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NECESSARY STABILITY CONDITION FOR FIELD-REVERSED THETA PINCHES

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ABSTRACT

Toroidal systems of arbitrary cross section without toroidal magnetic field are analyzed via the double adiabatic fluid equations. Such systems are shown to be unstable if there exists one closed field line on which the average of $\kappa r B^2$ is positive, where κ is the curvature. A similar criterion is derived for linear systems and is applied to a noncircular z-pinch.

I. Introduction

Field-reversed theta pinches have been observed to exist for many Alfven transit times[1,2] in spite of the fact that many fluid calculations for particular equilibria have found instability. This state of affairs has led to the pursuit of two possible resolutions of this conflict. The first is the search for large gyroradius effects which may cause stabilization of the system. The second is the search for particular equilibria which might be fluid stable.

In particular, we mention Newcomb's examination[3] of the linear magnetohydrodynamic equations for a field-reversed theta pinch near the null. He showed that if the flux function has nonvanishing second derivatives at the null, then the system is unstable, with a growth time equal to the magnetohydrodynamic transit time around the field line divided by 2π. However, one could argue that Newcomb's result does not resolve the question of fluid stability, since it does not apply, for example to systems with vanishing current at the null.

In the present paper we examine the fluid stability question by analyzing the energy principle[4] for the linearized double adiabatic equations[5] in the limit of large toroidal mode number. We find that the stability of the system depends on the properties of the integral of κrB^2, where κ is the curvature, around a closed field line. If over some region this integral increases with distance from the magnetic null, the field lines in that region are unstable. As discussed in more detail later, this implies that a system with one closed field line on which the curvature does not change sign must be stable. Naturally, since the present result is based on the double adiabatic equations, the comparison theorems[4,6,7] imply it also for the adiabatic (or Kruskal-Oberman) and magnetohydrodynamic equations.

We also show how this result applies to linear geometries such as EXTRAP[8], which is a z-pinch of noncircular cross section. We find that a field line is unstable if it exists entirely within a region where the determinant of the second derivative matrix of the flux function, $\det(\underset{\sim}{\nabla\nabla}A)$, is negative. ($\mathrm{Det}(\underset{\sim}{\nabla\nabla}A)$ is also known as the Gaussian curvature[9] of the flux surface.) This criterion is particularly easy to apply, as we show by considering a simple model of EXTRAP.

II. Potential Energy of a Pure Displacement

For azimuthally symmetric systems with no torodial field one may introduce the orthogonal field-line coordinate system[4] (ψ, χ, ϕ), where $\nabla\chi$ is parallel to $\underset{\sim}{B}$, and ϕ is the toroidal angle. Figure 1 illustrates this coordinate system.

We use the linearized double adiabatic equations to analyze the stability of this system. In doing so we assume the equilibrium perpendicular and parallel pressures to be equal. Working from Eq. (3.41) of Ref. 4, we find that the change in potential energy for a displacement, $\underset{\sim}{\xi} = (X \cos m\phi/rB)\hat{\psi} + (BZ \cos m\phi)\hat{b} + (rY \sin m\phi/m)\hat{\phi}$, is given by

$$W = \frac{\pi}{2}\int d\chi d\psi J\left\{ \frac{1}{r^2B^2J^2} \left(\frac{\partial X}{\partial \chi}\right)^2 + B^2\left(\frac{\partial X}{\partial \psi} + Y\right)^2 + P'X\left(\frac{\partial X}{\partial \psi} + Y\right)\right.$$
$$+ P'X\left[\frac{1}{J}\frac{\partial}{\partial \psi}(JX) + Y\right]$$
$$+ \gamma P\left[\frac{1}{J}\frac{\partial}{\partial \psi}(JX) + Y + \frac{1}{J}\frac{\partial Z}{\partial \chi}\right]^2$$
$$+ \alpha P\left[\frac{1}{J}\frac{\partial}{\partial \psi}(JX) + Y + \frac{1}{J}\frac{\partial Z}{\partial \chi} - 3q\right]^2 \qquad (1)$$

for large m, where $\gamma = 5/3$, $\alpha = 1/3$, and $q \equiv \hat{b}\cdot\underset{\sim}{\nabla}\underset{\sim}{\xi}\cdot\hat{b} = \kappa X/rB + \partial(BZ)/\partial s$. The variable s denotes arc length along χ and is related to χ according to $JBd\chi = ds$. To obtain the corresponding magnetohydrodynamic potential energy, one simply sets $\alpha = 0$.

This form for the potential energy can be algebraically minimized with respect to Y. To do so, it is useful to note that W can be written in the form

$$W = \frac{\pi}{2}\int d\chi d\psi J \left\{ \frac{1}{r^2B^2J^2}\left(\frac{\partial X}{\partial \chi}\right)^2 + B^2\left(\frac{\partial X}{\partial \psi} + Y\right)^2 + P'X\left(\frac{\partial X}{\partial \psi} + Y\right) + P'X\left[\frac{1}{J}\frac{\partial}{\partial \psi}(JX) + Y\right]\right.$$

$$+ \mu P\left[\frac{1}{J}\frac{\partial}{\partial \psi}(JX) + Y + \frac{1}{J}\frac{\partial Z}{\partial \chi} - \frac{3\alpha q}{\mu}\right]^2$$

$$\left. + 9\frac{\alpha\gamma}{\mu} P q^2 \right\},$$

where $\mu \equiv \alpha+\gamma$. We see that the result for the double adiabatic model minimized over Y is obtained from the magnetohydrodynamic result by replacing γ with μ, $J^{-1}\partial Z/\partial\chi$ with $J^{-1}\partial Z/\partial\chi - 3\alpha q/\mu$, and adding the term $9\alpha\gamma Pq^2/\mu$. This result is

$$W = \frac{\pi}{2}\int d\psi \frac{ds}{B}\left\{ \frac{1}{r^2}\left(\frac{\partial X}{\partial s}\right)^2 + \frac{2\kappa P'}{rB^2}X^2 + \frac{\mu P B^2}{\mu P + B^2}\left[\frac{2\kappa X}{rB} + B\frac{\partial Z}{\partial s} - \frac{3\alpha}{\mu}q\right]^2 + 9\frac{\alpha\gamma}{\mu}Pq^2\right\}, \quad (2)$$

where $\kappa \equiv -\hat{\psi}\cdot(\hat{b}\cdot\nabla)\hat{b}$ is the field line curvature. As in the magnetohydrodynamic case, there is no coupling between different values of ψ, and so W may be minimized on each field line.

We wish to insert a particular trial function, a pure displacement, into Eq. (2). We are motivated by Newcomb's finding[3] that a pure displacement is the minimizing perturbation near the null if the current does not vanish there. Let us define the tangential angle ψ according to Fig. 1, such that $\cos\theta = \hat{b}\cdot\hat{z} = \hat{\psi}\cdot\hat{r}$ and $\sin\theta = \hat{\chi}\cdot\hat{z} = -\hat{b}\cdot\hat{r}$. A pure displacement is one for which its projection in the r-z plane, $\underset{\sim}{\xi}_{rz}$ is constant, i.e.

$$\underset{\sim}{\xi}_{rz} = C_r\hat{r} + C_z\hat{z} ,$$

where C_r and C_z are constants. Hence, a constant displacement of unit magnitude has the form,

$$X = rB\cos(\theta + \theta_o) \tag{3a}$$

$$Z = -B^{-1}\sin(\theta + \theta_o), \tag{3b}$$

where θ_o is an arbitrary constant. By differentiation of $\cos\theta = \hat{b}\cdot\hat{z}$ one can show that $d\theta/ds = \kappa$.

We now insert the pure displacement (3) into our expression (2) for W. We note that for a pure displacement, q=0. We find the result,

$$W = \oint \frac{ds}{B}\left\{\frac{1}{r^2}\left[\left(\frac{\partial(rB)}{\partial s}\right)^2\cos^2(\theta+\theta_o) - 2\kappa rB\frac{\partial rB}{\partial s}\cos(\theta+\theta_o)\sin(\theta+\theta_o) + \kappa^2 r^2 B^2\sin^2(\theta+\theta_o)\right]\right.$$
$$+ \frac{\mu P}{\mu P + B^2}\left[\left(\frac{\partial B}{\partial s}\right)^2\sin^2(\theta+\theta_o) + 2\kappa B\frac{\partial B}{\partial s}\cos(\theta+\theta_o)\sin(\theta+\theta_o) + \kappa^2 B^2\sin^2(\theta+\theta_o)\right]$$
$$\left. + 2\kappa rB\frac{\partial P}{\partial\psi}\cos^2(\theta+\theta_o)\right\} ,$$

for W on a particular field line. Since the quantity $\mu P/(\mu P+B^2)$ multiplies a positive term, we can obtain an upper bound on W by making the replacement $\mu P/(\mu P+B^2) \to 1$. In addition we note that miminum value of W with respect to θ_o must be less than its average, i.e.

$$W_{min} < (2\pi)^{-1} \int_0^{2\pi} d\theta_o W(\theta_o)$$

We therefore obtain the bound

$$W < \oint \frac{ds}{B} \left\{ \frac{1}{2r^2} \left[\frac{\partial (rB)}{\partial s}\right]^2 + \frac{1}{2} \left(\frac{\partial B}{\partial s}\right)^2 + \kappa^2 B^2 + \kappa r B \frac{\partial P}{\partial \psi} \right\}.$$

Finally, we use the condition of equilibrium, $r\partial P/\partial\psi = -\kappa B - \hat{\psi}\cdot\nabla B$, to obtain the upper bound

$$W < - \oint \frac{ds}{B} \left\{ \kappa B \hat{\psi}\cdot \nabla B - \frac{1}{2r^2} \left[\frac{\partial (rB)}{\partial s}\right]^2 - \frac{1}{2} \left(\frac{\partial B}{\partial s}\right)^2 \right\}. \quad (4)$$

The last two terms in the expression (4) for W are stabilizing. W can be negative only when the first term dominates, which occurs when κ is mostly positive and B^2 increases rapidly with ψ. Thus, we might expect W to be negative only if some field line average of B is increasing. In fact, as we now show W is bound by the inequality

$$W < V \equiv - \frac{1}{2} \frac{\partial}{\partial \psi} \oint ds \kappa r B^2. \quad (5)$$

Since $\kappa ds = d\theta$, this inequality implies that W is bounded by the average of rB^2 over the angle $d\theta$.

To obtain the inequality (5), we note that the quantity V is given by

$$V = -\frac{1}{2}\frac{\partial}{\partial\psi}\oint d\chi \frac{\partial\theta}{\partial\chi} rB^2 = -\frac{1}{2}\oint d\chi\left[\frac{\partial^2\theta}{\partial\chi\partial\psi} rB^2 + \frac{\partial\theta}{\partial\chi}\frac{\partial}{\partial\chi}(rB^2)\right] . \tag{6}$$

Integrating by parts on χ and using $\partial/\partial\psi = (rB)^{-1}\hat{\psi}\cdot\nabla$ we find

$$V = \frac{1}{2}\oint ds\left[\frac{\partial\theta}{\partial\psi}\frac{\partial(rB^2)}{\partial s} - \frac{\kappa}{rB}\hat{\psi}\cdot\nabla(rB^2)\right] . \tag{7}$$

The quantity $\partial\theta/\partial\psi$ is given by $-h_\chi^{-1}\partial h_\psi/\partial\chi$ for an orthogonal coordinate system with metric $dr^2 + dz^2 = h_\psi^2 d\psi^2 + h_\chi^2 d\chi^2$, as one can see from Fig. 2. Using $h_\psi = (rB)^{-1}$ and $h_\chi d\chi = ds$, we obtain

$$V = -\oint \frac{ds}{B}\left\{\kappa B\hat{\psi}\cdot\nabla B - \frac{1}{2}\left[\left(\frac{\partial B}{\partial s}\right)^2 + \frac{1}{r^2}\left(\frac{\partial rB}{\partial s}\right)^2\right]\right\} + \oint ds \frac{B}{r^2}\sin^2\theta . \tag{8}$$

The inequality (5) implies instability for any system where the integral,

$$I(\psi) \equiv \oint ds\kappa rB^2 ,$$

is positive somewhere. For such systems, the fact the I vanishes at the null implies (by the mean value theorem) that $\partial I/\partial\psi$ must be positive somewhere, and, hence, W must be negative there. Naturally, if a system has a closed field with everywhere positive curvature, then it is unstable.

III. Application to Straight Systems

Let us consider the application of this analysis to straight systems, for which the flux function $A(x,y)$ is independent of z and the magnetic field is given by $\underset{\sim}{B} = \nabla A \times \hat{z}$. For this geometry, Eq. (5) is replaced by

$$W < -\frac{1}{2}\frac{\partial}{\partial A}\oint ds \;\; \kappa B^2 \quad .$$

However, a simple criterion may be obtained from the result,

$$W < -\oint \frac{ds}{B}\left[\kappa B\;\hat{A}\cdot\nabla B - \left(\frac{\partial B}{\partial s}\right)^2\right] , \tag{9}$$

where $\hat{A} \equiv \underset{\sim}{\nabla} A/|\underset{\sim}{\nabla} A|$. Eq. (9) is the analogue of Eq. (4) for straight geometries. Eq. (9) may be rewritten as the weighted integral of the determinant of the $\underset{\sim}{\nabla\nabla} A$ tensor upon noting the relations, $\kappa B = \hat{b}\cdot\underset{\sim}{\nabla\nabla}A\cdot\hat{b}$, $\hat{A}\cdot\underset{\sim}{\nabla}B = \hat{A}\cdot\underset{\sim}{\nabla\nabla}A\cdot\hat{A}$, and $\partial B/\partial s = \hat{A}\cdot\underset{\sim}{\nabla\nabla}A\cdot\hat{b}$. The result is

$$W < -\oint \frac{ds}{B}\,\det(\underset{\sim}{\nabla\nabla}A) \quad .$$

The quantity $\det(\underset{\sim}{\nabla\nabla}A)$ is the Gaussian curvature[9] of the surface $z = A(x,y)$. Hence, instability can be detected by examining the three-dimensional plot of $A(x,y)$. Field lines entirely within a region where $A(x,y)$ has positive curvature, like a bowl, are unstable. A necessary condition for stability on a field line is that it be at least partially within a region where the surface $z = A(x,y)$ has negative curvature, like a saddle.

As an example of the application of this criterion, we consider the flux function, $A(x,y) = (a/2)(x^2 + y^2) + (b/12)(x^4 - 6x^2y^2 + y^4)$, which corresponds to an equilibrium with constant $\partial P/\partial A$. For this flux function, $\det(\nabla\nabla A) = a^2 - b^2(x^2 + y^2)^2$. Hence, all field lines totally inside the radius $r = (b/a)^{1/2}$ are unstable. The contours of this flux function and the unstable region are shown in Fig. 3. We note that a large fraction of the region inside the separatrix is unstable.

IV. Discussion

The fact remains that the systems under consideration are observed to be stable for several Alfven transit times. Several possible resolutions of this contradiction exist. It remains a possibility, though perhaps an unlikely one, that one can construct peculiar equilibria for which the line integral of $\kappa r B^2$ decreases with ψ. Or, perhaps such systems do not have isotropic pressure, a case not discussed here. Alternatively, finite Larmor radius effects or other kinetic effects may be large enough to stabilize the system everywhere. A likely possibility is that the effects of large orbit particles are stabilizing near the magnetic axis, and that the outer field lines are fluid stable.

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Figure Captions

Fig. 1. Flux contours for an axisymmetric field-reversed configuration. The angle θ is the tangential angle of the text. Only $z > 0$ is shown since the equilibrium is symmetric.

Fig. 2. Illustration of the equality $\partial\theta/\partial\psi = -(\partial h_\psi/\partial\chi)/h_\chi$ for an orthogonal coordinate system. One can calculate the distance $\Delta\ell$ by either of two methods: $\Delta\ell = h_\psi(\psi,\chi + \Delta\chi)\Delta\psi - h_\psi(\psi,\chi)\Delta\psi$ or $\Delta\ell = -\Delta\theta h_\chi(\psi,\chi) = -(\partial\theta/\partial\psi)h_\chi(\psi,\chi)\Delta\psi$.

Fig. 3. Contours of the flux function $A = a(x^2 + y^2)/2 + b(x^4 - 6x^2y^2 + y^4)/12$ for $bA/a^2 = 0.15, 0.3, 0.45, 0.58$, and 0.75. Flux contours, i.e. field lines, contained entirely within the shaded region $(r^2 < a/b)$ are unstable.

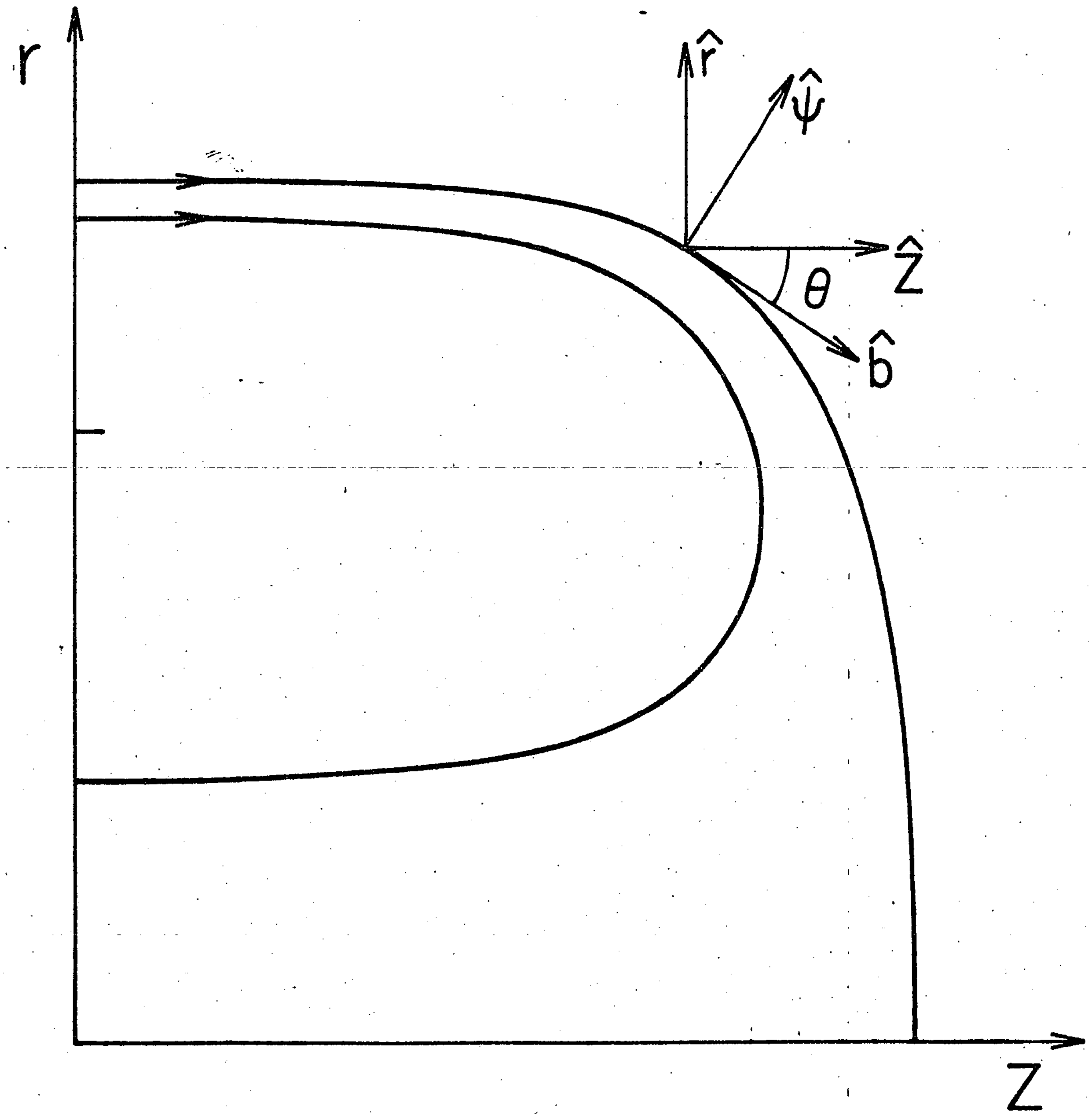

FIG. 1

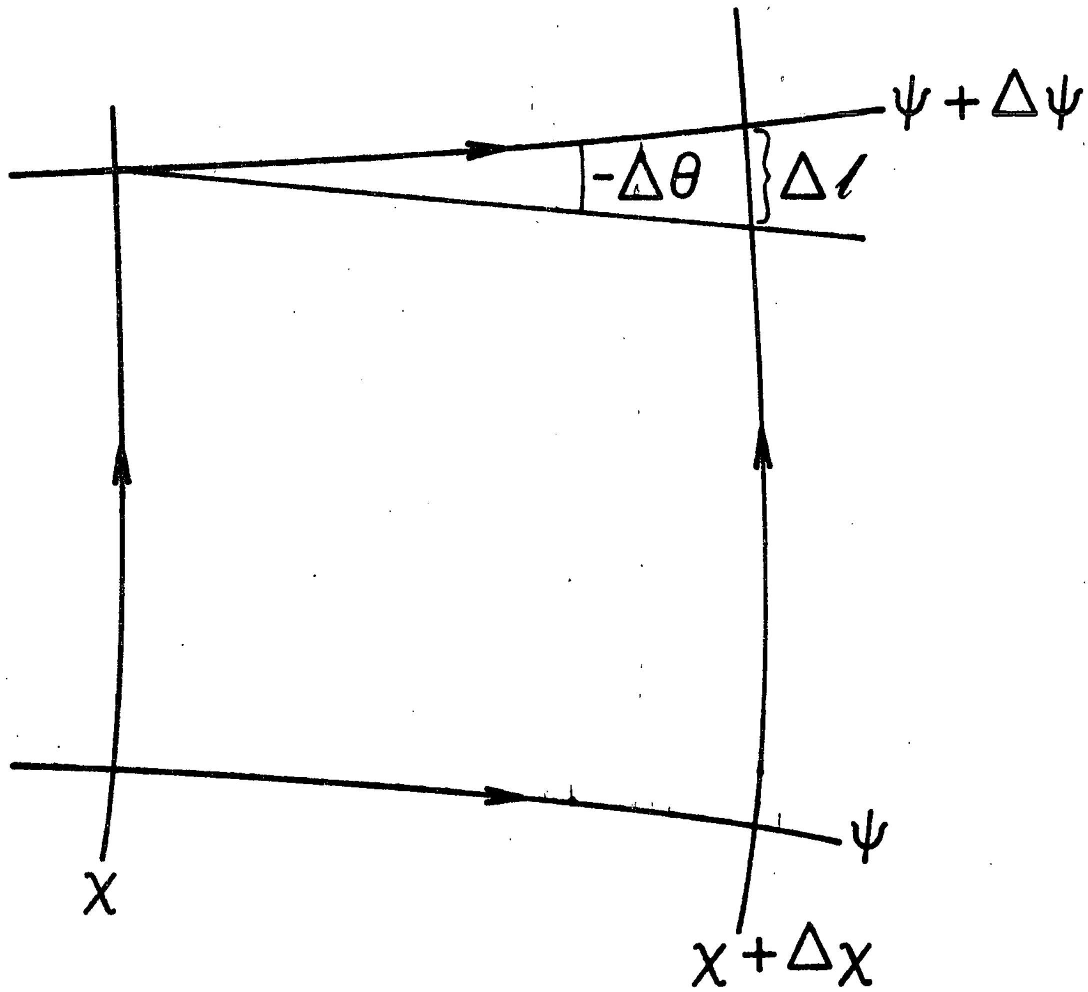

FIG. 2

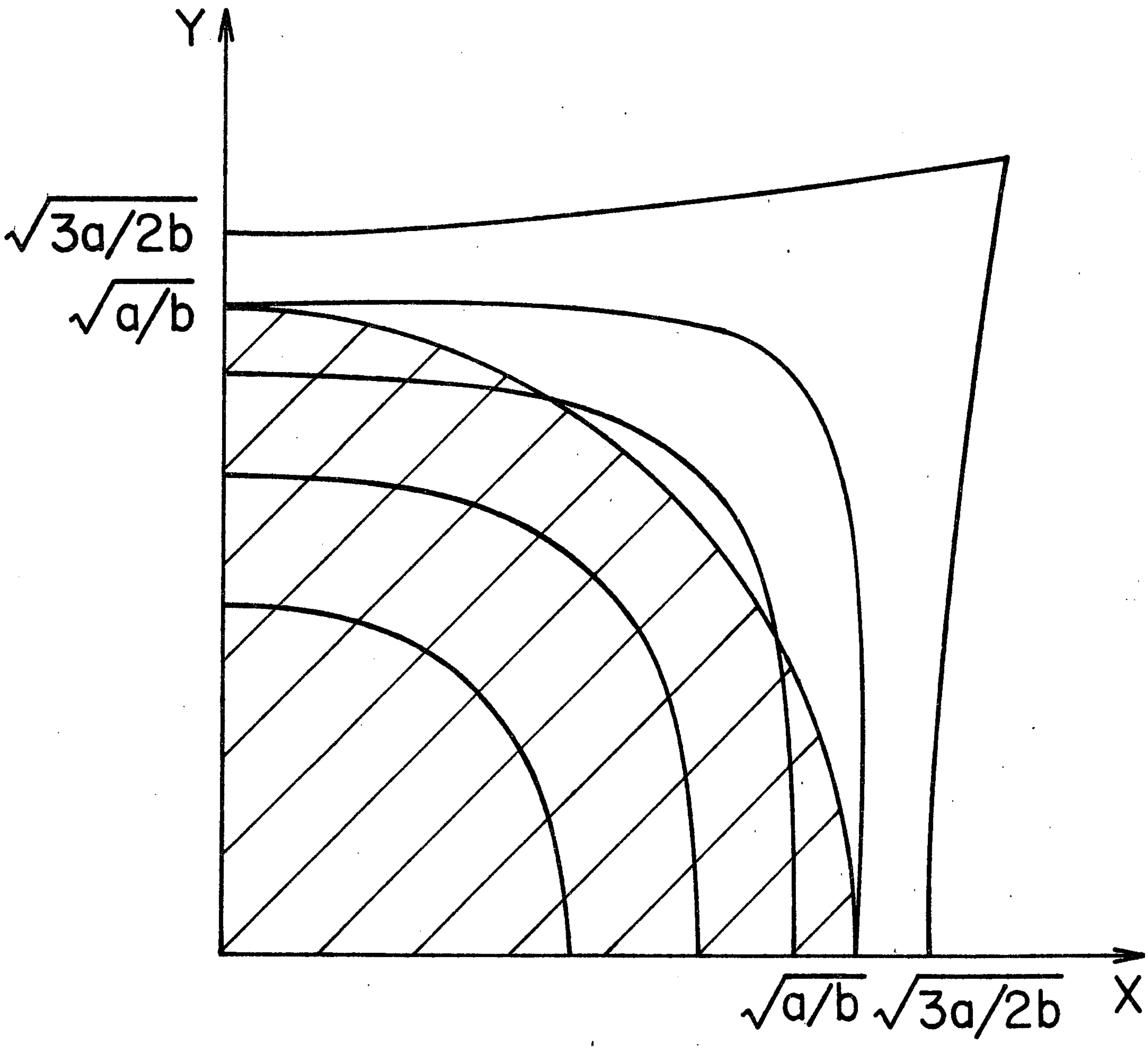

FIG. 3