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Abstract

The stability of the electron drift wave is investigated in the presence of $\underset{\sim}{E}\times\underset{\sim}{B}$ plasma rotation typical of the central cell plasma in tandem mirrors. It is shown that a rotationally-driven drift wave may occur at low azimuthal mode numbers. Conditions for rotational instabilities are derived. Quasilinear formulas are given for the anomalous transport associated with the unstable fluctuations.

I. INTRODUCTION

In this work the effect of plasma rotation on the stability of the electron drift wave is considered. The problem is natural to the tandem mirror system which uses the ambipolar potential of two high temperature mirror plasmas to provide an electrostatic trap for the confinement of ions. In the tandem mirror system, the radial electric field produces an azimuthal plasma drift that is greater in magnitude than the electron diamagnetic drift velocity and is in the direction of the ion diamagnetic drift velocity. In the laboratory both the electron and ion drift modes, analyzed by Horton[1,2] in the tandem mirror, appear as low frequency oscillations rotating in the ion diamagnetic direction. In addition to the well-known Doppler shift of the drift wave fluctuations, however, the plasma rotation can destabilize the drift waves. The effect of plasma rotation may also be relevant to the interpretation of drift waves in tokamaks.

Recently, Hooper[3] et al. reported on the characteristics of low frequency oscillations of the drift wave type in the tandem mirror experiment. The dominant oscillations have low m numbers and rotate with the $\underset{\sim}{E}\times\underset{\sim}{B}$ angular velocity which is faster near the axis than at the periphery of the plasma. The experiments are performed in a low beta, collisionless plasma.

The plasma rotation produces two physically different effects on the drift waves, each of which can influence its low frequency stability.[4] Due to rotation in the cylindrical geometry and the finite ion inertia, the ion component of the plasma experiences a centrifugal acceleration. The centrifugal acceleration adds to the acceleration from magnetic curvature and is proportional to the square of the rotational speed and the length of the central cell plasma. The growth rate of the drift wave is increased by a term proportional to the square of the azimuthal wave number m and the angular rotational velocity $\Omega = v_E/r$.

In addition to the centrifugal force effect, the drift wave is influenced by shear in the angular rotation of the plasma. For weak shear we find that the effect of sheared rotation on the drift waves is sub-dominant to the usual finite ion-inertial destabilization except at the lowest azimuthal wavenumbers and for substantial gradients in rotational velocity.

In Section III we give a simple nonlinear model for the development of the rotational drift wave instabilities. We give the constants of the motion for the model but do not attempt to solve the evolution equations.

We note that rotational instabilities are considered extensively in the plasma literature for the magnetohydrodynamic equations.[4-7] The effects of finite Larmor radius terms[4-7] are also well-known for the hydrodynamic rotational instabilities. In these studies the axial wavelength is either infinite or long compared to the electron transit length in one wave period. In contrast, the present study considers the regime where the electron transit length in one wave period is long compared with the parallel wavelength.

The drift wave regime of the rotational modes can be unstable where the magnetohydrodynamic rotational modes of Refs. 5-7 are stable. When only the drift waves are unstable, the plasma remains macroscopically confined but experiences a faster than classical transport due to the finite amplitude of the nondestructive drift wave fluctuations. Physically, the drift wave instabilities are characterized by the electron distribution remaining approximately in the Boltzmann distribution $\delta n_e/n_e \sim e\delta\phi/T_e$ on the space and time scales of the fluctuations in contrast to the MHD behavior where $\delta n_e/n_e \ll e\delta\phi/T_e$.

In Section II we derive the drift wave equations using a fluid description for the cross-field ion dynamics and kinetic theory for the parallel dynamics. In Section III we develop nonlinear models for the fastest-growing rotational instabilities and give the constants of the motion. In Section IV we derive the quasilinear equations for the anomalous transport produced by the fluctuations. In Section V we discuss the conclusions.

II. DRIFT WAVES WITH $\underset{\sim}{E}\times\underset{\sim}{B}$ ROTATION

In this section the radial drift wave eigenmode equation in the presence of an arbitrary equilibrium electrostatic potential $\Phi(r)$ is derived. The ion motion is described by the fluid velocity $\underset{\sim}{v}$ determined through the perpendicular component of the momentum balance equation. A fictitious radial gravity $g\hat{e}_r$ is included to represent the ion acceleration due to the curvature $\underset{\sim}{\kappa} = \hat{b}\cdot\nabla\hat{b}$ of the magnetic field lines

$\hat{b} = \underset{\sim}{B}(\underset{\sim}{x})/B$ of the mirror system within the cylindrical model. The analysis with the field line curvature $\underset{\sim}{\kappa}(\underset{\sim}{x})$ in the drift mode equations is given in Ref. 2. By including the fictitious gravity $\underset{\sim}{g}$ we may synthesize the results obtained here with those given in Ref. 2 to obtain a generalized eigenvalue problem governing the stability of drift modes in the tandem mirror system.

The reduced ion momentum balance equation in the cylindrical plasma with a constant axial magnetic field $B\hat{e}_z$ is

$$m_i n_i \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} \right) = e_i n_i \left(E_r + \frac{v_\theta B}{c} \right) - \frac{\partial p_i}{\partial r} + m_i n_i g \tag{1}$$

$$m_i n_i \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} \right) = e_i n_i \left(E_\theta - \frac{v_r B}{c} \right) - \frac{\partial p_i}{r\,\partial \theta} . \tag{2}$$

In the equilibrium ∂_t , ∂_θ and v_r vanish and the radial momentum balance equation yields the rotational velocity

$$v_\theta = -\frac{cE_r}{B} - \frac{m_i c}{e_i B} \left(g + \frac{v_\theta^2}{r} \right) \tag{3}$$

where we define $\Omega(r) = v_\theta/r \ll \omega_{ci} = e_i B/m_i c$. For regimes where the rotational effects are strong we have $e\Phi \sim T_e$ so that in the lowest order the rotational velocity v_θ is dominated by the electric drift with $v_\theta \simeq v_E = -cE_r/B = cd\Phi/Bdr$. The effect of the ion temperature T_i through the ion pressure gradient and the finite Larmor radius terms, omitted in Eqs. (1) and (2) for brevity, is taken from Refs. 4-7.

The first order fluctuating part of the ion momentum balance equation is

$$m_i n_i \left(\frac{d\underset{\sim}{v}}{dt} \right)^{(1)} + m_i n_i^{(1)} \left(\frac{d\underset{\sim}{v}}{dt} \right)^{(0)} =$$

$$e_i (n_i \underset{\sim}{E})^{(1)} + \frac{e_i}{c} \underset{\sim}{\Gamma}_i^{(1)} \times \underset{\sim}{B} + m_i n_i^{(1)} g \hat{e}_r \qquad (4)$$

where we define

$$\underset{\sim}{\Gamma}_i = n_i \underset{\sim}{v} \quad \text{and} \quad \left(\frac{d\underset{\sim}{v}}{dt} \right)^{(0)} = -\frac{v_\theta^2}{r} \hat{e}_r \ . \qquad (5)$$

The vector cross product of Eq. (4) with $B\hat{e}_z$ yields the cross-field ion current as

$$\underset{\sim}{\Gamma}_i^{(1)} = -\frac{c m_i n_i^{(1)}}{e_i B} \left(g + \frac{v_\theta^2}{r} \right) \hat{e}_\theta + \frac{c\hat{e}_z}{B} \times \left[-(n_i \underset{\sim}{E})^{(1)} + \frac{m_i N}{e_i} \left(\frac{d\underset{\sim}{v}}{dt} \right)^{(1)} \right] \qquad (6)$$

where $N(r)$ is the equilibrium density. The ion density fluctuation evolves according to $\partial_t n_i = -\nabla \cdot \underset{\sim}{\Gamma}_i$. In this work we restrict consideration to the evolution through electrostatic fields with $\underset{\sim}{E} = -\nabla\phi(\underset{\sim}{x},t)$.

In the cylindrical plasma model we decompose the fluctuations into the azimuthal mode m and axial mode k by writing

$$\phi(\underset{\sim}{x},t) = \phi_{mk}(r)\exp(im\theta + ikz - i\omega t) + c.c. \tag{7}$$

$$n_i(\underset{\sim}{x},t) = n_{mk}(r)\exp(im\theta + ikz - i\omega t) + c.c. \tag{8}$$

with analytic continuation of the response functions from $\mathrm{Im}(\omega)>0$. Now we compute separately the various contributions to the ion density fluctuations given by $\nabla\cdot\underset{\sim}{\Gamma}_i^{(1)}$ from Eq. (6).

A. Ion Density Fluctuation

The density fluctuation from the equilibrium acceleration is

$$i\omega n^{(g)} = -\nabla\cdot\left[\frac{cm_i n_i^{(1)}}{e_i B}\left(g + \frac{v_\theta^2}{r}\right)\hat{e}_\theta\right] = -\frac{imcm_i}{re_i B}\left(g + \frac{v_\theta^2}{r}\right)n_{mk}(r) \quad . \tag{9}$$

Field line curvature is analyzed in Ref. 2 and is modeled here through gravity with $g = c_s^2/R_c$ where R_c is the radius of curvature and is positive for the interchange instability and where $c_s = (T_e/m_i)^{1/2}$.

For the long-thin axisymmetric system the effective gravity is $g = (c_s^2 r/L^2)\hat{g}$ with $\hat{g} = (3B_0^2L/2)\int dz[B'(z)/B^2(z)]^2$ where $B(z)$ is the axial profile of the magnetic field. In general $\hat{g}$ is a positive or negative parameter of order unity as shown in the Appendix.

The density fluctuation from the $\underset{\sim}{E}\times\underset{\sim}{B}$ convection in the radial gradients is

$$i\omega n^{(E)} = \nabla\cdot(n_i \underset{\sim}{v}_E)$$

$$= -i \frac{mc}{rB}\left(\frac{dN}{dr}\phi_{mk} - n_{mk}\frac{d\Phi}{dr}\right) . \qquad (10)$$

The first order ion acceleration produces an ion density fluctuation proportional to the ion mass and given by

$$i\omega n_i^{(a)} = \nabla\cdot\left[\frac{cm_i N}{e_i B}\hat{e}_z \times \left(\frac{d\underset{\sim}{v}}{dt}\right)^{(1)}\right]$$

$$= \frac{cm_i}{e_i B}\nabla\cdot N\left[-i(\omega - m\Omega)\hat{e}_z \times \underset{\sim}{v}^{(1)} - \frac{2v_\theta v_\theta^{(1)}}{r}\hat{e}_\theta - \frac{v_r^{(1)}}{r}\frac{d(rv_\theta)}{dr}\hat{e}_r\right] . \qquad (11)$$

In the following analysis it is convenient to define the Doppler shift frequency $\tilde{\omega}$ by

$$\tilde{\omega} = \tilde{\omega}(\omega,m,r) = \omega - m\Omega(r) . \qquad (12)$$

Now, we reduce further the ion inertial contribution in Eq. (11) by approximating $\underset{\sim}{v}^{(1)}$ by $\underset{\sim}{v}_E = c\underset{\sim}{E}\times\underset{\sim}{B}/B^2 = c\hat{e}_z\times\nabla\phi/B$ which is the dominant velocity of the plasma.

With this approximation the contribution in Eq. (11) reduces to

$$i\omega n_{mk}^{(a)}(r) = \frac{im_i c^2}{e_i B^2} \left\{\frac{1}{r}\frac{d}{dr} rN \left[\tilde{\omega}\frac{d\phi_{mk}}{dr} + \frac{m}{r^2}\frac{d(rv_\theta)}{dr}\phi_{mk}\right]\right.$$

$$\left. - \frac{m^2 N\tilde{\omega}}{r^2}\phi_{mk} - \frac{2mNv_\theta}{r^2}\frac{d\phi_{mk}}{dr}\right\}$$

which upon introducing $\Omega = v_\theta/r$ reduces to

$$i\omega n_{mk}^{(a)}(r) = \frac{im_i c^2}{e_i B^2} \left\{ \tilde{\omega}\nabla_\perp\cdot(N\nabla_\perp\phi_{mk}) + \frac{mN}{r^3}\frac{d}{dr}\left(r^3\frac{d\Omega}{dr}\right)\phi_{mk}\right.$$

$$\left. + \frac{m}{r^2}\left(\frac{dN}{dr}\right)\frac{d(r^2\Omega)}{dr}\phi_{mk}\right\} \qquad (13)$$

where $\nabla_\perp = \hat{e}_r\partial_r + (im/r)\hat{e}_\theta$. Equation (13) shows that the familiar result for radial drift wave operator $\nabla_\perp\cdot(N\nabla_\perp\phi)$ is modified by two terms proportional to Ω.

In the presence of nonuniform rotation a new frequency

$$\Omega^{(2)} = \frac{1}{r}\frac{d}{dr}\left(r^3\frac{d\Omega}{dr}\right) \qquad (14)$$

occurs in Eq. (13). The frequency $\Omega^{(2)}$ measures the degree to which there is a nonvanishing viscous stress in the plasma rotation as shown by calculating $\mu\nabla^2\underset{\sim}{v}$ where

$$\nabla^2(v_\theta \hat{e}_\theta) = \left[\frac{1}{r}\frac{d}{dr}\left(r \frac{dv_\theta}{dr}\right) - \frac{v_\theta}{r^2}\right] \hat{e}_\theta$$

$$= \frac{1}{r^2}\frac{d}{dr}\left(r^3 \frac{d\Omega}{dr}\right) \hat{e}_\theta = \frac{\Omega^{(2)}}{r} \hat{e}_\theta \quad .$$

For an equilibrium with no viscous force, $\Omega^{(2)} = 0$ and the ambipolar potential must satisfy

$$\Omega(r) = \Omega_0 + \Omega_1 \left(\frac{a}{r}\right)^2 \quad , \tag{15}$$

where Ω_0 and Ω_1 are arbitrary constants.

The second new term in Eq. (13) describes the interchange of plasma density $N(r)$ and angular momentum $r^2\Omega(r)$ produced by the $\underset{\sim}{E}\times\underset{\sim}{B}$ convection.

Collecting the contributions from Eqs. (9), (10) and (13) and scaling the potential fluctuation to T_e/e , we find that the total ion density fluctuation is given by

$$n_{mk}^{(i)}(r) = (\omega - m\Omega - \omega_{Di})^{-1} \left[N\omega_{ne}\phi_{mk} + \tilde{\omega}\rho^2\nabla_\perp(N\nabla_\perp\phi_{mk}) + mN\left(\frac{\rho}{r}\right)^2\Omega^{(2)}\phi_{mk}\right.$$
$$\left. - \frac{N\omega_{ne}}{r\omega_{ci}}\frac{d}{dr}(r^2\Omega)\phi_{mk}\right] \tag{16}$$

where $\rho = c(m_i T_e)^{1/2}/eB$. In Eq. (16) we define electron drift wave frequency ω_{ne} by

$$\omega_{ne} = -\frac{m}{r}\frac{cT_e}{eB}\frac{1}{N}\frac{dN}{dr} = \frac{m}{r}\frac{cT_e}{eBr_n} = \frac{k_\theta \rho c_s}{r_n} ,$$

where the last formula follows from $k_\theta = m/r$ and $\rho c_s = cT_e/eB$ with $c_s = (T_e/m_i)^{1/2}$. For later use we also define

$$\omega_{ni} = \frac{m}{r}\frac{cT_i}{eBN}\frac{dN}{dr} = -\frac{T_i}{T_e}\omega_{ne} .$$

and the ion guiding-center drift frequency ω_{Di} by

$$\omega_{Di} = -\frac{m}{r}\left(\frac{cm_i}{e_i B}\right)\left(g + \frac{v_\theta^2}{r}\right) = -k_\theta \rho \frac{c_s}{r}\left[\frac{r}{R_c} + \left(\frac{v_\theta}{c_s}\right)^2\right] \qquad (17)$$

where $g = c_s^2/R_c$. For the tandem mirror system we recall that $r/R_c \simeq (r/L_t)^2$ where L_t is the axial length of the transitional region and that $v_\theta/c_s \simeq \rho/r_E$ where r_E is the radial gradient scale of $\Phi(r)$ defined by $r_E = -\Phi(r)/d\Phi/dr$. The centrifugal and magnetic curvature accelerations are comparable when $\rho L_t \sim r^2$.

B. Hydrodynamic Oscillations

There are two low-frequency modes for the plasma depending on the axial eigenfunction. One is the hydrodynamic regime where the electron density fluctuations is also given by $\tilde{E}\times\tilde{B}$ convection with

$$n_e^{(1)}/N \simeq (\omega_{ne}/\tilde{\omega})\,\phi_{mk}(r) \tag{18}$$

with $\tilde{\omega}$ given in Eq. (12) and requires a flute-like axial mode. In the hydrodynamic regime, the $\underset{\sim}{E}\times\underset{\sim}{B}$ drifts of the ions and electrons produce cancelling contributions to the quasi-neutrality equation and the ion inertial contribution in Eq. (13) balances the charge separation due to the radial equilibrium acceleration in Eq. (9) to give the well-known mode equation

$$\nabla_\perp\cdot\left(N\nabla_\perp\phi_{mk}\right) + \left[\ \frac{mc_1}{r\tilde{\omega}}\left(\frac{d}{dr}\left(\frac{N}{r}\frac{d(r^2\Omega)}{dr}\right)\right) + \frac{m^2c_2}{r^2\tilde{\omega}^2}\frac{dN}{dr}\left(g + \frac{v_\theta^2}{r}\right)\ \right]\phi_{mk} = 0 \tag{19}$$

for the perturbed potential. In Eq. (19) we introduce the constants c_1 and c_2 , whose values are unity in the hydrodynamic regime, to facilitate understanding the effect of these terms.

For uniform rotation, a Gaussian density profile $N(r) = N_o\exp(-r^2/a^2)$ and $g(r) = g_0(r/a)$, the radial eigenvalue problem may be solved exactly to obtain the dispersion relation

$$c_1\,\frac{2m\Omega}{\tilde{\omega}} + c_2\,\frac{m^2(\Omega^2 + g_0/a)}{\tilde{\omega}^2} + \nu = 0 \tag{20}$$

where ν is the eigenvalue of the reduced radial mode equation. For the unbounded plasma $\nu = m+2n$ with $n = 0,1,2,...$ being the radial mode number.

For the plasma bounded by a conducting wall at r=b the eigenvalue shifts up with $\nu = m+2n+f(b/a)$ with $f(x)>0$. For $b/a \geqslant 3$ the shift due to the wall is weak and unimportant except for the m=1 mode where the conducting wall removes the $\omega=0$ degeneracy of the translational invariance of the unbounded system as previously pointed out by Chen[8] and Rognlein.[9] For n = 0 and $g_0 = 0$, the frequencies and growth rate are

$$\omega = m\left(1 - \frac{c_1}{\nu}\right) \frac{|v_E|}{r}$$

and

$$\gamma = \frac{m}{\nu}\left(c_2 \nu - c_1^2\right)^{1/2} \left(\frac{|v_E|}{r}\right) \qquad (21)$$

For $g_0 \neq 0$, the complex frequency is

$$\omega = m\left(1-\frac{c_1}{\nu}\right)\Omega \pm \frac{m}{\nu}\left[\left(c_1^2 - c_2 \nu\right)\Omega^2 - c_2 \nu g_0/a\right]^{1/2} \qquad (22)$$

where $\Omega = |v_E|/r$. For $c_1 = c_2 = 1$ the equation is that given by Mikhailovskii.[4] The finite Larmor radius modifications of Eq. (19) are also given by Mikhailovskii and in Refs. 5-7. We include the FLR terms in the final dispersion relation in Eq. (25).

C. Drift Wave Oscillations

In the drift wave regime the electron density fluctuations are given by

$$\frac{n^e_{mk}(r)}{N(r)} \doteq \left[1 - \left(1 - \frac{\omega_{ne}}{\tilde{\omega}}\right)(\lambda - i\delta_e)\right] \phi_{mk}(r) \qquad (23)$$

when λ is the axial eigenvalue of the bounce averaging operator $\hat{L}$ defined in Refs. 1 and 2. In particular for odd axial modes $\lambda = 0$, for axial modes with several nodes $\lambda \ll 1$, and for the flutelike mode $\lambda = 1$ which recovers Eq. (18). The eigenvalue λ is an effective axial mode number for the bounce averaged plasma wave.

For $\lambda \ll 1$ the electrons establish a local Maxwell-Boltzmann distribution in the fluctuating electrostatic potential. The electron-wave resonance given by $i\delta_e$ in Eq. (23) is analyzed in Ref. 1. A simple approximate formula for the electron Landau resonance is $\delta_e \cong (\pi/2)^{1/2}(\omega_{ne}L_c/v_e) \cong (\pi/2)^{1/2}(k_\theta\rho)(L_c/r_n)(m_e/m_i)^{1/2}$ which applies in the plateau collisionality regime of the electron response. Other δ_e formulas for the trapped electron and the collisional response are given in Ref. 1.

The condition of quasi-neutrality $n_e = n_i$ and the formulas in Eqs. (16) and (23) for the fluctuating densities yield the radial eigenvalue equation

$$\tilde{\omega}\rho^2 \left[\frac{1}{r}\frac{d}{dr}\left(rN\frac{d\phi_{mk}}{dr}\right) - \frac{m^2N}{r^2}\phi_{mk}\right] + \left[\omega_{ne}\left(1 - \frac{c_1}{r\omega_{ci}}\frac{d}{dr}(r^2\Omega)\right)\right.$$

$$+ c_1\frac{m\rho^2\Omega^{(2)}}{r^2} - \tilde{\omega} + \omega_{Di} + \left(1 - \frac{\omega_{Di}}{\tilde{\omega}}\right)$$

$$\left.\times\left(\tilde{\omega} - \omega_{ne}\right)(\lambda - i\delta_e)\right] N\phi_{mk} = 0 \quad . \qquad (24)$$

with $\omega_{Di} = -c_2 m(\dot{\Omega}^2+g_0/a)/\omega_{ci}$. From Eq. (24) we note that the radial mode equation is singular at the resonant layer where $\omega = m\Omega(r_m)$ in the presence of nonuniform rotation.

For the radial profiles used in Eq. (20) the eigenvalues of Eq. (24) are determined by

$$A\tilde{\omega}^2 + B\tilde{\omega} + C = 0 \tag{25}$$

with

$$A = 1-\lambda+i\,\delta_m+2\alpha^2\nu_m$$

$$B = 2\,\alpha^2[2m\Omega-\omega_{ni}(\nu_m-1)] - (\omega_{ne}+\omega_{Di})(1-\lambda+i\,\delta_m)$$

$$C = -\,\omega_{Di}\omega_{ne}(\lambda-i\,\delta_m)$$

where $\alpha=\rho_s/a$ and $\delta_m=\delta_0 m$. In Eq. (25) the finite Larmor radius terms are included through $\omega_{ni} = -T_i\omega_{ne}/T_e$. For $\lambda=1$ and $\delta_0=0$ the dispersion relation (25) reduces to Eq. (20). In Figs. 1 and 2 we show the frequencies $\omega_\pm(m)$ and growth rates $\gamma_\pm(m)$ obtained from Eq. (25) for various parameters similar to those of Hooper et al.[3] For reference parameters we choose values representative of the tandem mirror experiments with $\alpha=\rho/a=1/20$, $L/a=20$, $T_i=T_e$ and $b/a=3$.

In general the drift wave growth rate depends in complex manner on the parameters Ω and g. In Figs. 3 and 4 we show the variations of the m=1 and m=10 growth rates with the dimensionless $\hat{\Omega}$ and $\hat{g}$ defined by $\Omega = (c/reB)(d\Phi/dr) = (cT_e/eBa^2)\hat{\Omega}$ and $g_0 = (c_s^2 a/L^2)\hat{g}$. The dimensionless unit of frequency is $cT_e/eBa^2 = \rho c_s/a^2$. In terms of these dimensionless units $\omega_{ne} = 2m$, $\omega_{Di} = mv_{Di} = -m(\hat{g}/A^2+\alpha^2\hat{\Omega}^2)$, the MHD growth rate is $\gamma_{mhd} = m^{1/2}v_{Di}^{1/2}/\alpha = m^{1/2}\hat{g}^{1/2}/\alpha A$ and the ion-acoustic frequency is $\omega_s = 1/\alpha A$. The behavior of the electron drift wave growth rate is given approximately by

$$\gamma(m) \cong \delta_o m^2\{\alpha^2\hat{\Omega}^2 + 4\alpha^2[m+\hat{\Omega}+(m-1)\frac{T_i}{T_e}] + \frac{\hat{g}}{A^2}\} .$$

Variation of the drift wave growth rate with $\hat{\Omega}$ and $\hat{g}$ are shown in Figs. 3 and 4 as given by the unstable root of the quadratic Eq. (25) with $\lambda = n = 0$. The variations agree adequately with the simple approximation given here. Fig. 5 gives the growth rate for the m=1 and m=2 flute modes (λ=1) showing the stable rotational window[5] from FLR effects and the magnetic well.

The radial mode equation (24) along with boundary conditions that $\phi_{mk}(r) \simeq r^m$ for $r \to 0$ and $\phi_{mk}(b) = 0$ where $r = b$ is the radius of the edge plasma ($\omega_{pi} \lesssim \omega_{ci}$) determines the radial eigenvalue problem. There are two types of radial plasma modes given by Eq. (24). One mode is the electron drift wave for which $\omega_k \simeq m\bar{\Omega} + (\omega_{ne} + \omega_{Di})/(1 + k_\perp^2\rho^2)$ and the resonant layer $\omega = m\Omega$ either does not occur or occurs with little effect outside the maximum of the wave function $\phi_k(r)$. The second mode is a Kelvin-Helmholtz[9-12] type of plasma instability produced by the differential rotation $\Omega(r)$. The eigenfunctions for the two types of modes are rather different as we show in the following analysis.

The radial analysis shows that the drift wave eigenfunction $\phi_{mk}(r)$ is peaked near the maximum of the local drift wave frequency $\omega(r) = m\Omega(r) + (\omega_{ne} + \omega_{Di})/(1 + k_\theta^2\rho^2)$ with the fastest-growing radial mode being the lowest order (n = 0) wave function. The Kelvin-Helmholtz mode is driven by the radial gradient of the vorticity $\zeta = 1/r\ \partial(rv_\theta)/\partial r$ and is peaked about the maximum of $\zeta(r)$. The radial wave function has a node near the resonant layer in the region of the strongest sheared rotation. The maximum growth rate is of order

$$\gamma_{max} \simeq \frac{1}{2} r \left|\frac{d\Omega(r)}{dr}\right| ,$$

and occurs for $k_\theta r_\Omega \sim 1$ where $r_\Omega = \Omega dr/d\Omega$.

To analytically determine the characteristics of the mode driven by the sheared rotation it is convenient to transform Eq. (24) for the potential fluctuation $\phi_{mk}(r)$ to an equation for the Lagrangian fluid displacement $\xi(r)$ defined by $d\xi/dt = v_{Er} = cE_\theta/B$ where $d/dt = \partial_t + \hat{e}_z \times \nabla\phi \cdot \nabla$. In the linear approximation

$$\xi(r) = \frac{mc\,\phi_{mk}(r)}{rB[\omega - m\Omega(r)]} \tag{26}$$

valid for

$$\left|\frac{mc}{rB}\frac{\partial\phi_k}{\partial r}\right| < |\omega - m\Omega(r)| . \tag{27}$$

At finite amplitude oscillations the $\underset{\sim}{E}\times\underset{\sim}{B}$ motion of the plasma is strongly nonlinear in the trapping layer of width Δr_t about the resonant layer r_m where

$$\Delta r_t = \left(\frac{c\,\phi_k}{r\Omega' B}\right)^{1/2} .$$

as follows from Eq. (27) by expanding about r_m where $\Omega(r_m)=\omega/m$.

The $\underset{\sim}{E}\times\underset{\sim}{B}$ motion of the plasma is shown in Fig. 6. Outside the resonant layer the radial eigenvalue equation for $\xi(r)$ obtained from Eq. (24) is

$$\frac{\rho^2}{r^3}\frac{d}{dr}\left(r^3 N\,\tilde{\omega}^2\frac{d\xi}{dr}\right) + \Big\{ \frac{\rho^2(1-m^2)\tilde{\omega}^2}{r^2} + \frac{\rho^2}{r}\frac{1}{N}\frac{dN}{dr}\,\tilde{\omega}\,(\tilde{\omega}+2c_1 m\Omega) + m\,\frac{\rho^2}{r}\,\tilde{\omega}\,\Omega^{(2)}\left(c_1-1\right)$$

$$+ \tilde{\omega}\left(\omega_{ne}+\omega_{Di}-\tilde{\omega}\right)(1-\lambda) + \omega_{Di}\,\omega_{ne}\,\lambda - i(\tilde{\omega}-\omega_{ne})(\tilde{\omega}-\omega_{Di})\,\delta_e\Big\}\, N\xi = 0 . \qquad (28)$$

In the limit of a strong shearing layer $\Omega(r)$ the displacement field $\xi(r)$ remains finite and continuous across the resonant layer. Integrating Eq. (27) across the resonant layer leads to the condition

$$r^3 N(r)\left[\omega - m\Omega(r)\right]^2 \frac{d\xi}{dr}\Bigg|_{r_m - \Delta r_t}^{r_m + \Delta r_t} = 0 . \qquad (29)$$

expressing the continuity of the radial current across the resonant layer. Using the approximate outer solutions for $\xi(r)$ to evaluate the change in $d\xi/dr$, we obtain the approximate dispersion relation

$$(m+1)\,N_+\left(\omega - m\Omega_+\right)^2 + (m-1)N_-\left(\omega - m\Omega_-\right)^2 = 0 \qquad (30)$$

valid for strong gradients with $k_\theta r_\Omega = mr_\Omega/r_m \ll 1$. For $N_+ = N_-$ the two complex roots are

$$\omega = \frac{m}{2}\left(\Omega_+ + \Omega_-\right) + \frac{1}{2}\left(\Omega_+ - \Omega_-\right) \pm i\frac{(m^2-1)^{1/2}}{2}\,|\Omega_+ - \Omega_-| \tag{31}$$

where $\Omega_\pm$ are the angular rotational speeds at $r_m \pm \Delta r_t$, respectively. The potential fluctuation $\phi_k(r)$ for this mode has a node in the resonant layer where $\Omega(r_m) = \omega/m = 1/2(\Omega_+ + \Omega_-)$ and opposite phases at its two amplitude maxima at $r_m \pm \Delta r_t$. The growth rate increases with m until $k_\theta \sim r_\Omega^{-1}$ where $\gamma \simeq r_m|\Omega_+ - \Omega_-|/2r_\Omega \sim r_m|d\Omega/dr|$. For larger k_θ the oscillations are stable.

In the limit of a narrow layer with a linear gradient in $\Omega(r)$ it is straight forward to derive the linear growth rate γ_k and the eigenfunction $\phi_k(x=r-r_m)$ taking the density constant across the layer. For $kx_o = 0.64$ where $2x_o$ is the width of the sheared flow and for $e\delta\phi_k/T_e \sim 0.1$ $[e\Phi(r_o)/T_e]$ or $\Delta r_t \sim x_o/3$, the $\underset{\sim}{E}\times\underset{\sim}{B}$ flow of the plasma is shown in Fig. 6. In the region of trapping the linear approximation fails and the nonlinear vortex flow mixes the velocity gradient to zero giving rise to an anomalous viscosity.

In the general case of nonuniform rotation both the centrifugal force drive in Eq. (20) and the sheared drive rotation in Eq. (31) contribute to instability and the eigenvalues must be found numerically. Ronglien[9] investigated this problem for a linear gradient connecting Ω_- to Ω_+ and shows the conbined effects on the wave functions and eigenvalues for a particular choice of parameters.

III. NONLINEAR EQUATIONS

In order to investigate the nonlinear stage of the rotaional instabilites we derive the dynamical equations for $n(\underset{\sim}{x},t)$ and $\phi(\underset{\sim}{x},t)$ in the two dimensional, quasineutral limit. We find it useful to study the equations both in the laboratory inertial reference frame and in the reference frame rotating with the plasma. In this section we generalize the external gravitational field by writing

$$\underset{\sim}{g} = -\nabla U(r,\theta,t)$$

and consider that we may use a localized $\underset{\sim}{g}(\underset{\sim}{x},t)$ as a probing force on the plasma.

A. Dynamical Equations in the Laboratory and Rotating Frames

Computing the nonlinear $\underset{\sim}{E}\times\underset{\sim}{B}$ particle flux and the nonlinear charge flux $\underset{\sim}{J}$ as in Sec. II, the quasineutral field equations are

$$\frac{\partial n}{\partial t} + \underset{\sim}{v}\cdot\nabla n = \frac{\partial n}{\partial t} + \hat{e}_z \times \nabla\phi\cdot\nabla n = 0 \tag{32}$$

and

$$\nabla \cdot \underset{\sim}{J} = 0 \tag{33}$$

where in appropriate units

$$\underset{\sim}{J} = - n\left(\frac{\partial}{\partial t} + \hat{e}_z \times \nabla\phi \cdot \nabla\right)\nabla\phi + \hat{z}\times\nabla U.$$

Reducing the system of Eqs. (32) and (33) yields the field equations in the laboratory frame

$$\frac{\partial n}{\partial t} - [n,\phi] = 0 \tag{34}$$

$$\nabla\cdot(n\nabla\partial_t\phi) - \nabla\cdot\left(n[\nabla\phi,\phi]\right) + [n,U] = 0 \tag{35}$$

where

$$[f,g] = \hat{e}_z \cdot \nabla f\times\nabla g = \frac{1}{r}\left(\frac{\partial f}{\partial r}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial r}\right) .$$

In a frame of reference rotating with the angular velocity Ω the velocity field is $\underset{\sim}{v}_{rot} = \underset{\sim}{v}_{lab} - r\Omega\hat{e}_\theta$ and the coordinate transformation is $r_{rot} = r_{lab}$, $\theta_{rot} = \theta_{lab} - \Omega t$. The laboratory potential ϕ_{lab} and the rotating frame potential are related by

$$\phi_{lab} = \frac{1}{2}\Omega r^2 + \phi_{rot}.$$

In the rotating frame the plasma experiences the centrifugal force and the coriolis force.

We carry out the transformation from the laboratory frame to the rotating frame, and subsequently drop the subscript "rot", to find that the equations in the rotating frame are

$$\frac{\partial n}{\partial t} - [n,\phi] = 0 \tag{36}$$

$$\nabla\cdot(n\nabla\partial_t\phi) - \nabla\cdot\big(n[\nabla\phi,\phi]\big) - 2\Omega[n,\phi] + \Big[n,U - \frac{r^2\Omega^2}{2}\Big] = 0\ . \tag{37}$$

The rotating frame equations (36) and (37) clearly show that the effective plasma potential is $U_{eff} = U - r^2\Omega^2/2$ and the coriolis force introduces a convective drift separating charge as in the Rossby waves given by $2\Omega[n,\phi]$. We recall that the nonlinear Rossby wave equation[13,14] in the β-plane approximation is $\nabla^2\partial_t\psi + [\psi,\nabla^2\psi] + \beta\hat{e}_z\times\nabla\psi\cdot\hat{e}_x = 0$ where ψ is the velocity stream function for the two dimensional atmospheric flow.

To explain physically the form of Eq. (37) for the rotating plasma we reconsider the calculation of the charge current $\underset{\sim}{J}$ in the rotating frame of reference. In a rotating frame the particles experience the centrifugal force $m_j\Omega^2r\hat{e}_r$ and the coriolis force $2m_j\underset{\sim}{v}_E\times\Omega\hat{e}_z$ both which produce an m_j/e_j dependent drift of the particle. The charge current density due to these non-inertial frame forces is

$$\underset{\sim}{J}_{rot} = -\sum_j \frac{cm_jn_j\hat{e}_z}{B}\times(\Omega^2r\hat{e}_r + 2\underset{\sim}{v}_E\times\Omega\hat{e}_z).$$

A short calculation of $\nabla\cdot\underset{\sim}{J}_{rot}$ shows that these inertial drifts explain the new terms in Eq. (37). Note that as $\Omega\to 0$ the coriolis force is the dominant correction to the inertial frame of reference.

The obvious advantage of the nonlinear equation in the rotating frame is that the potential ϕ is first order in the amplitude of the fluctuation. Thus, the $\nabla\cdot\left(n[\nabla\phi,\phi]\right)$ is second order in the rotating frame and contributes to mode coupling but not to the linear modes, whereas in the laboratory frame this term contains several linear contributions. It is straight forward to linearize Eqs. (36) and (37) to obtain Eq. (19).

B. Constants of Motion

Here we give the constants of the motion that follow for an <u>isolated</u> system governed by Eqs. (33) and (34). Since the plasma density is convected by an incompressible flow $\underset{\sim}{v} = \hat{e}_z\times\nabla\phi$, all functions $G\left(n(\underset{\sim}{x},t)\right)$ of the density integrated over all r, θ are constants of the motion

$$S = \int d\underset{\sim}{x}\; G\left(n(\underset{\sim}{x},t)\right) = \text{const.} \tag{38}$$

The total energy of the plasma is the sum of the kinetic energy $\frac{1}{2}m_i n v_E^2$ and the potential energy $m_i nU$. Multiplying Eq. (34) by $1/2(\nabla\phi)^2$, subtracting Eq. (35) multiplied by ϕ, and integrating over all r,θ shows that

$$E = \int d\underset{\sim}{x}\; \left[\frac{n}{2}(\nabla\phi)^2 + nU\right] = \text{const.} \tag{39}$$

The angular momentum of each charge species j has two parts; the mechanical angular momentum $n_j m_j r v_\theta$, and the magnetic angular momentum $n_j e_j B r^2/2c$. In the quasi-neutral limit the total magnetic angular momentum $(\sum_j n_j e_j) B^2 r^2/2c$ cancels, and thus the total angular momentum density $L = \sum_j n_j m_j r v_\theta$ or neglecting the electron mass is

$$L = m_i n_i r v_\theta$$

and satisfies

$$\frac{\partial L}{\partial t} + \underset{\sim}{v} \cdot \nabla L = -r J_r B/c \tag{40}$$

where $T_\theta = -r J_r B/c$ is the torque acting on the plasma due to the radial curent J_r. For a model system with no end losses, $J_{end} = 0$, the constraint of $\nabla \cdot \underset{\sim}{J} = 0$ integrated over the plasma surface of radius r leads to $\int_0^{2\pi} r J_r d\theta = 0$ and for the isolated system Eq. (40) yields

$$L = \int d\underset{\sim}{x}\, n\, r v_\theta = \int d\underset{\sim}{x}\, n\, r \frac{\partial \phi}{\partial r} = \text{const.} \tag{41}$$

C. Nonlinear Drift Wave Equation

In the drift wave regime of the quadratic equation (25) the oscillation branch $\omega \simeq \omega_+(k) \sim \omega_{ne}$ approximately decouples from the $\omega_-(k)$ branch. We use this feature and the assumption that for saturation at low amplitudes the linear electron response in Eq. (23) remains valid to derive a single nonlinear partial differential equation for $\phi(r,\theta,t)$. The analysis follows that given by Terry and Horton[15] for dissipative systems and Hasegawa and Mima[16] for dissipationless systems.

The general equations for the drift wave system are

$$\nabla \cdot \underset{\sim}{J}_{\perp} = - \frac{\partial J_{\parallel}}{\partial z} \tag{42}$$

where

$$\underset{\sim}{J}_{\perp} = - n \left(\frac{\partial}{\partial t} + \hat{e}_z \times \nabla\phi \cdot \nabla\right) \nabla\phi \tag{43}$$

and

$$\frac{\partial n}{\partial t} + \hat{e}_z \times \nabla\phi \cdot \nabla n = \frac{\partial J_{\parallel}}{\partial z} \tag{44}$$

in the quasineutral limit. We write the approximate electron susceptibility given in Eq. (23) as

$$n = N(r)\left[1 + (1 + \hat{\mathscr{L}})\phi\right] \tag{45}$$

valid for $k_\parallel v_e > \Delta\omega_k = 1/\tau_c > \Delta k_\parallel (e\phi/m_e)^{1/2}$. Using Eq. (43) we calculate the divergence of the fluctuating parallel current from Eqs. (44) and (45)

$$\frac{\partial J_\parallel}{\partial z} = N(1+\hat{\mathscr{L}})\,\frac{\partial\phi}{\partial t} + N\,v_{de}\,\frac{\partial\phi}{r\partial\theta} + \left[\phi,\ N(1+\hat{\mathscr{L}})\,\phi\right] \tag{46}$$

where $v_{de} = \rho\, c_s/r_n$. Substituting Eq. (46) into Eq. (42), collecting terms, and making a few simplifications we obtain

$$\left[N(1+\hat{\mathscr{L}})-\rho^2\nabla_\perp\cdot N\nabla\right]\frac{\partial\phi}{\partial t} + N\left[v_{de}+v_{Di}(1+\hat{\mathscr{L}})\right]\frac{\partial\phi}{r\partial\theta} + 2\Omega\rho^2[N,\phi]$$
$$+\ \rho c_s\left[\phi,\ N\hat{\mathscr{L}}\phi - \rho^2\nabla\cdot(N\nabla\phi)\right] = 0 \tag{47}$$

where $v_{Di} = -(g+v_\theta^2/r)/\omega_{ci}$. Linearization of Eq. (47) yields the branch $\omega = \omega_+ = -B/A$ of Eq. (25).

The energy balance equation follows from multiplying by ϕ and integrating over space to obtain

$$\frac{d}{dt}\int \frac{N}{2}\left[\phi^2+(\nabla\phi)^2\right]d\underset{\sim}{x} = -\int\phi\ N\hat{\mathscr{L}}^{ah}\left(\frac{\partial\phi}{\partial t} + v_{Di}\,\frac{\partial\phi}{r\partial\theta}\right)d\underset{\sim}{x} \tag{48}$$

where $\hat{\mathscr{L}}^{ah}$ is the anti-hermitian part of the non-adiabatic electron response (45).

The nonlinear drift wave equation (47) reduces to the Terry-Horton model[15,17] in the limit N=const and $\Omega = v_{Di} = 0$. Extensive simulations with this model are reported by Waltz[17] and Terry and Horton.[18] Under appropriate conditions[17,18] the nonlinear equation (47) predicts broad band, low amplitude turbulence.

IV. Quasilinear Equations

In the weakly nonlinear steady state the quasilinear equations determine the effects of the density fluctuations δn and potential fluctuaions $\delta\phi$ on the background plasma $N(r,t) = \langle n\rangle$ and $\Phi(r,t) = \langle\phi\rangle$. We define the average $\langle\ \rangle$ over θ and a time T containing many wave oscillations but short compared with the time for the evolution of the background plasma,

$$\langle\ \rangle = \int \frac{d\theta}{2\pi} \int \frac{dt}{T} \,.$$

The nonlinear fluxes controlling the evolution of $N(r,t) = \langle n(\underset{\sim}{x},t)\rangle$ and $\Phi(r,t) = \langle\phi(\underset{\sim}{x},t)\rangle$ follow from by writing $n = \langle n\rangle + \delta n$ and $\phi = \langle\phi\rangle + \delta\phi$ and taking the average of Eqs. (34) and (35). The quasilinear equations are

$$\frac{\partial N}{\partial t}(r,t) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\Gamma(r,t)\right) = S^{ext}(r,t) \tag{49}$$

$$\frac{\partial}{\partial t}\Big(N(r,t)rV_\theta(r,t)\Big) + \frac{1}{r}\frac{\partial}{\partial r}\Big(r\Pi(r,t)\Big) = rF_\theta^{ext}(r,t)/m_i \tag{50}$$

where $S^{ext}(r,t)$ is the source and sink of plasma density due to processes such as end losses, gas puffing ionization and rF_θ^{ext} is the external torque density acting on the plasma due to processes such as neutral beam injection and end-loss currents J_{end} which give rise to $rF_\theta = -(Br^2/cL_c)J_{end}$ due to the compensating J_r. Eqs. (49) and (50) apply to fluctuations with general values of λ and $i\delta_e$.

The anomalous fluxes Γ and Π in Eqs. (41) and (42) are given by

$$\Gamma = \langle \delta n \delta v_r \rangle = \sum_{m,n} i k_\theta \delta\phi_k^* \delta n_k = - \sum_{m,n} \frac{2k_\theta^2 \gamma_k |\delta\phi_k(r)|^2}{(\omega - m\Omega)^2 + \gamma_k^2} \frac{dN}{dr} \tag{51}$$

since $\langle v_r \rangle \equiv 0$ and where $k_\theta \equiv m/r$. The angular momentum flux Π per unit of mass is given by

$$\Pi = \langle n r v_\theta \delta v_r \rangle = r V_\theta \Gamma + r N \langle \delta v_\theta \delta v_r \rangle \ . \tag{52}$$

From the linear equation (23) we compute that

$$\frac{1}{r} \frac{\partial}{\partial r} r N \langle \delta v_\theta \delta v_r \rangle = - \sum_{m,n} \frac{2k_\theta^2 \gamma_k |\delta\phi_k(r)|^2}{(\omega - m\Omega)^2 + \gamma_k^2} \, r \frac{d}{dr} \left[\frac{N}{r} \frac{d}{dr} (r V_\theta)\right]$$

where k is a short notation for the azimuthal and radial mode numbers m and n.

In the absence of external driving forces S^{ext} and F_θ^{ext} the quasilinear equations (49)-(50) describe the diffusion of the background plasma to a uniform density and vorticity state, i.e. solid body rotation. In the presence of S^{ext} and F_θ^{ext} the state steady solutions in general are non-uniform containing a balance of the anomalous fluxes Γ and Π with the external sources.

We do not attempt to solve the quasilinear equations in this work. We note, however, a simple limit to further illustrate their content. In the limit where S^{ext} is localized near the axis the steady state flux is $\Gamma(r) = \Gamma_0 a/r$ and the outward flow of plasma slows down the angular velocity $\Omega(t)$ of the plasma just as by extending her arms a spinning ice skater slows down. For solid body rotation $\langle \delta v_\theta \delta v_r \rangle = 0$ and $\Pi = r^2 \Omega(t) \Gamma$. Substituting the anomalous angular momentum flux Π into Eq. (50) leads to $d\Omega/dt = -2\Omega\Gamma_0 a/Nr^2$ for the rate of slowing down due to the particle flux Γ.

IV. CONCLUSIONS

The stability of a strongly magnetized plasma with nonuniform density $N(r)$, and potential $\Phi(r)$ to electrostatic perturbations of the drift wave type is analyzed. The analysis is appropriate for the low beta regime of the central cell plasma in the tandem mirror trap where $e\Phi \sim T_e \gtrsim T_i$. In the tandem mirror experiments low frequency electrostatic oscillations of the drift wave type are reported recently by Hooper et al.[3] The stability analysis shows that both dN/dr and $d\Phi/dr$ influence the stability of the oscillations.

For the curvature stable flute modes there is a stable window of rotation centered about the dimensionless rotational frequency $\hat{\Omega} = \Omega a^2/\rho c_s = T_i/T_e = -\omega_{ni}/2m$ in the direction of electron diamagnetic current. In the limit of negligible curvature the stable window spans the region

$$\frac{\omega_{ni}}{2m}(\sqrt{\nu_m} - 1) \leqslant \hat{\Omega} \leqslant -\frac{\omega_{ni}}{2m}(\sqrt{\nu_m} + 1)$$

The narrowest window occurs for the m=1, n=0 mode with $b/a \to \infty$ where $\nu_m \to 1+0^+$, gives the sufficient condition of stability $0 \leqslant \hat{\Omega} \leqslant -\omega_{ni}/m$ given by Freidberg and Pearlstein.[5] For strong favorable curvature the stability window expands both directions; however, the typical tandem mirror rotation velocity is well outside this stable window. For n=0, ν_m=m, the growth rate varies as

$$\gamma = [\hat{\Omega}^2 m - (\hat{\Omega} + (m-1)\frac{T_i}{T_e})^2 + \frac{m\hat{g}}{A^2\alpha^2}]^{1/2} \qquad [\frac{\rho c_s}{a^2}]$$

when $m < m^* = |T_e\hat{\Omega}/T_i - 1|^2$.

The stability of the electron drift modes driven by the electron-wave dissipation $i\delta_o$ is influenced by the charge separation produced by both the coriolis force (Ω) and the centrifugal force (Ω^2). For the drift wave the coriolis force is destabilizing for $\Omega > 0$ and stabilizing for to $\Omega < 0$. The most favorable rotational frequency is $\hat{\Omega}_m = -2$ where there the growth rate in Figs. 3 and 4 show a a minimum. Rotation at $m\Omega = -\omega_{ne}$ stabilizes the m=1 drift wave for $b/a = \infty$; however, the higher m modes remain unstable. For $m \lesssim 1+(1+\hat{\Omega}/2)^2$ the electron drift growth rate increases as $(m\Omega+\omega_{ne})^2$ as observed in Figs. 3 and 4. For $m < 1/\alpha^2$ the dimensionless drift wave growth rate is given by

$$\gamma^{dw} = \delta_o m^2[\alpha^2(\hat{\Omega}+2)^2+4\alpha^2(m-1)(1+T_i/T_e)+\hat{g}/A^2] \left[\frac{\rho c_s}{a^2} \right]$$

for the n = 0 radial mode and b/a = ∞. Comparison of γ and γ^{dw} shows that the FLR effect has an opposite influence on the flute and drift mode. The influence is stabilizing for high m flute modes while destabilizing for drift waves.

Applying the results derived here to the experiment of Hooper et al.[3] We conclude that both the flute modes and the drift waves are predicted to be unstable. From the experimental information available it would appear difficult to rule out the presence of either instability.

The nonlinear equations for the evolution of the density and potential are given in Sec. III. By a transformation to a frame of reference rotating with the plasma the physical origin of the three terms in the potential equation arising from the polarization current, the coriolis force and the centrifugal drift are made clear. The rotating frame is shown to have an advantage for the small amplitude expansion of the nonlinear equations. The effective potential determining the interchange stability is shown to be $U_{eff}(r) = U_{\Delta B}(r)-\Omega^2r^2/2$ where $U_{\Delta B}(r)$ is the effect of the radial magnetic well produced by the quadrupole fields.

For the isolated system the nonlinear equations conserve the number of particles, the energy and the angular momentum of the plasma.

In an open system with a steady state driven by the injection of particles, energy and angular momentum, we derive the quasilinear formulas for the anomalous component of the system's losses. The anomalous particle flux is $\Gamma = \langle \delta n \delta v_r \rangle$ and the angular momentum flux per unit mass is $\Pi = r\langle n v_\theta \delta v_r \rangle = rV_\theta \Gamma + nr\langle \delta v_\theta \delta v_r \rangle$. Quasilinear aproximations for Γ and Π are given in Eqs. (51) and (52) in terms of the slowly varying fluctuation spectrum $I(k,r,t) = \langle |\delta\phi_k(r,t)|^2 \rangle$. These quasilinear formulas show that the fluctuations act through $\underset{\sim}{E}\times\underset{\sim}{B}$ convection to produce an effective anomalous particle diffusion D_A and anomalous viscosity μ_A. The anomalous transport coefficients D_A and μ_A may be estimated from these formulas with knowledge of the potential fluctuation spectrum.

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Appendix

Effective Gravity for Cylindrical Model of Tandem Mirror

In the analysis of the rotational instabilities we use a cylindrical model with an effective radial gravity $g(r) = g_o(r/a)$ in Eq. (1) to represent the average magnitude of the field line curvature. Here we derive the formula for the value of g_o, and the dimensionless $\hat{g}$ used in the dispersion relation relations in terms of the axial magnetic field B(z).

The physical origin of g_o is the charge separation produced by the pressure (or density $\delta p = T\delta n$) perturbation in the presence of the curvature $\underset{\sim}{\kappa} = (\hat{b}\cdot\nabla)\hat{b}$ of the magnetic field. Introducing the centrifugal force by $\underset{\sim}{F} = m_i\Omega^2\underset{\sim}{r}$, the charge separation is given by

$$\nabla\cdot\delta\underset{\sim}{j} = c\underset{\sim}{B} \times \nabla\delta p\cdot\nabla \left(\frac{1}{B^2}\right) + \frac{c\underset{\sim}{F}\times\underset{\sim}{B}}{B^2} \cdot \nabla\delta n$$

$$= - \frac{imcm_i\,\delta n}{B} \left(\frac{2c_s^2}{rr_c} + \Omega^2\right) \qquad \text{(A1)}$$

where we use $\nabla_\perp \ell n B = (\hat{b}\cdot\nabla)\hat{b} = -\hat{n}/r_c$. In the long-thin approximation where $\psi = r^2B(z)/2 = \text{const}$ we compute r_c from $r_c^{-1} = -d^2r/dz^2$ and obtain for the equivalent cylindrical gravity

$$\frac{g_o(z)}{a} = \frac{2c_s^2}{r_c} = c_s^2 \left(\frac{\ddot{B}}{B} - \frac{3}{2}\frac{\dot{B}^2}{B^2}\right) . \qquad \text{(A2)}$$

In terms of $g_o(z)$ Eq. (A1) becomes

$$\nabla \cdot \delta \underset{\sim}{j} = - \frac{imcm_i \delta n}{B} \left(\frac{g_o(z)}{a} + \Omega^2 \right)$$

which is the generalization of Eq. (9).

The average radial mode equation for the systems follows from the local equation $\nabla \cdot \delta \underset{\sim}{j} = i\omega\delta\rho_Q = 0$ by taking the $\int dz/B(z)$ average. Performing the average we derive Eq. (19) with the effective value of g_o for constant Ω given by

$$g_o/a + \Omega^2 = \left(\int \frac{dz}{B^2}\right)^{-1} \int \frac{dz}{B^2} \left(\frac{g_o(z)}{a} + \Omega^2\right) \qquad (A3)$$

with $g_o(z)$ given by Eq. (A2).

Introducing the dimensionless parameters defined in Sec. II.C we obtain for the definition of $\hat{g}$

$$\hat{g} = \frac{3L^2}{2 \int_1^2 \frac{dz}{B^2}} \int_1^2 \frac{dz}{B^4} \left(\frac{dB}{dz}\right)^2 \qquad (A4)$$

required for the comparison of the relative strength of the curvature and centrifugal force driving mechanisms.

For a simple parabolic mirror field with $B(z) = B_c(1+z^2/L^2)$ the unit of bad curvature obtained from definition (A4) is $\hat{g} = 3/4$. For the hyperbolic tangent model[2] of the tandem mirror field the value of $\hat{g}$ is less than for the parabolic mirror field. For the field in Eq. (2) of Ref. 2, with R_c the central cell mirror ratio, $2L_1$ the length of the central cell and ΔL_1 the length of the mirror region at the end of the central cell we find

$$\hat{g} \simeq (L_c/\Delta L_1)(R_c-1)^2/(1+R_c)^4$$

valid for $R_c \gg 1$ and $\Delta L_1 < L_1$ and thus $\hat{g} \lesssim 1$.

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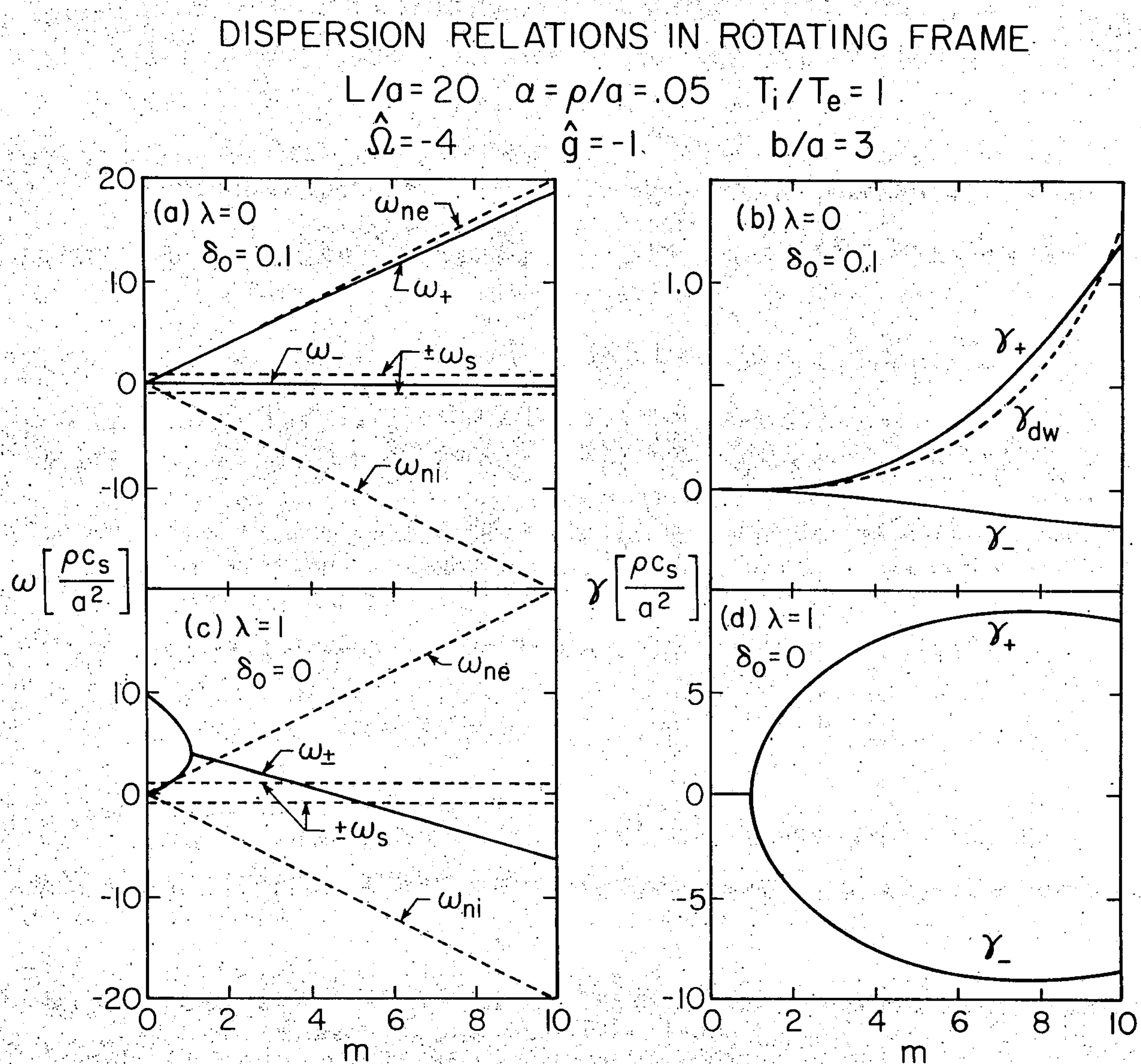
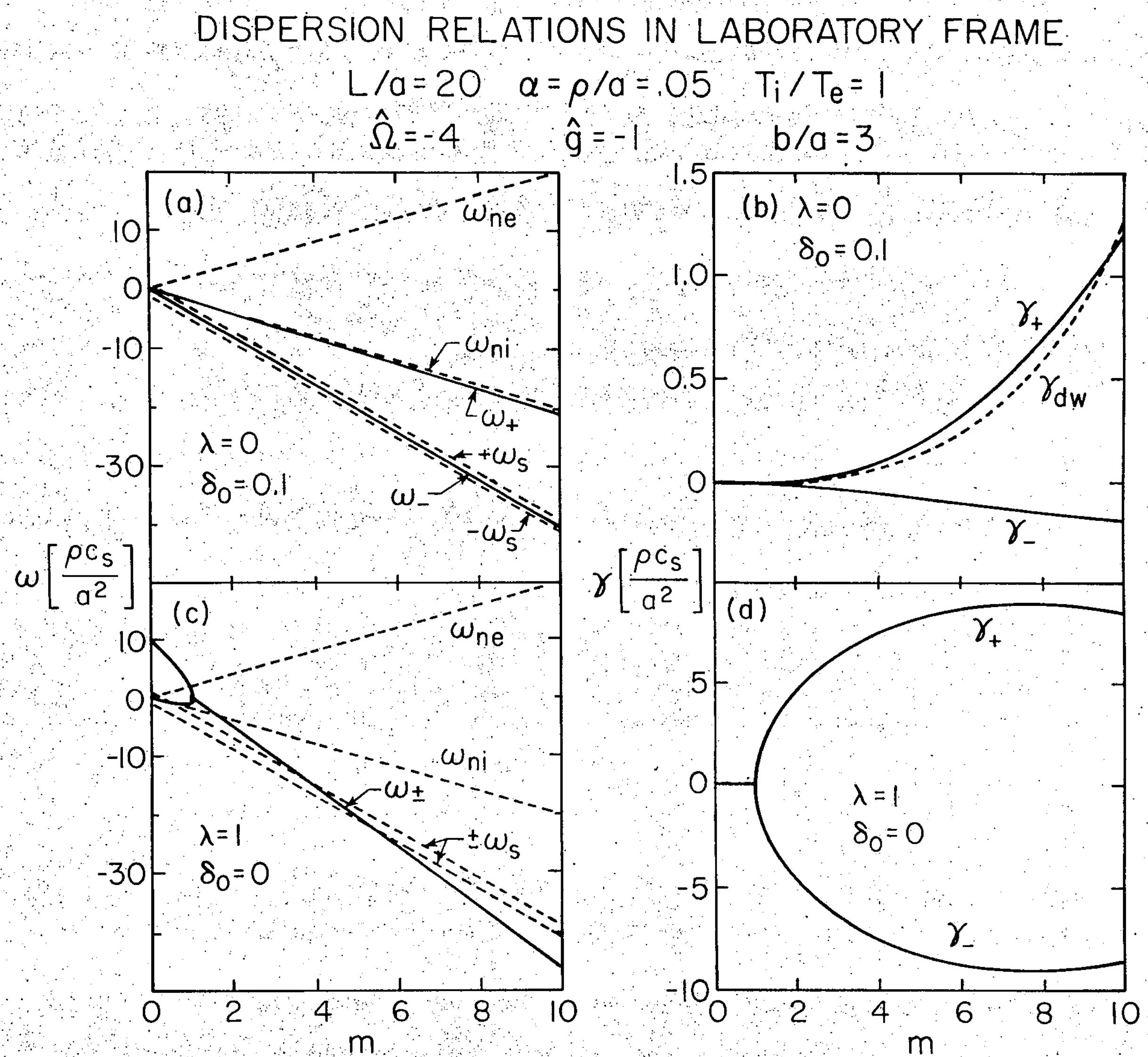
Figure Captions

1. Typical frequency and growth rate in the reference frame rotating with $\Omega = -4[\rho c_s/a^2]$ for the drift wave $\lambda = 0$ and the flute mode $\lambda = 1$.
2. Same frequency and growth rate given in Fig. 1 now shown in the laboratory reference frame.
3. Variation of the m=1 drift wave (λ=0) growth rate with $\hat{\Omega}$ and $\hat{g}$ defined in Section II.
4. Variation of the m=10 drift wave growth rate with $\hat{\Omega}$ and $\hat{g}$.
5. Variation of the m=1,2 flute mode growth rates with $\hat{\Omega}$ and $\hat{g}$.
6. Perturbed flow of the plasma produced by the instability due to the gradient of $\Omega(r)$ in a layer of width $2x_o$.



DISPERSION RELATIONS IN ROTATING FRAME
L/a = 20 α = ρ/a = .05 T_i/T_e = 1
$\hat{\Omega}$ = -4 $\hat{g}$ = -1 b/a = 3
(a) λ = 0
δ_0 = 0.1
ω_{ne}
ω_+
ω_-
$\pm\omega_s$
ω_{ni}
(b) λ = 0
δ_0 = 0.1
γ_+
γ_{dw}
γ_-
$\omega\left[\frac{\rho c_s}{a^2}\right]$
$\gamma\left[\frac{\rho c_s}{a^2}\right]$
(c) λ = 1
δ_0 = 0
ω_{ne}
$\omega_\pm$
$\pm\omega_s$
ω_{ni}
(d) λ = 1
δ_0 = 0
γ_+
γ_-
20
10
0
-10
-20
1.0
5
-5
-10
2
4
6
8
m

FIGURE 1



DISPERSION RELATIONS IN LABORATORY FRAME
L/a=20 α=ρ/a=.05 Ti/Te=1
Ω̂=-4 ĝ=-1 b/a=3
(a)
λ=0
δo=0.1
ωne
ωni
ω+
+ωs
ω−
-ωs
(b) λ=0
δo=0.1
γ+
γdw
γ−
(c)
λ=1
δo=0
ωne
ωni
ω±
±ωs
(d)
λ=1
δo=0
γ+
γ−
ω [ρcs/a²]
γ [ρcs/a²]
m

FIGURE 2

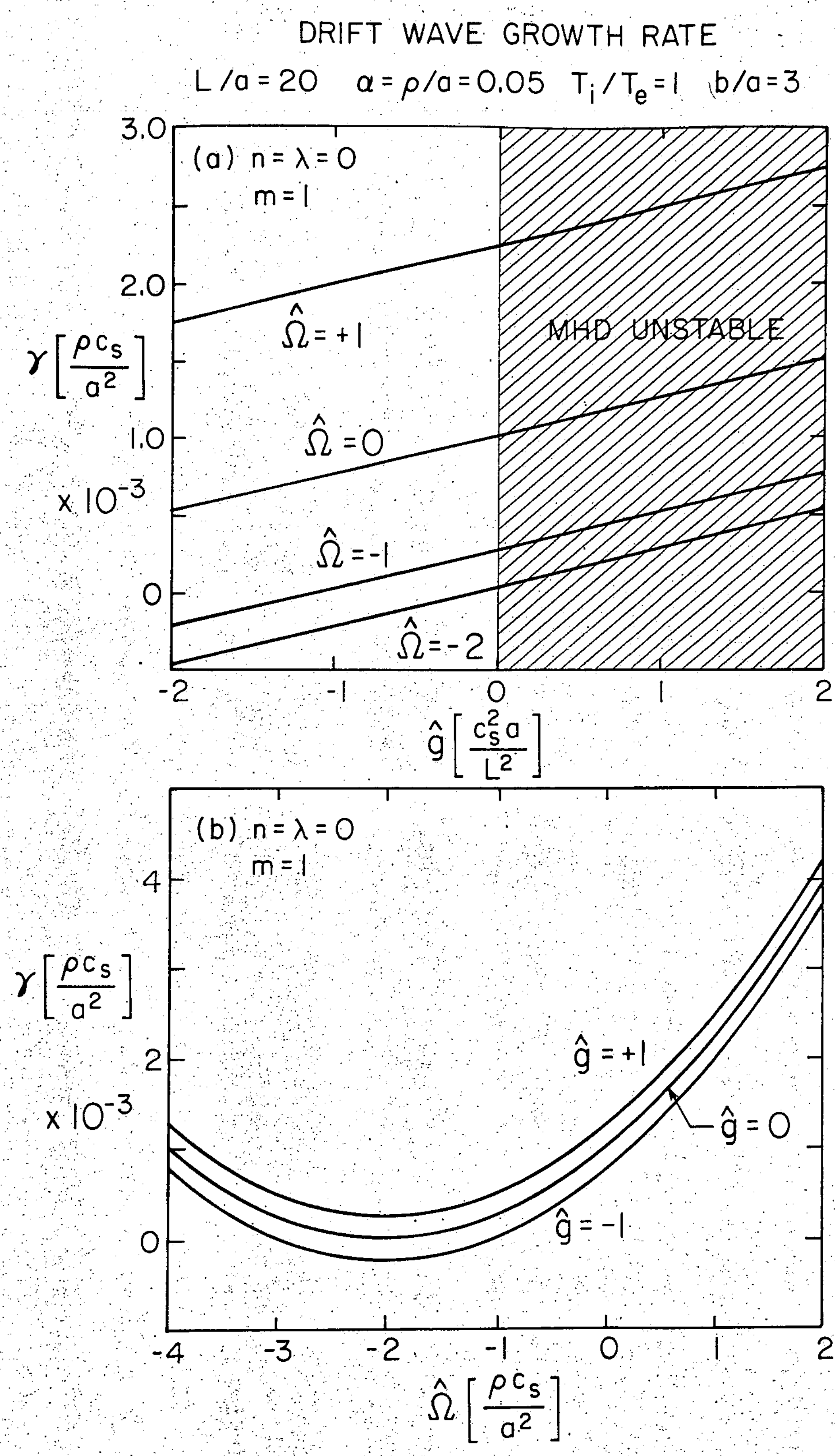

DRIFT WAVE GROWTH RATE
L /a = 20   α = ρ/a = 0.05 T_i/T_e = 1   b/a = 3
(a) n = λ = 0
m = 1
3.0
2.0
1.0
0
x 10^{-3}
γ [ρc_s/a^2]
$\hat{\Omega}$ = +1
$\hat{\Omega}$ = 0
$\hat{\Omega}$ = -1
$\hat{\Omega}$ = -2
MHD UNSTABLE
-2
-1
0
1
2
ĝ [c_s^2a/L^2]
(b) n = λ = 0
m = 1
4
2
0
x 10^{-3}
γ [ρc_s/a^2]
ĝ = +1
ĝ = 0
ĝ = -1
-4
-3
-2
-1
0
1
2
$\hat{\Omega}$ [ρc_s/a^2]

FIGURE 3

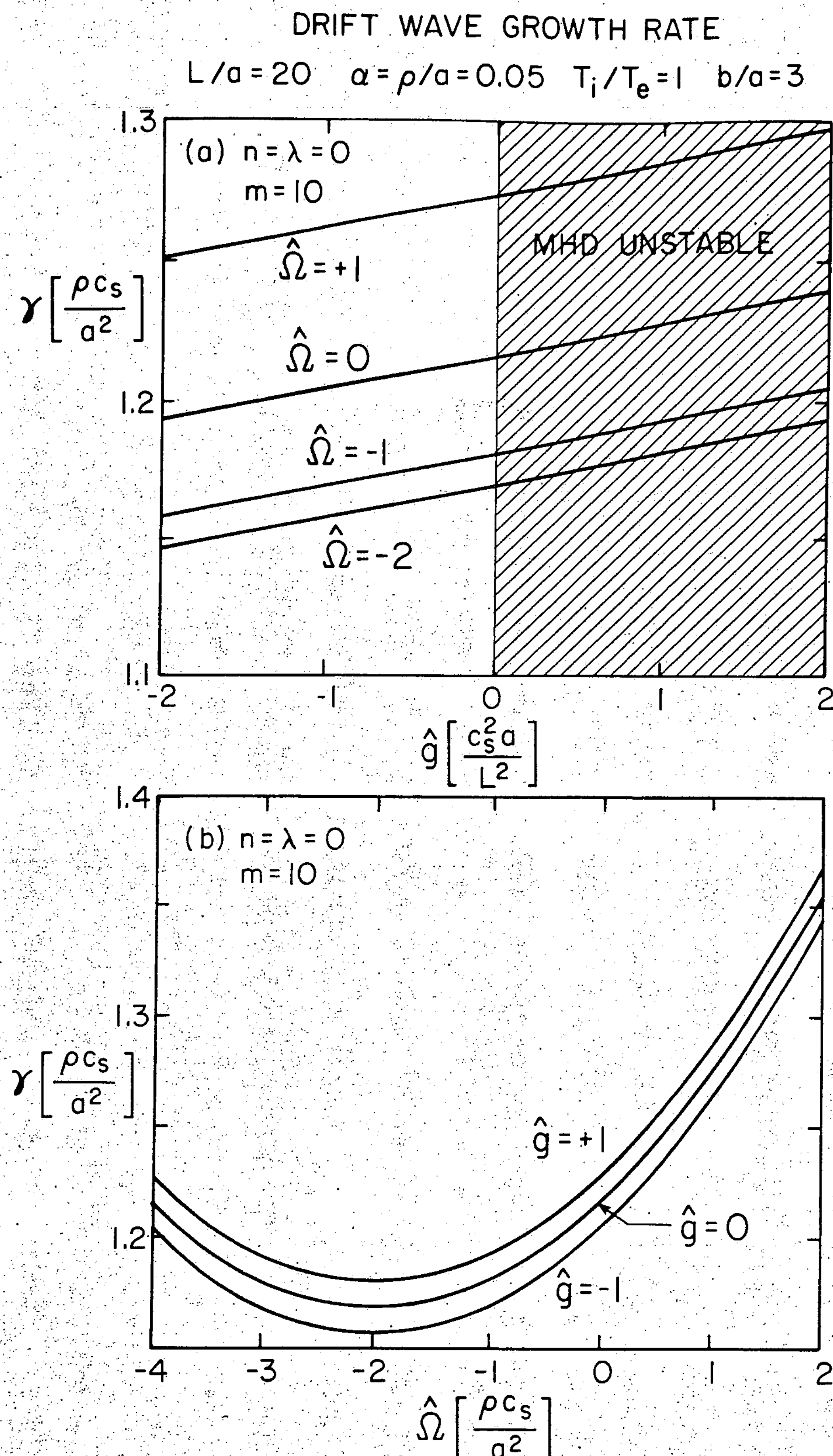

DRIFT WAVE GROWTH RATE
L/a=20 α=ρ/a=0.05 Ti/Te=1 b/a=3
(a) n=λ=0
m=10
MHD UNSTABLE
Ω̂=+1
Ω̂=0
Ω̂=-1
Ω̂=-2
γ [ρcs/a²]
ĝ [cs²a/L²]
(b) n=λ=0
m=10
ĝ=+1
ĝ=0
ĝ=-1
γ [ρcs/a²]
Ω̂ [ρcs/a²]

FIGURE 4

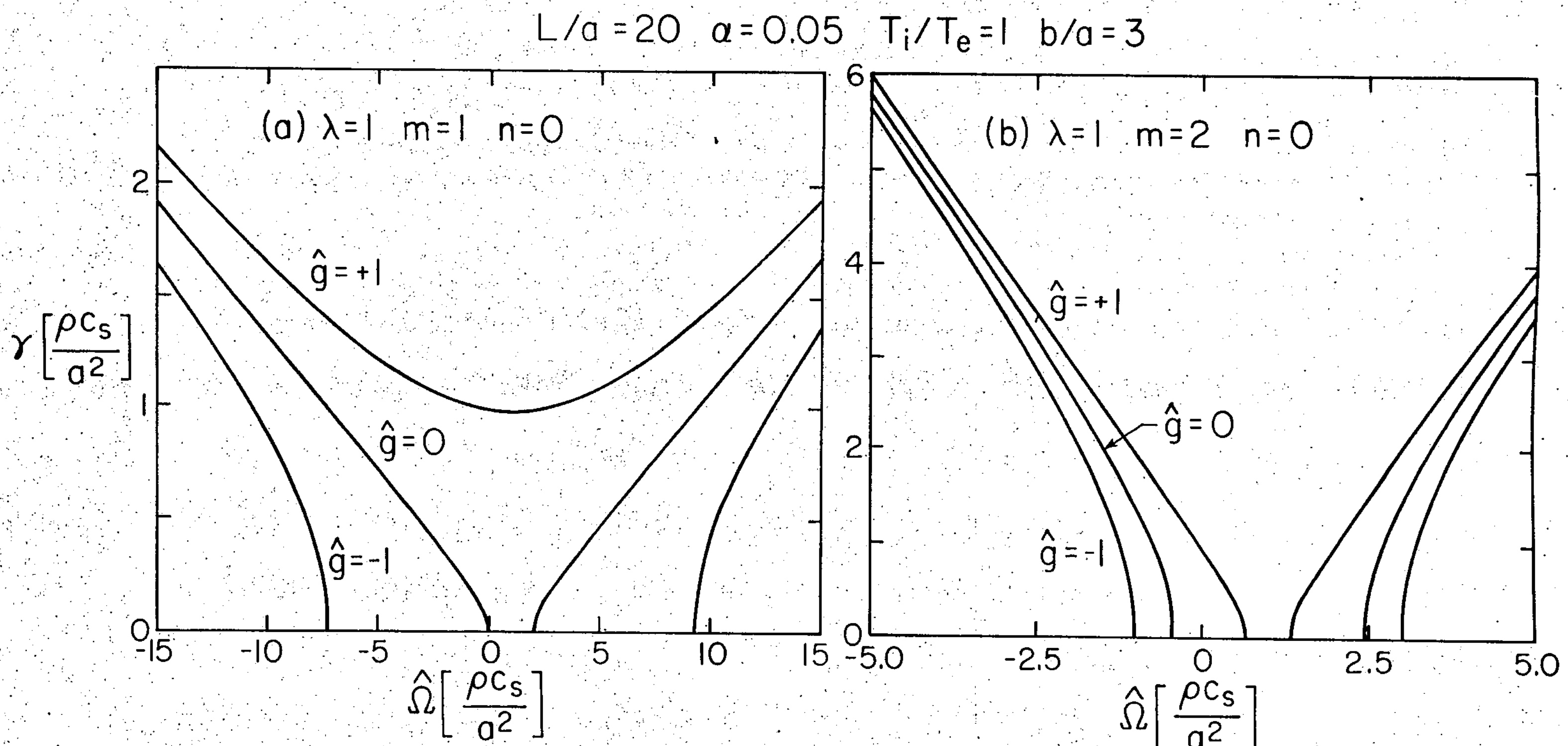


FIGURE 5

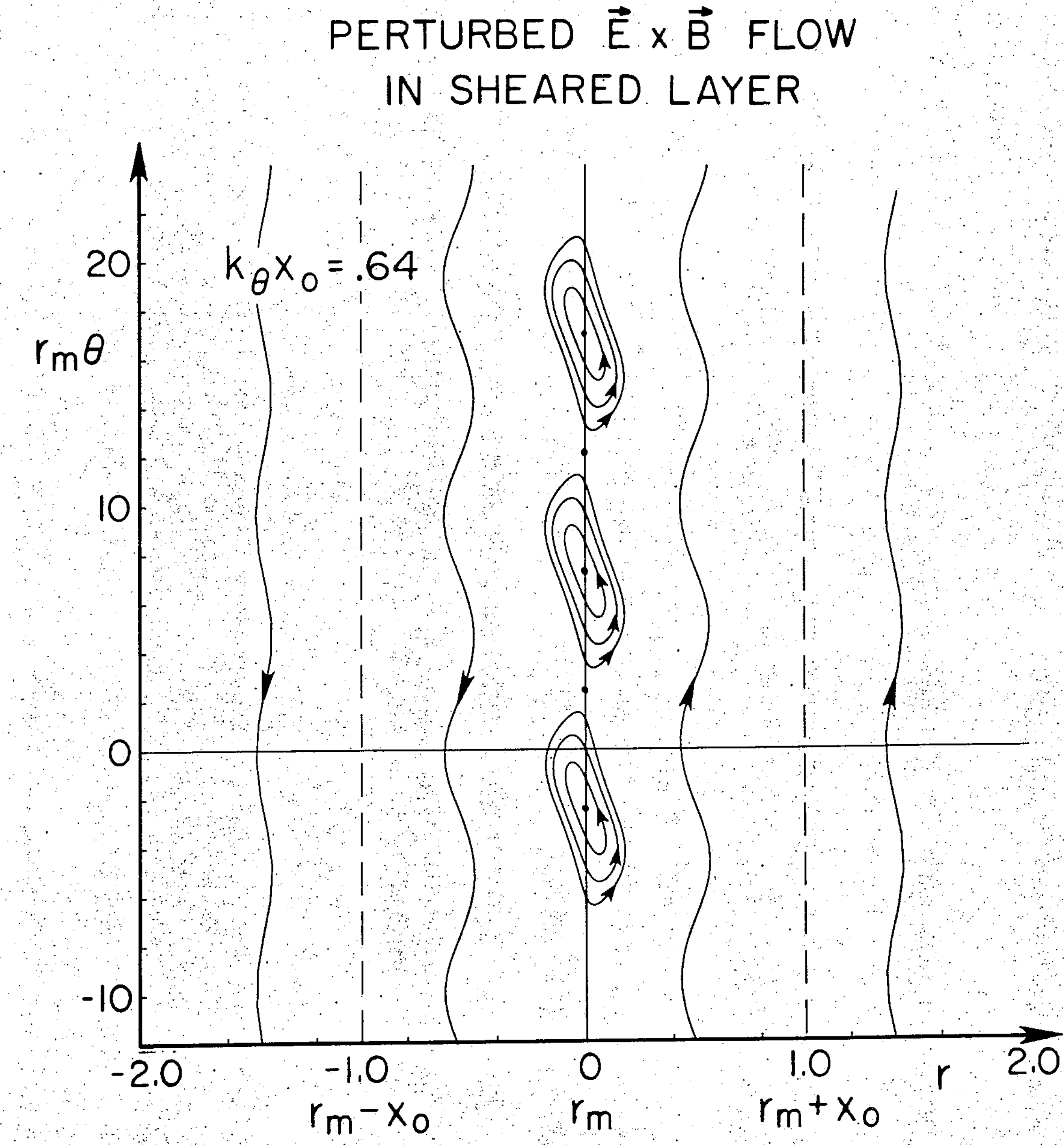

PERTURBED $\vec{E} \times \vec{B}$ FLOW
IN SHEARED LAYER
$k_\theta x_o = .64$
$r_m\theta$
20
10
0
-10
-2.0
-1.0
0
1.0
2.0
r
$r_m - x_o$
r_m
$r_m + x_o$

FIGURE 6