \hat{n}}{\partial t} + \nabla_{\parallel} v_{\parallel e} + \nabla \cdot (n \, \underset{\sim}{v}_{\perp e}) = 0 \tag{1}$$

$$\frac{dv_{\parallel e}}{dt} + \nu_{ei} v_{\parallel e} = v_{T_e}^2 \, \nabla_{\parallel} \left(\frac{e\hat{\varphi}}{T_e} - \hat{n} \right) \tag{2}$$

where we have assumed that $v_{\parallel e} >> v_{\parallel i}$, and

$$n = n_o + \hat{n} \quad \text{and} \quad \underset{\sim}{v}_{\perp e} = \underset{\sim}{v}_{de} \tag{3}$$

where

$$\underset{\sim}{v}_E = - \frac{c}{B_o} \nabla \varphi \times \hat{n}_o$$

$$\underset{\sim}{v}_{de} = - \frac{cT_e}{eB_o} (n \times \nabla \, \ell n \, B).$$

We substitute Eqs. (2) and (3) into Eq. (1). Noting that

$$\nabla \cdot (n \, v_{\perp e}) \cong v_{de} \cdot \nabla \left(\hat{n} - \frac{|e|\hat{\varphi}}{T_e} \right) + \frac{cT_e}{|e|B} \frac{d}{dx} \ell n \, N_o \, \nabla_{\vartheta} \left(\frac{|e|\varphi}{T_e} \right) + \underset{\sim}{v}_E \cdot \nabla \hat{n} \quad ,$$

we write Eq. (1) in terms of the nonadiabatic electron density $H_e = \hat{n} - \frac{|e|\hat{\varphi}}{T_e}$. Since the ExB drift does not convect the adiabatic density $(v_e \cdot \nabla e\hat{\varphi}/T_e = 0)$, it follows that the electron dynamics are described by:

$$\frac{\partial}{\partial t} H_e - \frac{v_{T_e}^2}{\nu_{ei}} \nabla_\parallel^2 H_e + v_{de} \cdot \nabla H_e$$

$$- \frac{c}{B_o} \nabla\varphi \times \hat{n} \cdot \nabla\hat{H}e = - \frac{|e|}{T_e} \left(\frac{\partial\hat{\varphi}}{\partial t} + v_D \nabla_\vartheta \hat{\varphi}\right) \tag{4}$$

where $v_D = \frac{cT_e}{eB} \frac{d}{dx} \ell n\ N$.

We consider toroidal geometry and apply the ballooning transformation:

$$\binom{\hat{\varphi}}{H_e} = \sum_n \exp(in\varphi) \sum_m \exp(-im\vartheta) \int d\eta \exp(i[m-nq(r)])\eta \binom{\hat{\varphi}_n(\eta)}{H_n(\eta)}$$

where φ is the toroidal angle and η is the variation in the direction of the magnetic field. Using this transformation and Fourier transforming in time, Eq. (4) is:

$$-i(\omega-\omega_{de}) H_{\substack{n\\ \omega}} - \frac{v_{T_e}^2}{\nu_{ei}(Rq)^2} \frac{\partial^2}{\partial\eta^2} H_{\substack{n\\ \omega}} + N_{\substack{n\\ \omega}} = \frac{i|e|}{T_e} (\omega-\omega_{*e}) \hat{\varphi}_{\substack{n\\ \omega}} \tag{5}$$

where

$$\omega_{de} = \frac{cT_e}{eB} \frac{k_\vartheta}{R} (\cos\eta + \hat{s}\eta \sin\eta),$$

$$\omega_{*e} = v_D k_\vartheta, \qquad k_\vartheta = nq/r, \qquad \hat{s} = rq'/q$$

and

$$N_{\substack{n\\ \omega}} = \sum_{n'} \sum_m \frac{c}{B_o} k_\vartheta k_\vartheta' (2\pi m\hat{s}) \exp[2\pi\ in'q(r)m]\ \hat{\varphi}_{\substack{-n'\\ -\omega'}}(\eta+2\pi m) H_{\substack{n+n'\\ \omega+\omega'}}(\eta) \ . \tag{6}$$

The ballooning representation provides a compact form of the E×B nonlinearity, in which the interaction is between n and n′ at r along η.[4,10] Reviewing Eq. (5) the model contains a magnetic curvature drift convection in toroidal angle φ represented by ω_{de}, parallel viscous diffusion, $[v_{T_e}^2/(Rq)^2\nu_{ei}]\partial^2/\partial\eta^2$, the E×B nonlinear mixing and the usual drift wave density gradient source term.

As previously mentioned, we consider warm, low frequency ion dynamics. Parallel ion motion which gives ion-sound shear damping and diamagnetic drifts which introduce toroidal coupling and permit quasi-bounded eigenstates for the toroidicity induced branch[11] are retained to determine the mode structure. The ions provide a sink of fluctuation energy, and hence a potential saturation mechanism, in that the combined effects of nonlinearity and parallel ion dynamics can result in (nonlinear) ion heating.

The linear theory of dissipative drift waves is discussed in Ref. 11. It is instructive to review the scalings of growth rate with respect to collisionality and to qualitatively estimate the size of the nonlinear instability for regimes relevant to the tokamak edge. The growth rate scalings may be succinctly obtained from a variational integral representation of the electron part of the linear dielectric. This alternate derivation of the linear growth rate relies on the fact that the toroidicity induced branch is localized in local potential wells induced by finite toroidal coupling. Outward convection of wave energy is impeded by the potential barrier and occurs only through tunnelling. Shear damping is therefore negligible. The dissipative mechanism is collisional parallel diffusion of the electron density and this determines the linear growth rate. The growth rate is proportional to the imaginary part of the electron contribution to the dielectric, which is given by

$$\Delta = \int_{-\infty}^{\infty} d\eta\ \hat{\varphi}_{\substack{n\\ \omega}}(\eta)\ H_{\substack{-n\\ -\omega}}(\eta) \tag{7}$$

where $H_{\substack{-n\\ -\omega}}(\eta)$ is the solution of the linearized form of Eq. (7). We note that Δ is variational in $\hat{\varphi}$, recovering the electron part of the linear eigenmode equation upon variation with $\hat{\varphi}$. Evaluation of Eq. (7) with $\hat{\varphi}$ determined from the full eigenmode equation gives the electron part of the dielectric. The imaginary part of the electron dielectric is well approximated by retaining only the dissipative mechanism associated with collisional viscosity in the propagator for $H_{\substack{-n\\ -\omega}}(\eta)$:

$$\mathrm{Im}\Delta \cong \mathrm{Im}\ \frac{i|e|}{T_e}\left(1 - \frac{\omega_{*e}}{\omega}\right) \int_{-\infty}^{\infty} d\eta\ \hat{\varphi}_{\substack{n\\ \omega}}(\eta)$$

$$\left[-i - \frac{v_{T_e}^2}{(Rq)^2 \nu_{ei}\omega}\ \frac{\partial^2}{\partial\eta^2}\right]^{-1} \hat{\varphi}_{\substack{-n\\ -\omega}}(\eta). \tag{8}$$

Expressing $\hat{\varphi}_{\substack{n\\ \omega}}(\eta)$ in terms of the Fourier transformed eigenfunction $\hat{\varphi}_{\substack{n\\ \omega}}(\eta) = \int d\kappa\ \exp(-i\kappa\eta)\hat{\varphi}_{\substack{n\\ \omega}}(\kappa)$, Δ is expressed as

$$\Delta = \frac{i|e|}{T_e}\left(1 - \frac{\omega_{*e}}{\omega}\right) \int_{-\infty}^{\infty} \frac{d\kappa \left(\hat{\varphi}_{\substack{n\\ \omega}}(\kappa)\right)^2}{\left[-i + \dfrac{v_{T_e}^2\,\kappa^2}{(Rq)^2 \nu_{ei}\omega}\right]}, \tag{9}$$

where we have used the fact that $\hat{\varphi}(\kappa)$ is even in κ. For convenience we define $\alpha = v_{T_e}^2/(Rq)^2\nu_{ei}$. With the transformation $\kappa = \sqrt{\omega/\alpha}\ y$,

$$\Delta = \frac{i|e|}{T_e}\left(1 - \frac{\omega_{*e}}{\omega}\right)\left(\frac{\omega}{\alpha}\right)^{1/2} \int_{-\infty}^{\infty} dy\ \frac{\left[\hat{\varphi}_n(\sqrt{\omega/\alpha}\ y)\right]^2 \omega}{[i + y^2]} , \tag{10}$$

and two limits are apparent assuming toroidicity-induced mode structure with its extent in κ expressed by $\Delta\kappa$. For $\sqrt{\omega/\alpha}\ (\Delta\kappa)^{-1} < 1$, the y variation of the integrand is controlled by the denominator and the mode structure has little effect on the growth rate scaling. This limit is equivalent to $k_\parallel^2 v_{T_e}^2/(\omega_{*e}\ \nu_{ie}) > 1$ and corresponds to an adiabatic electron regime. Evaluating Eq. (10), we obtain

$$\mathrm{Im}\Delta \cong \frac{|e|}{T_e}\left(1 - \frac{\omega_{*e}}{\omega}\right)\left(\frac{\omega}{\alpha}\right)^{1/2} |\varphi(0)|^2 \frac{\pi}{\sqrt{8}} \tag{11}$$

which implies that the growth rate scales as $\nu_{ei}^{1/2}$:

$$\frac{\gamma}{\omega} \sim \left(\frac{\omega\nu_{ei}}{v_{T_e}^2/(Rq)^2}\right)^{1/2} . \tag{12}$$

We note that replacing ν_{ei} with ω recovers the proper scaling of collisionless toroidicity-induced drift waves.

The opposite limit, $\sqrt{\omega/\alpha}\ (\Delta\kappa)^{-1} > 1$ implies that $k_\parallel^2 v_{T_e}^2/\omega_{*e}\ \nu_{ei} < 1$ and corresponds to the hydrodynamic electron regime. In this limit the mode structure controls the y variation of the integrand of Eq. (10) yielding,

$$\mathrm{Im}\Delta \cong \frac{|e|}{T_e}\left(1 - \frac{\omega_{*e}}{\omega}\right)\left(\frac{\omega}{\alpha}\right)^{1/2} \int_{-\infty}^{\infty} dy\, y^2 \left[\hat{\varphi}_{\underset{\omega}{n}}(\sqrt{\omega/\alpha}\; y)\right]^2$$

$$\cong \frac{|e|}{T_e}\left(1 - \frac{\omega_{*e}}{\omega}\right)\left(\frac{\alpha}{\omega}\right)(\Delta\kappa)^2 \frac{\sqrt{\pi}}{2} .$$

The growth rate thus scales as ν_{ei}^{-1} :

$$\frac{\gamma}{\omega} \sim \frac{v_{T_e}^2}{(Rq)^2 \nu_{ei}\omega} . \tag{13}$$

Because $\omega_{*e} > k_{\parallel}^2 v_{T_e}^2/\nu_{ei}$ for this regime, and because incoherent emission tends to produce an uncertainty in frequency $\Delta\omega$ on the order of ω for toroidicity-induced structure[10], the uncertainty in phase velocity induced by the nonlinear instability exceeds the parallel flow velocity, $\Delta\omega > k_{\parallel}^2 v_{T_e}^2/\nu_{ei}$. This is suggestive of a strong electron mode coupling process requiring a renormalized electron response. The nonlinear destabilization is large and linear structure is probably irrelevant. In the adiabatic regime, parallel flow velocity increases relative to the uncertainty in the phase velocity. The associated parallel viscosity decreases the lifetime of incoherent fluctuations and the electron nonlinearity is effectively weakened in comparison to the hydrodynamic regime. Weak turbulence is thus possible in the adiabatic regime provided the parallel viscosity is sufficiently large. Weak and strong turbulence limits will be distinguished in the next section. We anticipate that in the hydrodynamic regime MHD instabilities such as rippling modes become relevant. For most of the remainder of this paper we will restrict ourselves to the adiabatic regime.

III. PHYSICS OF COHERENT AND INCHOHERENT MODE COUPLING

The incoherent fluctuations, which are responsible for broad frequency spectra, are produced by the mode coupling associated with the nonlinearity. The nonlinearity, as a convolution in Fourier transform space, naturally provides a component of the perturbed distribution at wavenumber k which is proportional to the potential at some other wavenumber k'. This is the incoherent component. In nonperturbative treatments of mode coupling equations, i.e., mode simulations, the incoherent mode coupling is handled numerically. Analytic treatments of mode coupling rely on renormalization techniques. These are usually applied to one-point equations for the evolution of perturbed densities and fields. Such theories are intrinsically coherent and do not retain the incoherent component. In order to treat incoherent fluctuations, it is necessary to use a two-point equation. Dupree has shown that the renormalized two-point equation has a response exhibiting distinct behavior at opposite asymptotic limits of the relative separation variable.[5] At large scales, the correlation time is identical with the eigenmode lifetime, the eigenmode lifetime being defined as the decay time of the mode at a point, as obtained from one-point renormalized theory. The closure of the one-point equation is intrinsically coherent so the eigenmode lifetime is a coherent relaxation time. Thus the two-point density correlation at large scale is coherent. At small scales, however, the correlation time exceeds the eigenmode lifetime, increasing logarithmically to infinity as the scale diminishes to zero. This strong scale dependence arises as a consequence of the spatial correlation in the turbulent field at small scale. Since the one-point theory is incapable of treating two-point correlations it naturally retains only the coherent, scale independent response. The component of two-point correlation which outlives the eigenmodes is the incoherent response. Because this possiblity only occurs for small scales, the incoherent correlation is strongly peaked at the small scales and falls to zero at the scale where the two-point correlation time asymptotes to the coherent eigenmode lifetime. The incoherent density is thus grainy and the graininess has been referred to as "clumps".[5] In the steady state, when the decay of two-point correlation is offset by a driving source, the steady state correlation is given approximately as the product of the source term with the lifetime. The coherent part of the correlation may be extracted by subtracting the coherent correlation time from the lifetime, leaving the incoherent response.

The two-point equation is derived from the one-point equation. For incoherent fluctuations in the electron species we multiply the one-point equation, Eq. (4), by the nonadiabatic density at a second point, ensemble average and symmetrize. The resulting equation is

$$\left[\frac{\partial}{\partial t} + v_{de}(1)\cdot\nabla_1 + v_{de}(2)\cdot\nabla_2 - \frac{v_{T_e}^2}{\nu_{ei}}\left(\nabla_{\parallel_1}^2 + \nabla_{\parallel_2}^2\right)\right] \langle H_e(1)H_e(2)\rangle$$

$$- \frac{c}{B_o}\langle\nabla_1\hat{\varphi}(1)\times\hat{n}\cdot\nabla_1 H_e(1)H_e(2)\rangle - \frac{c}{B_o}\langle\nabla_2\hat{\varphi}(2)\times\hat{n}\cdot\nabla_2 H_e(2)H_e(1)\rangle$$

$$= \frac{-|e|}{T_e}\left(\langle H_e(2)\left(\frac{\partial\hat{\varphi}(1)}{\partial t} + V_D\nabla_{\vartheta_1}\hat{\varphi}(1)\right)\rangle + \langle H_e(1)\left(\frac{\partial\hat{\varphi}(1)}{\partial t} + V_D\nabla_{\vartheta_2}\hat{\varphi}(2)\right)\rangle\right) . \tag{14}$$

Employing the ballooning representation, we rewrite Eq. (14) as

$$
\Big[\frac{\partial}{\partial t} + v_{de}(\eta_1)\,\frac{q(r_1)}{r_1}\,\frac{\partial}{\partial \phi_1} + v_{de}(\eta_2)\,\frac{q(r_2)}{r_2}\,\frac{\partial}{\partial \phi_2}
- \frac{v_{T_e}^2}{(Rq)^2 \nu_{ei}} \Big(\frac{\partial^2}{\partial \eta_1^2} + \frac{\partial^2}{\partial \eta_2^2}\Big) \Big] \langle H(\eta_1,\varphi_1) H(\eta_2,\varphi_2)\rangle
$$

$$
+ \sum_{\substack{n\\ \omega}} \sum_{\substack{n'\\ \omega'}} \sum_{\substack{n''\\ \omega''}} \langle \exp(in\phi_1)\,\exp(in'\phi_1)\,\exp(in''\phi_2)\,\frac{c}{B_o}\sum_{m'} k_\vartheta k'_\vartheta\,\hat{s}(2\pi m')
$$

$$
\exp(-i2\pi n' q m')\,\hat{\varphi}_{\substack{n'\\ \omega'}}(\eta_1+2\pi m')\,H_{\substack{n\\ \omega}}(\eta_1) H_{\substack{n''\\ \omega''}}(\eta_2)\rangle
$$

$$
= \Big\{ \sum_{\substack{n'\\ \omega'}} \frac{i|e|}{T_e}\exp[in'(\phi_1-\phi_2)]\,(\omega'-\omega'_{*e})\,\langle H(\eta_2)\hat{\varphi}(\eta_1)\rangle_{\substack{n'\\ \omega'}} + (1 \leftrightarrow 2)\Big\} \qquad (15)
$$

where Φ is the toroidal angle.

Equation (15) posesses the necessary properties for the formation of density granulations. The turbulent E×B mixing goes to zero as the relative spatial separation of the two points diminishes to zero. The right hand side of Eq. (15) describes the rearrangement of the mean density by the turbulent potential, and in contrast to the E×B mixing, remains finite as the relative separation vanishes. Correlations of all scales are driven by this source. Turbulent mixing quickly destroys all but small scale correlations on the time scale of the correlation time. These, too, decay with time, but for sufficiently small scale they persist beyond the lifetime of the turbulent fields and are turbulently scattered as macroscopic fluid elements. We note then that it is the spatial variation of the turbulent mixing process which gives rise to the strong scale dependence of the two-point corelation time. Dissipative parallel viscosity is included in this problem. Unlike the turbulent mixing, it produces a diffusion which stays finite as the relative separation goes to zero. This scale independent decay mechanism reduces small scale correlation.

A. Free Energy Relaxation

The right-hand side of Eq. (15), or source as it is commonly called, plays a fundamental role in determining the incoherent spectrum. The source term is proportional to the rate of relaxation of mean density, and hence the expansion free energy provides the driving energy for incoherent fluctuations. The growth of incoherent fluctuations under this nonlinear instability induces finite amplitudes in the eigenmodes because they shield the granulations. For eigenmodes already at finite amplitude due to linear instability, incoherent fluctuations excite the modes to higher levels through the shielding response. The damping of these overdriven modes necessary to obtain a saturated state is the width $\Delta\omega_k$ of the frequency spectrum. A spectrum balance relation which describes the role of these processes in the steady state is obtained from the solution of Eq. (15) when the source term is expressed solely in terms of the incoherent part of the two-point correlation. Rewriting the source from Eq. (15),

$$S = \sum_{n',\omega'} \frac{i|e|}{T_e} \exp[in'(\phi_1-\phi_2)]\ (\omega'-\omega'_{*e})\ \langle H(\eta_2)\hat{\varphi}(\eta_1)\rangle_{n',\omega'} \qquad (16)$$

we express $H(\eta_2)$ as a decomposition into its coherent component $H^c(\eta_2)$ and its incoherent component $\tilde{H}(\eta_2)$:

$$S = \sum \frac{i|e|}{T_e} \exp[in'(\phi_1-\phi_2)](\omega'-\omega'_{*e})[\langle H^c(\eta_2)\hat{\varphi}(\eta_1)\rangle_{\substack{n'\\ \omega'}} + \langle\tilde{H}(\eta_2)\varphi(n_1)\rangle_{\substack{n'\\ \omega'}}] \quad . (17)$$

Because the coherent density is phase coherent with the potential, it may, in general, be written in terms of response function

$$H^{(c)}_{\substack{n\\ \omega}}(\eta_2) = R_{\substack{n\\ \omega}} \frac{|e|\varphi_n}{T_e} \quad . \tag{18}$$

In the present calculation, the response function includes both linear and nonlinear contributions. To make contact with the present model, we write the linear part of the response which is given by the formal operator

$$R^L_{\substack{n\\ \omega}} = \frac{(\omega_{*e}-\omega)}{\left(\omega - \dfrac{v^2_{T_e}}{(Rq)^2 \; im\nu_{ei}} \dfrac{\partial^2}{\partial\eta^2} - \omega_{de}\right)} \quad .$$

The function $\varphi_{\substack{n\\ \omega}}$ is the potential fluctuation, determined from the quasineutrality condition

$$n^I - n^E = L^{ion}_{\substack{n\\ \omega}} \frac{e\varphi_{\substack{n\\ \omega}}}{T_e} - \frac{e\varphi_{\substack{n\\ \omega}}}{T_e} - \frac{R_{\substack{n\\ \omega}} e\varphi_{\substack{n\\ \omega}}}{T_e} - \tilde{H} = 0 \quad , \tag{19}$$

where $L^{ion}_{\substack{n\\ \omega}}$ is the ion response function. The coherent part of the quasineutrality condition specifies the eigenfunction-operator $L_{\substack{n\\ \omega'}}$

$$L^{ion}_{\substack{n\\ \omega}} \frac{e\hat{\varphi}_{\substack{n\\ \omega}}}{T_e} - \frac{e}{T_e}\hat{\varphi}_{\substack{n\\ \omega}} - R_{\substack{n\\ \omega}} \frac{e\hat{\varphi}_{\substack{n\\ \omega}}}{T_e} \equiv L_{\substack{n\\ \omega}} \frac{e\hat{\varphi}_{\substack{n\\ \omega}}}{T_e} , \tag{20}$$

with the full quasineutrality condition expressing the shielding of the incoherent fluctuations by the eigenmodes,

$$L_{\substack{n\\ \omega}}(\eta) \frac{e\hat{\varphi}_{\substack{n\\ \omega}}}{T_e} = \tilde{H}(\eta) . \tag{21}$$

This last relationship allows us to write the shielding potential in terms of the shielded clumps, so that Eq. (17) becomes

$$S = \sum_{\substack{n'\\ \omega'}} \frac{i|e|}{T_e} \exp[in'(\phi_1-\phi_2)](\omega'-\omega'_{*e})$$

$$\left[\frac{T_e}{|e|} R_{\substack{-n'\\ -\omega'}}(\eta_2) L^{-1}_{\substack{-n'\\ -\omega'}}(\eta_1) \langle\tilde{H}(\eta_2)\ \tilde{H}(\eta_1)\rangle_{\substack{n'\\ \omega'}} + \frac{T_e}{|e|} L^{-1}_{\substack{n'\\ \omega'}}(\eta_1)\langle\tilde{H}(\eta_2)\tilde{H}(\eta_1)\rangle\right] . \tag{22}$$

We multiply the last term by the unit operator $L_{\substack{-n\\ -\omega}}(\eta_2) L^{-1}_{\substack{-n'\\ -\omega'}}(\eta_2) = (L^{ion}_{\substack{-n'\\ -\omega'}} - 1 - R_{\substack{-n'\\ -\omega'}}) L^{-1}_{\substack{-n'\\ \omega'}}(\eta_2)$ and note that the nonadiabatic electron response completely cancels out of the expression. Accordingly the source becomes

$$S = \sum_{\substack{n'\\ \omega'}} i(\omega'-\omega'_{*e})(L^{ion}_{\substack{-n'\\ -\omega'}} - 1) L^{-1}_{\substack{-n'\\ -\omega'}}(\eta_2) L^{-1}_{\substack{n'\\ \omega'}}(\eta_1)\ \langle\tilde{H}(\eta_2)\tilde{H}(\eta_1)\rangle_{\substack{n'\\ \omega'}} .$$

Since S is real,

$$S = \sum_{\substack{n'\\ \omega'}} (\omega'-\omega'_{*e})(L^{ion}_{\substack{n'\\ \omega'}})_{Im}\ L^{-1}_{\substack{-n'\\ -\omega'}}\ L^{-1}_{\substack{n'\\ \omega'}} \langle\tilde{H}(\eta_2)\tilde{H}(\eta_1)\rangle_{\substack{n'\\ \omega'}} , \tag{23}$$

where $L^{ion}_{-n', -\omega'} = R_e(L^{ion}_{n', \omega'}) - i\,Im(L^{ion}_{n', \omega'})$.

The scaling of the source term for electron small scale correlation with ion dissipation is a familiar feature of clump phenomena. This derivation clearly shows the scaling to be a result of quasineutrality, or equivalently, ambipolarity. In particular, since the electron density, $H(\eta) = R\, e\varphi/T_e + \tilde{H}$ is shielded by the neutralizing ion density $(L^{ion})|e|\varphi/T_e$, the relaxation of the electron density distribution function (by scattering of electron clumps down the density gradient) is constrained by the quasi-neutrality condition, which requires that the ions respond to neutralize the scattered electron clump. This constraint underlies the result that electron relaxation is proportional to ion dissipation. Note the eigenmode frequency is less than ω_{*e} and when the shielding distribution is energy absorbing $[Im(L^{ion}) < 0]$, a source is available which permits free energy extraction from the equilibrium gradient through an electron clump channel. The energy goes from the gradient to electron small scale correlation, to the modes VIA coupling, and finally to the ions which damp the modes through the collective resonance.

The quasineutral shielding process is most transparent in the fluid description where the quasineutrality condition, Eq. (19) allows for a direct replacement in Eq. (17) of $H^{(c)} + \tilde{H}$ by $L^{ion} - 1$. This holds rigorously for the general nonlinear response R_n. In the kinetic description, where the density is distributed on a velocity continuum as well as in configuration space, the analysis is more complicated. There the ballistic nature of clumps selects a particle velocity at the wave-particle resonance. The nonlinear response, however, broadens the resonance and the replacement of $H^{(c)} + \tilde{H}$ by L^{ion} at the ballistic velocity is not in general exact.

This free energy extraction mechanism is a significant channel for the relaxation of expansion free energy, particularly when the modes are linearly unstable from electron dissipation. In the steady state the ion damping is necessarily large and negative in order to balance the linear growth and incoherent excitation. There is thus an enhanced source for small scale correlation.

We complete the source calculation by writing the inverse eigenfunction operator $L^{-1}_{\substack{n\\ \omega}}(\eta)$ in terms of the Green's function $\hat{\varphi}_{n,\omega}(\eta)$ and dielectric response $\varepsilon(n,\omega)$,

$$L^{-1}_{\substack{n\\ \omega}}(\eta)\ F(\eta) = \frac{1}{\varepsilon(n,\omega)}\ \hat{\varphi}_{\substack{n\\ \omega}}(\eta) \int d\eta'\ \varphi_{\substack{n\\ \omega}}(\eta')F(\eta') \ . \tag{24}$$

The source is then given by

$$S = \sum_{\substack{n'\\ \omega'}} (\omega'-\omega_{*e}^{\prime})\ \frac{\varepsilon^{ion}_{IM}(n',\omega')}{|\varepsilon(n',\omega')|^2}\ \ |\hat{\varphi}_{\substack{-n'\\ -\omega'}}(\eta_2)\hat{\varphi}_{\substack{n'\\ \omega'}}(\eta_1)|\ \ \langle\tilde{\tilde{H}}^2\rangle_{\substack{n'\\ \omega'}} \tag{25}$$

where $$\langle\tilde{\tilde{H}}^2\rangle_{\substack{n'\\ \omega'}} = \int d\eta_1 \int d\eta_2\ \hat{\varphi}_{\substack{n'\\ \omega'}}(\eta_1)\ \hat{\varphi}_{\substack{-n'\\ -\omega'}}(\eta_2)\ \langle\tilde{H}(\eta_2)\tilde{H}(\eta_1)\rangle_{\substack{n'\\ \omega'}} \tag{26}$$

and $\varepsilon^{ion}_{IM}(n',\omega')$ is the imaginary part of the ion contribution to the shielding dielectron response.

B. Evolution of Small Scale Correlation

In this section the solution of the two-point equation, Eq. (15), is presented. <H(1)H(2)> evolves according to:

$$[\frac{\partial}{\partial t} + V_{de}(\eta_1)\frac{q(r_1)}{r_1}\frac{\partial}{\partial \phi_1} + V_{de}(\eta_2)\frac{q(r_2)}{r_2}\frac{\partial}{\partial \phi_2}$$

$$- \frac{V_{T_e}^2}{(Rq)^2}\frac{1}{\nu_{ei}}(\frac{\partial^2}{\partial \eta_1^2} + \frac{\partial^2}{\partial \eta_2^2})] \langle H(1)H(2)\rangle + T_{12} = S \tag{27}$$

where S is given by Eq. (25) and T_{12}, the triplet ExB nonlinearity is

$$T_{12} = \sum_{\substack{n\\ \omega}} \sum_{\substack{n'\\ \omega'}} \sum_{\substack{n''\\ \omega''}} \langle \exp(in\phi_1)\exp(in'\phi_1)\exp(in''\phi_2)\frac{c}{B_o}$$

$$\sum_{m'} k_\vartheta k_\vartheta' \hat{S}(2\pi m')\exp(-iz\pi n'qm')\hat{\varphi}_{\substack{n'\\ \omega'}}(\eta_1+2\pi m')\langle H_{\substack{n\\ \omega}}(\eta_1)H_{\substack{n''\\ \omega''}}(\eta_2)\rangle$$

It is necessary to renormalize Eq. (27), by finding a closure for the triplet correlation in the nonlinearity. We use standard weak coupling methods associated with the direct interaction approximation. Writing the toroidal angle ϕ in terms of relative angle $\phi_- = 1/2(\phi_1-\phi_2)$ and composite angle $\phi_+ = \phi_1+\phi_2$, the ensemble average is accomplished by integrating over ϕ_+ a Kronnecker delta $\delta_{n+n'+n'',0}$ being the result. The wave-wave coupling contribution arising from the induced field $\hat{\varphi}_{n+n'}$ is neglected, so that

$$T_{12} = \frac{c}{B_o} \sum_{\substack{n \\ \omega}} \sum_{\substack{n' \\ \omega'}} \{\exp(in\phi\_) \sum_{m'} k_\vartheta k'_\vartheta \hat{s}(2\pi m') \exp(2\pi i n' q m')$$

$$< \varphi_{\substack{-n' \\ -\omega'}} (\eta_1 + 2\pi m') H_{\substack{n+n' \\ \omega+\omega'}}(1) H_{\substack{-n \\ -\omega}}(2)> - \exp(in\phi\_ + in'\phi\_)$$

$$\sum_{m'} k_\vartheta k'_\vartheta \hat{s}(2\pi m') \exp(2\pi i n' q m') < \hat{\varphi}_{\substack{-n' \\ -\omega'}}(\eta_1 + 2\pi m') H_{-n'}(1) H_{n+n'}(2)>\} \qquad (28)$$

To obtain driven density $H_{n+n'}$, we solve Eq. (5) to second order, writing $n+n'$ for n,

$$H_{n+n'} = -L^{-1}_{n+n} \frac{c}{B_o} \sum_{n''} \sum_{m'} k''_\vartheta (k_\vartheta + k'_\vartheta) \hat{s}(2\pi m')$$

$$\exp(2\pi i n'' q m') \hat{\varphi}_{-n''}(\eta + 2\pi m') H_{n+n'+n''}(\eta) \qquad (29)$$

where

$$L_{n+n'} = \omega + \omega' - \frac{v^2_{T_e} \frac{\partial^2}{\partial \eta_2}}{i m_e \nu_{ei}} - \tilde{\omega}_{de}(n+n') . \qquad (30)$$

From the sum over n'' in Eq. (29) the directly interacting triplet ($n'' = -n'$) is selected for substitution into the first term in Eq. (28) and ($n'' = -n$) for substitution into the second term. We neglect the renormalization of the potential from the choice ($n'' = -n$) for the first term and ($n'' = -n'$) for the second. Under these approximations the triplet is

$$T_{12} = \frac{-c^2}{B_o^2} \sum_n \sum_{n'} L_{\substack{n+n' \\ \omega+\omega'}}(1) \sum_m (2\pi m)^2 k_\vartheta^2 k_{\vartheta}'^2$$

$$\times \hat{s}^2 (2\pi m)^2 \{\exp(in\phi_-) <\hat{\varphi}^2(1)>_{\substack{n' \\ \omega'}} <H(1)H(2)>_{\substack{n \\ \omega}}$$

$$+ \exp[i(n+n')\phi_-] \exp(2\pi imr_-/\Delta) <\hat{\varphi}(1)\hat{\varphi}(2)>_{\substack{n' \\ \omega'}} <H(1)H(2)>_{\substack{n \\ \omega}}\} \quad (31)$$

where $\Delta = 1/k\hat{s}$. One further approximation, a Markovian approximation, will be made so that $L_{n'}$ will replace $L_{n+n'}$. The results, obtained from the solution of the two-point equation, Eq. (27), are not sensitive to the details of the renormalization. Hence, the simple closure scheme used above is adequate. This is because the renormalization affects the decay of small scale correlation but not the driving. It has been argued that the net effect of approximations in the renormalization is to produce a change in the two-point correlation time of at most a factor of three.[12] Because the most important neglected processes pertain to selfbinding, omitted in this closure, the actual correlation time is longer than the approximate correlation time, so that nonlinear growth rates, fluctuation levels and transport coefficients obtained from this theory are lower bounds.

The renormalized triplet, Eq. (31) approximates the E×B mixing process as a diffusion in toroidal angle. The critical property, in so far as small scale correlated triplet is concerned, is preserved in the approximated triplet, i.e. the diffusion vanishes as the relative separation goes to zero. In the coordinates of the relative separation, (r_-, φ_-, η_-), we may write T_{12} as

$$T_{12} = D_- \frac{\partial^2}{\partial y_-^2} \tag{32}$$

where

$$D_- = 2D - D^{(1,2)} - D^{(2,1)}, \tag{33}$$

$$D = \frac{c^2}{B_o^2} \sum_{\substack{k' \\ \omega'}} k_{\vartheta}'^2 \, \hat{s}^2 (R_e \, L_{\substack{k' \\ \omega'}}) \sum_m (2\pi m)^2 \, \langle \hat{\varphi}(\eta + 2\pi m)^2 \rangle_{\substack{k' \\ \omega'}} \tag{34}$$

and

$$D^{(1,2)} = D^{(2,1)} = \frac{c^2}{B_o^2} \sum_{\substack{k' \\ \omega'}} \exp(ik' y_-) \, k_{\vartheta}'^2 \, \hat{s}^2 \, (R_e \, L_{\substack{k' \\ \omega'}})$$

$$\sum_m (2\pi m)^2 \exp(2\pi i m r_- k' \hat{s}) \, \langle \hat{\varphi}(\eta_1 + 2\pi m) \varphi(\eta_2 + 2\pi m) \rangle_{\substack{k' \\ \omega'}} \tag{35}$$

and $y = r\phi_-/q$, and $k = nq/r$. The relative diffusion D_- is seen to consist of two parts, an independent diffusion D and correlated diffusion $D^{(1,2)}$, $D^{(2,1)}$. It is readily ascertainable that as $(r_-, y_-, \eta_-) \to 0$, $D^{(1,2)} \to D$ and D_- vanishes accordingly. In the opposite limit, as the separation becomes large, $D^{(1,2)} \to 0$. The relative diffusion is thus independent at large separation and accounts for the fact that turbulent correlation time asymptotes to the coherent correlation time or eigenmode lifetime $\tau_{E\times B} = (k^2 D)^{-1}$. $D^{(1,2)}$ and $D^{(2,1)}$ incorporate the small scale correlation and are responsible for a correlation time $\tau_{c\ell}$ which exceeds τ_c on the scale for which $D^{(1,2)}$ differs from zero. The scale for $D^{(1,2)} \neq 0$ defines the clump scale. For a spectrum which is Gaussian,

$$D_{-} \cong 2D[1 - \cos(k_o' y_{-}) \exp(-y_{-}^2/2\alpha^2) \exp(-\eta_{-}^2/\Delta\eta^2) \exp(-r_{-}^2 \hat{s}^2/2\alpha^2)] \qquad (36)$$

where k_o' corresponds to most probable wavenumber and α^{-1} corresponds to the wavenumber spread in then the k spectrum and is a measure of the root mean square wavenumber. At small scale, D_{-} is quadratic in the relative variables

$$D_{-} \sim 2D(y_{-}^2 k_o^2 + \frac{\eta_{-}^2}{\Delta\eta^2} + r_{-}^2 k_o^2 \hat{s}^2) \qquad (37)$$

where k_o^{-1} $[= (k_o'^2 + 1/2\alpha^2)^{-1/2}]$, $\Delta\eta$ and k_o^{-1} $\hat{s}^{-1}$ are the clump scales in y_{-}, η_{-} and r_{-} respectively.

Having renormalized the triplet nonlinearity, we substitute Eq. (32) into Eq. (27) and arrive at the renormalized two-point equation,

$$[\frac{\partial}{\partial t} + v_{de}'(\eta_{+})\eta_{-} \frac{\partial}{\partial y_{-}} - \frac{D_{\parallel}}{(Rq)^2} \frac{\partial^2}{\partial \eta_{-}^2} - D_{-} \frac{\partial^2}{\partial y_{-}^2}] \langle H(1)H(2)\rangle = S, \qquad (38)$$

where $D_{\parallel} = v_{T_e}^2/\nu_{ei}$. The diamagnetic drift term and viscous diffusion have been expressed in terms of coordinates η_{+}, η_{-}, assuming variation in η_{-} about a fixed η_{+}. The prime denotes differentiation with respect to η_{+}. Equation (38) states that the electron two-point correlation evolves by the balance of drive by average density relaxation (source) with decay by relative ExB diffusion (ExB shear stress), relative magnetic drift, and collisional diffusion.

We now solve Eq. (38). We seek the Green's function satisfying the homogeneous equation

$$\left(\frac{\partial}{\partial t} + v_{de}'(\eta_+)\eta_- \frac{\partial}{\partial y_-} - \frac{D_\parallel}{(Rq)^2}\frac{\partial^2}{\partial \eta_-^2} - D_- \frac{\partial^2}{\partial y_-^2}\right)$$

$$g(\eta_-, y_-, r_-, t \mid \eta_-', y_-', r_-', t') = 0 \qquad (39)$$

and the condition

$$g(\eta_-, y_-, r_-, t \mid \eta_-', y_-', r_-', t) = \delta(\eta_- - \eta_-')\delta(y_- - y_-')\delta(r_- - r_-') \qquad (40)$$

so that the solution is then

$$\langle H(1)\, H(2)\rangle = \int d\eta_-' dy_-' dr_-' dt'\; g(\eta_-, y_-, r_-, t \mid \eta_-', y_-', r_-', t')\; S(\eta_-', y_-', r_-') . \qquad (41)$$

We define a Reynold's number parameter to distinguish regimes where either linear behavior (parallel viscosity) or nonlinear behavior (turbulent diffusion) dominates

$$R_e \equiv \frac{Dk^2(Rq)^2\Delta y_c^2}{D_\parallel} = \frac{\tau_\parallel}{\tau_{E\times B}} , \qquad (42)$$

where D is the independent diffusion, Eq. (34), and $\Delta\eta_c$ is the parallel correlation length. As noted, the Reynold's number is the ratio of the parallel correlation time (viscous diffusion) to the coherent nonlinear relaxation rate. In the high Reynold's number regime the full coherent (one-point) relaxation rate, given by

$$\tau_c = \left(\frac{D_\parallel}{(Rq)^2}\frac{\partial^2}{\partial\eta_-^2} + D\frac{\partial^2}{\partial y_-^2}\right)^{-1}, \tag{43}$$

is dominated by turbulent E×B diffusion in toroidal angle so that $\tau_c \sim \tau_{E\times B} = (k_o^2 D)^{-1}$. The decay of two-point correlation is dominated by the scale dependent relative diffusion D_-. Consequently, the density granulation process is important and a large incoherent component of the density is anticipated. In this nonlinear regime, the separation of neighboring density elements is exponential, giving rise to a two-point correlation time τ_c which is bigger than the coherent correlation time τ_c by a factor which goes as the log of one over the relative separation. At the smallest scales, however, τ_c no longer diverges exponentially as it does in the collisionless case. The logarithmic divergence is cut off at the scale where $D_\parallel/(Rq)^2\Delta y_c^2 \sim D_- k_o^2$. In the low Reynold's number regime the coherent relaxation rate is controlled by the linear parallel viscous diffusion so that $\tau_c \sim \tau_\parallel$. Decay of two-point correlation is also dominated by the viscous diffusion. However, inside the clump scale it still holds that $D_\parallel^2/(Rq)^2\Delta\eta_c^2 + D_- k_o^2 < D_\parallel/(Rq)^2\Delta\eta_c^2 + Dk_o^2$ so that $\tau_{c\ell} > \tau_c$. Because the linear parallel viscosity predominates, a power law governs the separation of neighboring density elements and $\tau_{c\ell}$ no longer exhibits a logarithmic scale dependence. In this regime we will adopt a low Reynold's number perturbation treatment in order to solve the two-point equation and ultimately $\langle\tilde{H}^2\rangle$ will be proportional to $\tau_{\parallel}/\tau_{E\times B}$.

In general, an exact solution of the Green's function (Eq. (39)), or equivalently the two-point correlation, given the scale dependent diffusion, is very difficult. Previous papers have instead computed moments of the Green's function.[4,5,10,15] Defined as

$$\langle y_-^2\rangle = \int dy_-'d\eta_-'dr_-'dt' \; y_-^2 \; g(y_-,\eta_-,r_-,t \mid y_-',\eta_-',r_-',t') \, , \tag{44}$$

the y_-^2 moment gives the time evolution of the separation of two trajectories in toroidal angle. If the two-point correlation decays principally by toroidal diffusion, then the correlation time or clump lifetime $\tau_{c\ell}$ is given by the time it takes for two density elements, initially separated by $(y_-, \eta_-, r_-) < (k_o^{-1}, \Delta\eta, k_o^{-1}\hat{s}^{-1})$ to diverge to the clump scale. The clump lifetime enters the solution of the two-point equation as follows. From Eq. (38) the steady state solution of the two-point equation is formally

$$\langle H^2\rangle = \left(v_{de}'\,\eta_- \frac{\partial}{\partial y_-} - \frac{D_{\parallel}}{(Rq)^2}\frac{\partial^2}{\partial \eta_-^2} - D_- \frac{\partial^2}{\partial y_-^2}\right)^{-1} S$$

or approximately

$$\langle H^2\rangle \approx \tau_{c\ell}\, S \, . \tag{45}$$

A differential equation for $\langle y_-^2\rangle$ is obtained by taking the partial derivative with respect to time of Eq. (44). Equation (39),

$$\frac{\partial}{\partial t} g = -\left(v_{de}' \eta_- \frac{\partial}{\partial y_-} - \frac{D_\parallel}{(Rq)^2} \frac{\partial^2}{\partial \eta_-^2} - D_- \frac{\partial^2}{\partial y_-^2}\right) g$$

is used to replace the time derivative with the spatial operator, and partial integrations are used to transfer the operator from g to y_-^2. The resulting moment equations are

$$\frac{\partial}{\partial t} \langle y_-^2 \rangle = 2k_o^2 D \left[\langle y_-^2 \rangle + \frac{\langle \eta_-^2 \rangle}{k_o^2 \Delta\eta_c^2}\right] + 2v_{de}' \langle y_- \eta_- \rangle \tag{46}$$

$$\frac{\partial}{\partial t} \langle \eta_- y_- \rangle = v_{de}' \langle \eta_-^2 \rangle \tag{47}$$

$$\frac{\partial}{\partial t} \langle \eta_-^2 \rangle = \frac{2D_\parallel}{(Rq)^2} . \tag{48}$$

Combining Eqs. (46) through (48) yields the equivalent equation

$$\frac{\partial^3}{\partial t^3} \langle y_-^2 \rangle = 2k_o^2 D \frac{\partial^2}{\partial t^2} \langle y_-^2 \rangle + \frac{4\, v_{de}' D_\parallel}{(Rq)^2} . \tag{49}$$

Equation (49) is integrated using the initial conditions

$$\frac{\partial}{\partial t} \langle y_-^2 \rangle \Big|_{t=0} = 2k_o^2 D \left(y_-^2 + \frac{n_-^2}{k_o^2 \Delta\eta_c^2}\right) + 2v_{de}' \eta_- y_- \tag{50a}$$

$$\frac{\partial^3}{\partial t^3} \langle y_-^2\rangle|_{t=0} = 2k_o^2 \frac{\partial^2}{\partial t^2} \langle y_-^2\rangle|_{t=0} + \frac{4v'_{de}D_\parallel}{(Rq)^2} \quad (50b)$$

$$\frac{\partial^2}{\partial t^2} \langle y_-^2\rangle|_{t=0} = 2k_o^2 D \frac{\partial}{\partial t} \langle y_-^2\rangle|_{t=0} + \frac{4D\ D_\parallel}{\Delta\eta_c^2(Rq)^2} + 2v_{de}'^{\,2}\ \eta_-^2 \ . \quad (50c)$$

The resulting evolution equation for $\langle y_-^2\rangle$ is

$$\langle y_-^2\rangle = \{y_-^2 + \frac{\eta_-^2}{k_o^2\Delta\eta_c^2} + \frac{v'_{de}\eta_-y_-}{k_o^2 D} + \frac{D_\parallel}{k_o^4 D\ \Delta\eta_c^2(Rq)^2}$$

$$+ \frac{v_{de}'^{2}\eta_-^2}{2k_o^4D^2} + \frac{v_{de}'^{2}D_\parallel}{2k_o^6D^3(Rq)^2}\ \}\ \exp(2k_o^2Dt)$$

$$- \frac{v_{de}'^{2}D_\parallel}{(Rq)^2k_o^2D}\ t^2 - \{\ \frac{v_{de}^{2}D_\parallel}{k_o^4D^2(Rq)^2} + \frac{v_{de}'^{2}\eta_-^2}{k_o^2D} + \frac{2D_\parallel}{k_o^2\Delta\eta_c^2(Rq)^2}\ \}t$$

$$- \{\frac{\eta_-^2}{k_o^2\Delta\eta_c^2} + \frac{v'_{de}\eta_-y_-}{k_o^2D} + \frac{D_\parallel}{k_o^4D\Delta\eta_c^2(Rq)^2} + \frac{v_{de}'^{2}\eta_-^2}{2k_o^4D^2} + \frac{v_{de}'^{3}D_\parallel}{2k_o^6D^3(Rq)^2}\ \} \quad (51)$$

The $\langle\eta_-^2\rangle$ evolution, obtained from Eq. (48) is

$$\langle\eta_-^2\rangle = \eta_-^2 + \frac{2D_\parallel}{(Rq)^2}\ t \ . \quad (52)$$

From the inversion of Eqs. (51) and (52) we obtain decay rates corresponding to the diffusion in the toroidal and parallel directions respectively. The decay rate $\tau_{c\ell}$ of the two-point correlation is determined from these two processes. In the high Reynold's number limit the diffusion is predominately toroidal and $\tau_{c\ell}$ is well approximated by the toroidal decay time. In the low Reynold's number regime, however, though diffusion is predominantly parallel, it is not sufficient to approximate $\tau_{c\ell}$ with the parallel diffusion time. Incoherent fluctuations arise from scale dependence in the toroidal diffusion and are therefore a higher order effect. The zero order approximation $\tau_{c\ell} \approx \tau_{\parallel}$ thus includes no incoherent effects (clumps). In order for $\tau_{c\ell}$ to correctly represent small scale correlation, it is necessay to include both toroidal and poloidal evolution in the dissipation range ($R_e < 1$).

With these facts in mind we examine the toroidal and parallel relaxation times by inverting Eqs. (51) and (52). Equation (51), owing to its transcendental nature may not be inverted analytically for arbitrary Reynold's number. Approximations for the toroidal relaxation time in the high and low Reynold's number limit may be obtained as follows. For high Reynold's numbers $\tau_{E\times B} = k_o^{-2}D^{-1} < \tau_{\parallel}$. Within the clump scale ($y_-^2 < k_o^{-2}$, $\eta_-^2 < \Delta\eta_c^2$), $\tau_{c\ell} > \tau_{E\times B}$, and the exponential dependence dominates the power law dependence in Eq. (51). Inverting the exponential we obtain

$$\tau_y \approx \tau_{c\ell} = \tau_{E\times B}\, \ell n \left\{ \left[k_o^2\left(y_-^2 + \frac{\eta_-^2}{k_o^2 \Delta\eta_c^2} + \frac{v'_{de}\eta_- y_-}{k_o^2 D} + \frac{v_{de}'^2\,\eta_-^2}{2k_o^4 D^2} + \frac{D_{\parallel}}{k_o^4 D \Delta\eta_c^2 (Rq)^2} + \frac{v_{de}'^2\, D_{\parallel}}{2k_o^6 D^3 (Rq)^2} \right) \right]^{-1} \right\} \tag{53}$$

The last two terms are proportional to R_e^{-1} and are small for y_-, η_- near the clump scale. Inside the clump scale, the denominator decreases and $\tau_{c\ell}$ increases logarithmically until $y_- k_o$, $\eta_-/\Delta\eta_c \sim R_e^{-1}$. The $D_\parallel/D$ terms cut off the singularity which occurs in the collisionless theory and we have $\tau_{c\ell} \sim \ell n(R_e)$ for y_-, $\eta_- \to 0$. For $R_e \to \infty$ the collisionless results are recovered. We also note the similarity of Eq. (53) with the kinetic result.[4,5,10] Here the absence of v_- terms confirms the existence of density granulations in the fluid theory.

In the low Reynold's number regime $\tau_{E\times B} >> \tau_\parallel$, $\tau_y \geqslant \tau_c \sim \tau_\parallel$, implying that $\tau_y/\tau_{E\times B} < 1$. Therefore, the toroidal evolution in the low Reynold's number regime is approximated by expanding the exponential in Eq. (51) for small argument. Neglecting the diamagnetic drift terms yields a quadratic equation in t whose lowest order solution is

$$\tau_y = \left(\frac{\tau_\parallel \tau_{E\times B}}{2}\right)^{1/2} (1-k_o^2 y_-^2)^{1/2} . \tag{54}$$

The toroidal variation of the toroidal decay rate in both Reynold's number regimes is summarized in Fig. 1. It is interesting to note the Reynold's number similarity exhibited in Fig. 1 for the inertial range. The two-point correlation is equal for large Reynold's numbers with departures occuring on the scale where collisional viscosity becomes important. This scale decreases for increasing Reynold's nubmer. These are standard features of inertial flows (compare with Stewart and Townsend[13]). In the inertial range ($R_e > 1$) where the clump lifetime $\tau_{c\ell}$ is given by the toroidal decay time, we may proceed as in previous theories.[4,5,10] Writing $\langle\tilde{H}^2\rangle = (\tau_{c\ell}-\tau_c)S$, we shall obtain in Sec. V the spectrum balance equation, from which the nonlinear growth rate, spectrum width, fluctuation level and transport coefficients are derived. In the dissipation range neither τ_y or $\tau_\parallel$ correctly approximate $\tau_{c\ell}$. The appropriate combination of these rates to yield $\tau_{c\ell}$ is not obvious and the approximate solution $\langle\tilde{H}^2\rangle = (\tau_{c\ell}-\tau_c)S$ is not of use. In this regime it is most advantageous to directly solve Eq. (38) using a perturbation theory for small Reynold's numbers. The remainder of this section will be devoted to obtaining the moderate Reynold's number solution of Eq. (38).

With the neglect of the diamagnetic drift term and using Eq. (36) for D_- we seek the steady state solution of

$$\{\frac{\partial}{\partial t} - \frac{D_\parallel}{(Rq)^2}\frac{\partial^2}{\partial\eta_-^2} - D[1-\cos(k_0 y_-)\ \exp(-y_-^2/2\alpha^2)\ \exp(-\eta_-^2/\Delta\eta^2]\frac{\partial^2}{\partial y_-^2}\}$$

$$H(y_-,\eta_-,t) = S(\eta_- y_-) \qquad (55)$$

where $H(y_-,\eta_-,t) = \langle\tilde{H}(y_1,\eta_1,t)\tilde{H}(y_2,\eta_2,t)\rangle$. We undertake a perturbation solution for small Reynold's numbers with

$$H(y_-,\eta_-,t) = H^{(0)}+H^{(1)}+\ \ldots$$

where $H^{(0)}$ is the solution of

$$(\frac{\partial}{\partial t} - \frac{D_\parallel}{(Rq)^2}\frac{\partial^2}{\partial\eta_-^2})\ H^{(0)} = S \qquad (56)$$

and

$$\left(\frac{\partial}{\partial t} - \frac{D_{\parallel}}{(Rq)^2}\frac{\partial^2}{\partial \eta_-^2}\right) H^{(1)}$$

$$= D[1-\cos(k_o y_-)\ \exp(-y_-^2/2\alpha^2)\ \exp(-\eta_-^2/\Delta\eta^2)]\ \frac{\partial^2}{\partial y_-^2} H^{(0)} \qquad (57)$$

is the first order equation for $H^{(1)}$. The zero order solution is

$$H^{(0)}(y_-,\eta_-,t) = \int d\eta_-' \ \frac{1}{[\pi D_{\parallel} t/(Rq)^2]^{1/2}}$$

$$\exp\left\{\frac{-(\eta_- - \eta_-')^2}{4D_{\parallel} t/(Rq)^2}\right\}\ u(\eta_-',y_-) - \frac{(Rq)^2}{D_{\parallel}} \int^{\eta_-} d\eta_-' \int^{\eta_-'} d\eta_-''\ S(\eta_-'',y_-) \qquad (58)$$

where the first term represents a decaying transient and the second term is the steady state part driven by $S(\eta_-,y_-)$. We are concerned with the steady state and therefore ignore the transients. The first order solution is then

$$H^{(1)}(y_-,\eta_-,t) = \frac{(Rq)^4}{D_{\parallel}^2} D \int^{\eta_-} d\eta_-' \int^{\eta_-'} d\eta_-'' \ [1-\cos(k_o y_-)\exp(-y_-^2/2\alpha^2)$$

$$\exp(-\eta_-''^2/\Delta\eta^2)\ ]\ \frac{\partial^2}{\partial y_-^2} \int^{\eta_-''} d\eta_3 \int^{\eta_3} d\eta_4\ S(\eta_4,y_-) \qquad (59)$$

To obtain the incoherent two-point correlation $\tilde{H}(y_-,\eta_-,t) = \langle\tilde{H}(1)\tilde{H}(2)\rangle$ we subtract out the coherent correlation. The zero order correlation is coherent; there is no incoherent correlation in that order. The first order result is obtained by subtracting D from D_-:

$$\tilde{H}\,(y_-,\eta_-) = \frac{-(Rq)^4}{D_\parallel^2}\, D \int^{\eta_-} d\eta' \int^{\eta'} d\eta''\, \cos(k_o y)\, \exp(-y_-^2/2\alpha^2)$$

$$\exp(-\eta''^2/\Delta\eta^2)\, \frac{\partial^2}{\partial y_-^2} \int^{\eta''} d\eta_3 \int^{\eta_3} d\eta_4\, S(\eta_4, y_-) . \tag{60}$$

We Fourier transform the source term and substitute from Eq. (25)

$$S(\eta_4, y_-) = \int d\kappa\, \exp(i\kappa\eta_4)\, \exp(iky_-)\, S(\kappa, k), \tag{61}$$

$$S(\kappa, k) = \int d\omega(\omega-\omega_{*e})\, \varepsilon_{Im}^{Ion}(k,\omega) \frac{|\hat{\varphi}_{k\,\omega}(\kappa_1)\, \hat{\varphi}_{-k\,-\omega}(\kappa_2)|}{|\varepsilon(k,\omega)|^2} \langle\tilde{H}^2\rangle_{k\,\omega} . \tag{62}$$

Finally, we Fourier transform the left hand side of Eq. (60) and obtain the two-point, Fourier transformed incoherent spectrum:

$$\langle\tilde{H}^2(\kappa_- k_+)\rangle_k' = \frac{(Rq)^4}{D_\parallel^2}\, D\, (2)^{1/2}\, \pi\alpha\Delta\eta \int dk' \left(\frac{\delta\eta^2}{4} + \frac{1}{\Delta_{k'}^2}\right)^{1/2} \frac{k'^2}{\kappa_-^2} \exp\left[\frac{-\kappa_-^2\Delta\eta^2}{4}\right]$$

$$\{\exp[-(k-k'-k_o)^2\alpha^2/2] + \exp[-(k-k'+k_o)^2\alpha^2/2)] \int d\omega'(\omega'-\omega'_{*e})$$

$$\varepsilon_{Im\;k'\,\omega'}^{ion}(\kappa_+) \frac{|\hat{\varphi}^2(\eta_+)|_{k'\,\omega'}}{|\varepsilon(k',\omega')|^2} \langle\overline{\tilde{H}\tilde{H}}\rangle_{k'\,\omega'} , \tag{63}$$

where $\Delta_{k'}^{-1}$ is the shielding eigenmode width in the parallel direction. Several integrations have been performed to arrive at Eq. (63). Details may be found in Appendix I.

This result will be further simplified in the next section when we obtain from it the spectrum balance equation. We note at this time an important difference between the two-point correlation for fluids and the two-point phase space density correlation of kinetic theory. We observe that fluid incoherent fluctuations or density granulation are dependent on the coherent fluctuation level. This dependence enters through the decay process, not the source, and reflects the mixing of density granulations by the turbulent potential. A decay rate dependence of $D/D_{\parallel}^2$ for $R_e<1$ and $1/D$ for $R_e>1$ indicates the tendency of incoherent fluctuations to be smaller when the dominant decay mechanism is stronger. While a $1/D$ dependence also occurs in the decay rate for phase space density (kinetic) granulations the width in velocity of the granulation is given by the trapping velocity and is also proportional to D. In the process of integrating over velocity to obtain the incoherent density correlation the amplitude dependence is thus lost. Consequently, the spectrum balance, which describes the steady state balance between the shielding eigenmodes and driven incoherent fluctuations according to Poisson's equation, carries no explicit amplitude dependence in the kinetic case, while it does in the fluid case.

We turn now to a consideration of the spectrum balance and address the properties of the steady state turbulence in the presence of incoherent emission.

IV. THE SPECTRUM OF TOKAMAK EDGE TURBULENCE

A. Intermediate Reynold's Number Regime

1. Linewidth

The steady state solution of the two-point density correlation equation, Eq. (60), details the balance which occurs between driven incoherent fluctuations and the damped shielding eigenmodes when the incoherent fluctuations have steady amplitude. Having expressed the driving source in terms of the incoherent correlation it is possible to eliminate the incoherent correlation from Eq. (63) and solve to find the damping in the shielding response. This provides a formula for the frequency linewidth as well as a condition for saturation from which the wavenumber-spectrum and transport coefficients may be obtained. Proceeding, we construct from the left hand side of Eq. (63) the eigenfunction-projected correlation defined in Eq. (26). The projection is accomplished by multiplying $\langle\tilde{H}^2\rangle$ by the shielding response structure functions, $\hat{\varphi}_{\substack{k' \\ \omega'}}(\kappa_1)\hat{\varphi}_{\substack{-k' \\ -\omega'}}(\kappa_2)$, and integrating over κ_1 and κ_2. The integrations over κ_1 and κ_2 may be transformed to κ_+ and κ_-. The projection then allows the shielding response structure functions to sample the dependence of the clump on the two parallel scales: The fast scale sampling (κ_-) accounts for the evolution or clump decay and the slow scale sampling reflects the degree to which the mode structure in the parallel direction shields the clumps. For the wave functions $\varphi(\kappa_\pm)$ we assume a Gaussian of width Δ_k: $\varphi(\kappa_\pm) = (\Delta_k\sqrt{\pi})^{-1}\exp[-\kappa_\pm^2/\Delta_k^2]$. The integration over κ_+ is trivial; for the κ_- integration we use Eq. (A6) with u=0 and obtain

$$\langle\overline{\tilde{H}\tilde{H}}\rangle_k = \frac{-(Rq)^4}{(D_\parallel)^2} D(8\pi)^{1/2}\,\alpha\Delta\eta\,\frac{1}{\Delta_k}\left(\frac{\Delta\eta^2}{4}+\frac{1}{\Delta_k^2}\right)\int dk'k'^2\,\frac{\Delta k'}{(\Delta_k^2+\Delta_{k'}^2)^{1/2}}$$

$$[e^{-(k-k'-k_o)^2 \alpha^2/2} + e^{-(k-k'+k_o)^2\alpha^2/2}]\cdot$$

$$\int d\omega'(\omega'-\omega'_{*e}) \frac{\varepsilon_{Im}^{Ion}(k',\omega')}{|\varepsilon(k',\omega')|^2} \overline{\langle\tilde{H}\tilde{H}\rangle}_{\substack{k' \\ \omega'}} . \tag{64}$$

We expand the dielectric about the wavenumber $k_r(\omega')$ corresponding to the eigenmode frequency ω', provided the system is weakly turbulent, or equivalently, that the spectrum broadening is not too large. With this expansion, the k′ integration is performed by evaluating the residue at the pole corresponding to the eigenmode.

The two-time correlation $\overline{\langle\tilde{H}\tilde{H}\rangle}_{\substack{k_r(\omega') \\ \omega'}}$ is obtained from the equal-time correlation $\overline{\langle\tilde{H}\tilde{H}\rangle}_{k_r(\omega')}$ by multiplication with the one-particle propagator. Thus,

$$\overline{\langle\tilde{H}\tilde{H}\rangle}_{\substack{k_r(\omega') \\ \omega'}} = 2re\{[-i\omega' + \frac{T_e\Delta_k^2}{m\nu_{ei}(Rq)^2}]^{-1}\}\overline{\langle\tilde{H}\tilde{H}\rangle}_{k_r(\omega')}, \tag{65}$$

reflecting the coupling of turbulence with parallel diffusion. We substitute Eq. (65) into Eq. (64) and consider the integration over ω'. There are two decaying functions of ω' in the integrand and their relative widths determine the integration. One function is the propagator just given with width $D_{\parallel}\Delta_k^2/(Rq)^2$ and the other is the function $\{\exp[-(k-k_r(\omega')-k_o)^2 \alpha^2/2] + \exp[-(k-k_r(\omega')+k_o)^2 \alpha^2/2]\}$. This function describes the decay and modulation of two-point correlation arising from the spatial dependence of the relative diffusion. For drift waves with poloidal correlation on the order of ρ_s the width of this function in frequency is on the order of ω_{*e} as determined by expanding $k_r(\omega')$ about the eigenfrequency. For the adiabatic regime $\omega_{*e} < D_\parallel \Delta_k^2/(Rq)^2$ we may approximate the propagator by its value at $\omega'=0$, giving

$$\langle\overline{\tilde{H}\tilde{H}}\rangle_k = \frac{-(Rq)^4}{D_\parallel^2} D(2\pi)^{3/2} \frac{\alpha\Delta\eta}{\Delta_k} \left(\frac{\Delta\eta}{4} + \frac{1}{\Delta_k^2}\right) \frac{(Rq)^2}{D_\parallel \Delta_k^2} \int d\omega' \frac{\Delta_{k_r}(\omega')}{(\Delta_k^2 + \Delta_{k_r}(\omega')^2)^{1/2}}$$

$$\left[e^{-(k-k_r(\omega')-k_o)^2\alpha^2/2} + e^{-(k-k_r(\omega')+k_o)^2\alpha^2/2}\right]$$

$$(\omega'-\omega_{*e}) \frac{\varepsilon_{Im}^{Ion}(k_r(\omega'),\omega') \langle\overline{\tilde{H}\tilde{H}}\rangle_{k_r(\omega')}}{\varepsilon_{Im}(k_r(\omega'),\omega') \dfrac{\partial\varepsilon}{\partial k_r}} . \qquad (66)$$

We consider two limits in evaluating Eq. (66), depending on the value of k_o relative to α^{-1}. The most probable wavenumber k_o arises from the oscillation of the relative diffusion occurring as the remnant of the periodicity of the scattering waves in the turbulent spectrum. This oscillation may be viewed as a consequence of the fact that a density element is correlated not only with closely neighboring density elements but density elements whose separation is at multiples of the wavelength of the scattering field. If we assume, as in previous work, that $k_o < \alpha^{-1}$, then k_o drops out and the sum of exponentials becomes $2\exp[-(k-k_r(\omega'))^2 \alpha^2/2]$. Expanding $k_r(\omega')$ about the eigenfrequency and assuming that the rest of the integrand is slowly varying in ω' we obtain upon integration over ω'

$$\langle\overline{\tilde{H}\tilde{H}}\rangle_k = \frac{-(Rq)^6}{D_\parallel^3} D(4\pi)^{3/2} \frac{\alpha\Delta\eta}{\Delta_k^3} \left(\frac{\Delta\eta^2}{4} + \frac{1}{\Delta_k^2}\right) \left[\frac{\partial k_r}{\partial\omega_k}\alpha\right]^{-1}$$

$$k^2(\omega_k-\omega_{*e})\ \frac{\varepsilon^{Ion}_{Im}(k,\omega_k)}{\varepsilon_{Im}(k,\omega_k)\ \frac{\partial\varepsilon}{\partial k}}\ \langle\overline{\tilde{H}\tilde{H}}\rangle_k \ . \qquad (67)$$

The correlation now cancels out of the equation leaving an expression which relates the total dissipation in the eigenmodes to the ion contribution to the dissipation.

$$\varepsilon_{Im}(k,\omega_k) = C(k,\omega_k)\ \varepsilon^{Ion}_{Im}(k,\omega_k) \ , \qquad (68)$$

where

$$C(k,\omega_k) = \frac{(Rq)^6}{D_\parallel^3}\ D(4\pi)^{3/2}\ \frac{\alpha\Delta\eta}{\Delta_k^3}\ \left(\frac{\Delta\eta^2}{4} + \frac{1}{\Delta_k^2}\right)\ \left[\frac{dk_r}{d\omega_k}\ \alpha\right]^{-1}\ k^2(\omega_{*e}-\omega_k)\ \left(\frac{d\varepsilon}{\partial k}\right)^{-1} \ . \qquad (69)$$

From Chen et al.[11],

$$\omega(k) = \frac{(1-\pi\varepsilon_n\hat{s})v_{de}k}{1+k_\perp^2\rho^2(1+\hat{s}^2\pi^2/4)} \ , \qquad (70)$$

from which we obtain

$$\frac{dk_r}{d\omega_k} = \frac{k}{\omega_k}\ \left[\frac{1+k^2\rho^2(1+\hat{s}^2\pi^2/4)}{1-k^2\rho^2(1+\hat{s}^2\pi^2/4)}\right] \ . \qquad (71)$$

The function $d\varepsilon/dk$ is given approximately by

$$\frac{d\varepsilon}{dk} \simeq \frac{1}{k}[1+3k^2\rho^2(1+\hat{s}^2\pi^2/4)\ ]\ . \tag{72}$$

Substituting these results into Eq. (66) we have

$$C(k,\omega_k) = R_e(k\rho)^4 \left(\frac{Rq}{L_n}\right)^4 \bar{\nu}^2\ (4\pi)^{3/2} \frac{1}{\Delta\eta\Delta_k^3} \left(\frac{\Delta\eta^2}{4} + \frac{1}{\Delta_k^2}\right)$$

$$\times \frac{[1-k_\perp^2\rho^2(1+\hat{s}^2\pi^2/4)][1-\pi\varepsilon_n\hat{s}][\pi\varepsilon_n\hat{s}+k_\perp^2\rho^2(1+\hat{s}^2\pi^2/4)]}{[1+k_\perp^2\rho^2(1+\hat{s}^2\pi^2/4)]^3[1+3k^2\rho^2(1+\hat{s}^2\pi^2/4)]} \tag{73}$$

where

$$\bar{\nu} = \frac{\nu_{ei}}{\omega_{*e}}\ \frac{m_e}{m_i}\ .$$

Equation (68) describes the steady-state sheilding eigenmode response to incoherent emission. The total dissipation $\varepsilon_{Im}(k,\omega_k)$ is composed of electron and ion contributions, $\varepsilon_{Im} = \varepsilon_{Im}^{Elec} - |\varepsilon_{Im}^{Ion}|$, and is necessarily negative ($\varepsilon_{Im} < 0$) in order to balance the incoherent emission. Rewriting Eq. (65) to introduce the electron dissipation we obtain a saturation condition:

$$\varepsilon_{Im}^{Ion}(k,\omega_k) = \frac{\varepsilon_{Im}^{Elec}(k,\omega_k)}{[1-C(k,\omega_k)]}\ . \tag{74}$$

This expresses the balance in the steady state between the linear electron destabilization (Eq. (11)) of the collision driven toroidal dissipative drift wave with the enhancement by incoherent emission $[1-C(k,\omega_k)]^{-1}$ and the ion damping, both linear and nonlinear. A related expression;

$$\varepsilon_{Im}(k,\omega_k) = \frac{C(k,\omega_k)}{[1-C(k,\omega_k)]}\,\varepsilon_{Im}^{Elec}(k,\omega_k) \tag{75}$$

gives the total dissipative mode response to the incoherent emission process in terms of a numerical enhancement of the linear electron dissipation. The width of the frequency spectrum at fixed k is just $\varepsilon_{Im}/(\partial\varepsilon/\partial\omega)$ and is given by

$$\Delta\omega_k = \gamma_k^{elec}\,\frac{C(k,\omega_k)}{[1-C(k,\omega_k)]} \tag{76}$$

where γ_k^{elec} is given approximately in Eq. (12) and $\varepsilon(k,\omega_k)$ is given by Eq. (73). The physics of the shielding of incoherent fluctuations by the eigenmodes is reflected in the formula for $C(k,\omega_k)$. As previously discussed, the factor $(Rq)^4D/D_\parallel^2$ enters from the decay of two-point correlation by the viscous diffusion. The factor $(\frac{\Delta\eta^2}{4} + \frac{1}{\Delta_k^2})\Delta\eta/\Delta_k$ arises from the shielding of granulations of scale $\Delta\eta$ by eigenmode structures of scale Δ_k^{-1}. When the clump scale exceeds the mode width the shielding is only partial and the excitation of the modes is weaker than in cases where the mode width is comparable to or greater than the clump scale. The factor $(Rq)^2/[\Delta_k^2 D_\parallel]$ reflects coupling of the turbulence with parallel diffusion.

To determine the magnitude of the incoherent emission in this regime we evaluate the linewidth from Eq. (76). We use the parameter values obtained in measurements of the Pretext edge plasma.[3] These measurements indicate that L_n lies within a range, $0.5<L_n<3$, which because of the L_n^{-4} dependence of $C(k,\omega_k)$, corresponds to a large variation in possible values of $C(k,\omega_k)$. At an intermediate value of $L_n=2$ cm, and with $(k_\vartheta\rho_s)^2=0.1$, $\hat{s}=1$, $\nu_{ei}=10^6\sec^{-1}$, $\omega_{*e}=0.4\times10^6\sec^{-1}$, and $\Delta\eta\approx\Delta_k^{-1}=\pi/2$ we obtain $C(k,\omega_k)=1.38R_e$. The linewidth is an increasing function of R_e and becomes substantial $(\Delta\omega_k/\omega_{*e}\gtrsim1.0)$ for Reynold's numbers above 0.5. For lower Reynold's numbers the parallel viscosity is increasingly dominant in the decay of small scale correlation and the linewidth quickly decreases. For example, with $R_e=0.2$, $\Delta\omega_k/\omega_{*e}=0.15$. The dependence of the linewidth with Reynold's number is illustrated in Fig. 2. Because the low Reynold's number expansion and weak turbulence approximations break down in the region where $\Delta\omega_k/\omega_{*e}>1$, the linewidth formula is not reliable in regimes of very large linewidth. It does, however, indicate the range of Reynold's numbers for which the linewidth is broad, i.e., $R_e>0.5$, and it exhibits the dependence on mode structure, collisionality, and other parameters. Because of the L_n^{-4} dependence, broad linewidths below $R_e=0.5$ are possible for density scale lengths less than 2 cm. Finally, note that appreciable frequency broadening is possible for $R_e<1$.

We now consider the spectrum balance in the limit where the most probable wave number k_o exceeds the correlation length scale α^{-1}. In this case there is granulation at the small scale and at multiples of k_o^{-1}. Returning to Eq. (68) we expand $k_r(\omega')$ about $\omega_{k\pm k_o}$: