DOE/ET-53088-104 IFSR #104

LINEAR RELATIVISTIC GYROKINETIC EQUATION IN GENERAL MAGNETICALLY CONFINED PLASMAS

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August 1983

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Abstract

The gyrokinetic formalism for linear electromagnetic waves of arbitrary frequency in general magnetic field configurations is extended to include full relativistic effects. The derivation employs the small adiabaticity parameter ρ/L_0 where ρ is the Larmor radius and L_0 the equilibrium scale length. The effects of the plasma and magnetic field inhomogeneities and finite Larmor radii effects are also contained.

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1. INTRODUCTION

Hot electrons in the Elmo Bumpy Torus (Uckan et al., 1983) and Symmetric Tandem Mirror (Quon et al., 1982) experiments are already in the relativistic regime, and relativistic effects might also be significant in the recent idea proposed by Rosenbluth et al. (1983) to enhance tokamak stability with highly energetic particles. However, most theoretical studies of these devices have used a nonrelativistic treatment. Only quite recently, Berk et al. (1983) have considered a relativistic fluid model for the Elmo Bumpy Torus. Also, Ohsawa, Dawson, and Van Dam (1983) have used the relativistic gyrokinetic equation presented in this paper in the low-frequency limit to interpret the stability results observed in a particle simulation of the Elmo Bumpy Torus plasma.

This paper describes the fully relativistic generalization of the linear gyrokinetic formalism, which was recently extended to arbitrary frequencies (Chen and Tsai, 1983a and 1983b, hereafter referred to as Papers I and II). The same systematic approach as in Papers I and II will be employed here, namely, transforming to the guiding-center phase space and then averaging over the gyrophase angle, while expanding in the small adiabaticity parameter $\lambda = \rho/L_0 << 1$. Here ρ is the Larmor radius and L_0 is the scale length of the equilibrium plasma and magnetic field. The present work differs from Papers I and II by beginning from the relativistic Vlasov equation and by adopting the Lorentz gauge instead of the Coulomb gauge. A complete collisionless relativistic gyrokinetic equation for linear electromagnetic wave dynamics in general plasma equilibria is thereby obtained.

In Section 2, the derivation of the theoretical formulation is explained. In Section 3, the WKB eikonal ansatz is imposed, which greatly simplifies the equation, and then the general wave equations are presented. Section 4 contains a concluding discussion.

2. THEORETICAL ANALYSIS

The course of formulation and derivation given here will closely follow that of Paper II. We define the guiding-center phase space $(\underset{\sim}{X}, \underset{\sim}{U})$ which is related to the particle phase space $(\underset{\sim}{x}, \underset{\sim}{u})$ via the following transformation:

$$\underset{\sim}{X} = \underset{\sim}{x} + \underset{\sim}{u} \times \underset{\sim}{e}_{\parallel}/\Omega \ , \tag{1}$$

$$\underset{\sim}{U} = (\varepsilon, \mu, \alpha) \ , \tag{2}$$

where $\mu = u_{\perp}^2/2B$, $\varepsilon = c(u^2+c^2)^{1/2} + q\Phi_0/m_0 = c^2\gamma + q\Phi_0/m_0$; the relativistic velocities are $\underset{\sim}{u} = \gamma\underset{\sim}{v}$ and $\underset{\sim}{U} = \gamma\underset{\sim}{V}$; m_0 is the rest mass; $\Omega = qB/m_0c$, $\underset{\sim}{e}_{\parallel} = \underset{\sim}{B}/B$, α is the gyrophase angle, and

$$\gamma = (1+u^2/c^2)^{1/2} = (1-v^2/c^2)^{-1/2} = \frac{1}{c^2}(\varepsilon - q\Phi_0/m_0), \tag{3}$$

$$u_{\parallel}^2 = -2B\mu + (\varepsilon - q\Phi_0/m_o)^2/c^2 - c^2 \ , \tag{4}$$

$$\underset{\sim}{u}_{\perp} = u_{\perp}(\underset{\sim}{e}_1 \cos\alpha + \underset{\sim}{e}_2 \sin\alpha) \ , \tag{5}$$

with $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, and $\underset{\sim}{e}_{\parallel}$ being the local orthogonal unit vectors. Also, Φ_0 and $\underset{\sim}{B}$ are the equilibrium electrostatic potential and magnetic field, respectively.

The unperturbed Vlasov propagator in $(\underset{\sim}{x}, \underset{\sim}{u})$ particle phase space is

$$L_u = \gamma\, \partial/\partial t + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}_x + (\underset{\sim}{u}\times\underset{\sim}{\Omega})\cdot\underset{\sim}{\nabla}_u + (q/m_0)\gamma\underset{\sim}{E}_0\cdot\underset{\sim}{\nabla}_u\ , \tag{6}$$

and in $(\underset{\sim}{X}, \underset{\sim}{U})$ guiding-center phase space is

$$L_g = \gamma\, \partial/\partial t + u_\parallel \underset{\sim}{e}_\parallel\cdot\underset{\sim}{\nabla}_X + \underset{\sim}{u}\cdot(\underset{\sim}{\lambda}_{B_1}+\underset{\sim}{\lambda}_{B_2}) - \Omega\, \partial/\partial\alpha$$
$$+ (q/m_0)\gamma\underset{\sim}{E}_0\cdot[(\underset{\sim}{u}_\perp/B)(\partial/\partial\mu) + (\underset{\sim}{e}_\alpha/u_\perp)(\partial/\partial\alpha)] + \gamma v_E\cdot\underset{\sim}{\nabla}_X\ , \tag{7}$$

where

$$\underset{\sim}{\lambda}_{B_1} = \underset{\sim}{u}\times\underset{\sim}{\nabla}_x(\underset{\sim}{e}_\parallel/\Omega)\cdot\underset{\sim}{\nabla}_X\ , \tag{8}$$

$$\underset{\sim}{\lambda}_{B_2} = -\left[\frac{\mu}{B}\underset{\sim}{\nabla}_x B + \frac{u_\parallel}{B}\underset{\sim}{\nabla}_x\underset{\sim}{e}_\parallel\cdot\underset{\sim}{u}_\perp\right]\frac{\partial}{\partial\mu} + \left[(\underset{\sim}{\nabla}_x\underset{\sim}{e}_2)\cdot\underset{\sim}{e}_i + \frac{u_\parallel}{u_\perp^2}\underset{\sim}{\nabla}_x\underset{\sim}{e}_\parallel(\underset{\sim}{u}_\perp\times\underset{\sim}{e}_\parallel)\right]\frac{\partial}{\partial\alpha}\ , \tag{9}$$

$$\underset{\sim}{\nabla}_u = \underset{\sim}{\nabla}_U + \Omega^{-1}\,\underset{\approx}{I}\times\underset{\sim}{e}_\parallel\cdot\underset{\sim}{\nabla}_X \tag{10}$$

$$\underset{\sim}{\nabla}_U = \underset{\sim}{u}\gamma^{-1}\,\partial/\partial\varepsilon + (\underset{\sim}{u}_\perp/B)\,\partial/\partial\mu + (\underset{\sim}{e}_\alpha/u_\perp)\,\partial/\partial\alpha\ , \tag{11}$$

$$\underset{\sim}{E}_0 = -\underset{\sim}{\nabla}_x\Phi_0,\ u_\parallel = \underset{\sim}{u}\cdot\underset{\sim}{e}_\parallel,\ \underset{\sim}{v}_E = c\underset{\sim}{E}_0\times\underset{\sim}{e}_\parallel/B,\ \underset{\sim}{e}_\alpha = -\underset{\sim}{e}_1\sin\alpha + \underset{\sim}{e}_2\cos\alpha.$$

Assuming the following ordering,

$$|\rho\underset{\sim}{\nabla}_X F_g|/|F_g| \sim |v_E/v_t| \sim O(\lambda),$$

we can solve for the equilibrium distribution function, F_g, up to first order in $\lambda \equiv \rho/L_0 \ll 1$, where the subscript g denotes functions of the guiding-center variables $(\underset{\sim}{X}, \underset{\sim}{U})$. We have (Rutherford and Frieman, 1968; Taylor and Hastie, 1968)

$$F_g = F_{g0} + F_{g1} + \cdots \quad , \tag{13}$$

where

$$F_{g0} = F_{g0}(\varepsilon, \mu, \underset{\sim}{X}_\perp) \quad ; \tag{14}$$

that is, $\underset{\sim}{e}_\parallel \cdot \underset{\sim}{\nabla}_X F_{g0} = 0$,

$$\tilde{F}_{g1} = (\tilde{\beta}/B)(\partial F_{g0}/\partial\mu) \quad , \tag{15}$$

$$\tilde{\beta} = -\left\{\underset{\sim}{u}_\perp \cdot \underset{\sim}{u}_D + \int^{\alpha} (d\alpha'/\Omega)\left[u_\parallel(\underset{\sim}{u}_\perp \cdot \underset{\sim}{\nabla}_X \underset{\sim}{e}_\parallel \cdot \underset{\sim}{u}_\perp) - u_\perp^2 \underset{\sim}{\nabla}_X \cdot \underset{\sim}{e}_\parallel/2\right]\right\} \quad , \tag{16}$$

$$\underset{\sim}{u}_D = \underset{\sim}{u}_d + \gamma \underset{\sim}{v}_E \quad , \tag{17}$$

and

$$\underset{\sim}{u}_d = \underset{\sim}{e}_\parallel \times \left[(u_\perp^2/2)\underset{\sim}{\nabla}_X \ln B + u_\parallel^2 \, \underset{\sim}{e}_\parallel \cdot \underset{\sim}{\nabla}_X \underset{\sim}{e}_\parallel\right]/\Omega \quad . \tag{18}$$

Here, $\tilde{F}_{g1}$ is the α-dependent part of F_{g1}; for the purpose of the present work, it is sufficient to know F_{g0} and $\tilde{F}_{g1}$.

The perturbed distribution function, δF_g, obeys the linearized relativistic Vlasov equation

$$L_g \delta F_g = -(q/m_0)\left[\delta \underset{\sim}{a}_g \cdot \underset{\sim}{\nabla}_U + (\delta \underset{\sim}{a}_g \times \underset{\sim}{e}_\parallel/\Omega) \cdot \underset{\sim}{\nabla}_X\right] F_g \quad , \tag{19}$$

where

$$\delta\underset{\sim}{a}_g = \gamma\delta\underset{\sim}{E}_g + \underset{\sim}{u} \times \delta\underset{\sim}{B}_g/c \quad . \tag{20}$$

Adopting $\delta\Phi$ and $\delta\underset{\sim}{A}$ as field variables such that $\delta\underset{\sim}{B} = \underset{\sim}{\nabla}_x \times \delta\underset{\sim}{A}$ and $\delta\underset{\sim}{E} = -(\underset{\sim}{\nabla}_x\delta\Phi + \partial\delta\underset{\sim}{A}/c\partial t)$ and letting

$$\delta F_g = \delta F_{ag} + \delta G_g \tag{21}$$

with

$$\delta F_{ag} = \frac{q}{m_0}\left[\delta\Phi_g \frac{\partial}{\partial\varepsilon} + \left(\gamma\delta\Phi - \frac{u_\parallel \delta A_\parallel}{c}\right)_g \frac{1}{B}\frac{\partial}{\partial\mu} + \frac{\delta\underset{\sim}{A}_g \times \underset{\sim}{e}_\parallel}{c\Omega} \cdot \underset{\sim}{\nabla}_X\right]F_{g0} \quad , \tag{22}$$

we find that Eq. (19) becomes, after some algebra,

$$L_g\delta G_g = -R_g = -(q/m_0)(R_1+R_2+R_3+R_4) \quad , \tag{23}$$

where

$$R_1 = \left[(\partial\delta\psi_g/\partial t)\partial/\partial\varepsilon - (\underset{\sim}{\nabla}_x\delta\psi_g \times \underset{\sim}{e}_\parallel/\Omega)\cdot\underset{\sim}{\nabla}_X\right]F_{g0} \quad , \tag{24}$$

$$\delta\psi_g \equiv \gamma\delta\Phi_g - \underset{\sim}{u}\cdot\delta\underset{\sim}{A}_g/c \quad , \tag{25}$$

$$R_2 = (\partial F_{g0}/B\partial\mu)(\gamma\ \partial/\partial t + u_\parallel\ \underset{\sim}{e}_\parallel\cdot\underset{\sim}{\nabla}_x)\delta\psi_g \quad , \tag{26}$$

$$\begin{aligned} R_3 = \Big[&-\gamma(\underset{\sim}{\nabla}_U\widetilde{\beta})\cdot\underset{\sim}{\nabla}_X\delta\Phi_g - \gamma\delta\Phi_g\underset{\sim}{u}\cdot\underset{\sim}{\nabla}_x \ln B + (q/m_0c^2)\underset{\sim}{E}_0\cdot\underset{\sim}{u}\delta\Phi_g \\ &+\Omega(\delta\Phi_g\widetilde{\beta})'_\alpha(\partial/\partial\varepsilon+\gamma\partial/B\partial\mu) + \gamma\delta\Phi_g(\hat{u}_\parallel\underset{\sim}{e}_\parallel + \underset{\sim}{u}_D)\cdot\underset{\sim}{\nabla}_X \\ &-\gamma u_\parallel\cdot\underset{\sim}{\nabla}_X\delta\Phi_g\widetilde{\beta}\ \partial/\partial\varepsilon\Big]\partial F_{g0}/B\partial\mu + \delta\Phi_g(\hat{u}_\parallel\underset{\sim}{e}_\parallel + \underset{\sim}{u}_D)\cdot\underset{\sim}{\nabla}_X\partial F_{g0}/\partial\varepsilon, \end{aligned} \tag{27}$$

$$R_4 = \{-[u_\parallel(\underset{\sim}{u}\cdot\underset{\sim}{\nabla}_X\underset{\sim}{e}_\parallel)\cdot\delta\underset{\sim}{A}_g+\delta A_{\parallel g}(\underset{\sim}{u}\cdot\underset{\sim}{\nabla}_X\underset{\sim}{e}_\parallel)\cdot\underset{\sim}{u}]+[u_\parallel\delta A_{\parallel g}(\underset{\sim}{u}\cdot\underset{\sim}{\nabla}_X \ln B)$$

$$-(q/m_0)\gamma E_{0\parallel}\delta A_{\parallel g}]+(\nabla_U\tilde{\beta})\cdot[\underset{\sim}{\nabla}_X(\underset{\sim}{u}\cdot\delta\underset{\sim}{A}_g)-(\underset{\sim}{u}\cdot\underset{\sim}{\nabla}_X)\delta\underset{\sim}{A}_g]$$

$$-\Omega u_\parallel(\tilde{\beta}\delta A_{\parallel g})'_\alpha\ \partial/B\partial\mu - u_\parallel\delta A_{\parallel g}(\hat{u}_\parallel\underset{\sim}{e}_\parallel+\underset{\sim}{u}_D)$$

$$-u_\parallel\underset{\sim}{u}_\perp\cdot(\nabla_{X\parallel}\delta\underset{\sim}{A}_\perp)\tilde{\beta}\partial/B\partial\mu\}(cB)^{-1}\ \partial F_{g0}/\partial\mu$$

$$-(\gamma/c)(\partial\delta\underset{\sim}{A}/\partial t)\cdot\underset{\sim}{\nabla}_U(\tilde{\beta}\partial F_{g0}/B\partial\mu)\ , \tag{28}$$

$$\hat{u}_\parallel = u_\parallel + (u_\perp^2/2\Omega)\underset{\sim}{e}_\parallel\cdot\underset{\sim}{\nabla}_X \times \underset{\sim}{e}_\parallel\ , \tag{29}$$

and

$$(a)'_\alpha \equiv \partial a/\partial\alpha\ .$$

We adopt the following formal ordering for the perturbed quantities:

$$\gamma|\partial/\partial t| \sim u_t|\underset{\sim}{e}_\parallel\cdot\underset{\sim}{\nabla}_X| \sim u_t|\underset{\sim}{e}_\parallel\times\underset{\sim}{\nabla}_X| \sim |\Omega| \sim O(1)\ . \tag{30}$$

Correspondingly, $L_g\delta G_g$ then becomes

$$L_g\delta G_g = \{\overset{O(1)}{\gamma\partial/\partial t} + \overset{O(1)}{u_\parallel\underset{\sim}{e}_\parallel\cdot\underset{\sim}{\nabla}_X} - \overset{O(1)}{\Omega\partial/\partial\alpha} + \overset{O(\lambda)}{\underset{\sim}{u}\cdot(\underset{\sim}{\lambda}_{B1} + \underset{\sim}{\lambda}_{B2})}$$

$$+ \overset{O(\lambda)}{\gamma(q/m_0)\underset{\sim}{E}_0\cdot[(\underset{\sim}{u}_\perp/B)(\partial/\partial\mu) + (\underset{\sim}{e}_\alpha/u_\perp)(\partial/\partial\alpha)]}$$

$$+\overset{O(\lambda)}{\gamma\underset{\sim}{v}_E\cdot\underset{\sim}{\nabla}_X} + O(\lambda^2)\ \}\delta G_g\ , \tag{31}$$

where we have indicated the formal ordering above each term. Fourier decomposing the perturbed quantities in the gyrophase angle α, e.g.,

$$\delta G_g = \sum_{\ell=-\infty}^{\infty} \langle\delta G_g\rangle_\ell \exp(-i\ell\alpha)\ , \tag{32}$$

where

$$\langle \delta G_g \rangle_\ell = (1/2\pi) \int_0^{2\pi} d\alpha\ \delta G_g(\underset{\sim}{X},\ \mu,\ \varepsilon,\ \alpha)\ \exp(i\ell\alpha)\ ,$$

and assuming an $\exp(-i\omega t)$ time dependence, we can express Eq. (23) for all ℓ as

$$\langle L_g \delta G_g \rangle_\ell \simeq \langle L_g \rangle_\ell \langle \delta G_g \rangle_\ell = -\langle R_g \rangle_\ell \tag{33}$$

where

$$\langle L_g \rangle_\ell = (\hat{u}_\parallel\ \underset{\sim}{e}_\parallel + \underset{\sim}{u}_D)\cdot\underset{\sim}{\nabla}_X - i(\omega - \ell\Omega + \ell\omega_\alpha)\quad , \tag{34}$$

and

$$\omega_\alpha = \langle \underset{\sim}{u}\cdot\underset{\sim}{\nabla}_x\alpha \rangle_0 = u_\parallel \left[\underset{\sim}{e}_1\cdot(\underset{\sim}{e}_\parallel\cdot\underset{\sim}{\nabla}_x\underset{\sim}{e}_2) - \underset{\sim}{e}_\parallel\cdot(\underset{\sim}{\nabla}_x\times\underset{\sim}{e}_\parallel)/2\right]. \tag{35}$$

Thus, ω_α corresponds to the gyrophase-averaged correction to the cyclotron frequency. Furthermore, in Eq. (33), we have ignored both $O(\lambda^2)$ terms in L_g as well as the cyclotron-harmonic coupling by the same arguments as stated in Paper II. Letting

$$\langle \delta G_g \rangle_\ell = -(q/m_0)\langle \delta\psi_g \rangle_\ell (\partial F_{g0}/B\partial\mu) + \langle \delta H_g \rangle_\ell\ , \tag{36}$$

we can manipulate Eq. (33) into

$$\langle L_g \rangle_\ell \langle \delta H_g \rangle_\ell = -(q/m_0)\left(\langle S_1 \rangle_\ell + \langle S_2 \rangle_\ell\ \delta_{\ell,0}\right)\ , \tag{37}$$

where

$$\langle S_1\rangle_\ell = -i\left[\omega\partial F_{g0}/\partial\varepsilon + \ell\Omega\ \partial F_{g0}/B\partial\mu + i\underset{\sim}{\nabla}_X\ F_{g0}\cdot(\underset{\sim}{e}_\parallel/\Omega)\times\underset{\sim}{\nabla}_X\right]\langle\delta\psi_g\rangle_\ell\ , \tag{38}$$

$$\begin{aligned}\langle S_2\rangle_\ell = \{&u_\parallel\langle\underset{\sim}{e}_\parallel\cdot\underset{\sim}{\nabla}_X\ \delta\psi_g\rangle_\ell - u_\parallel\underset{\sim}{e}_\parallel\cdot\underset{\sim}{\nabla}_X\langle\delta\psi_g\rangle_\ell - \underset{\sim}{u}_D\cdot\underset{\sim}{\nabla}_X\langle\delta\psi_g\rangle_\ell\\ &-\left(\gamma\langle\underset{\sim}{u}_\perp\ \delta\Phi_g\rangle_\ell + \langle\underset{\sim}{u}\cdot\delta\underset{\sim}{A}_g/c\rangle_\ell\ u_\parallel\underset{\sim}{e}_\parallel - u_\parallel\langle\underset{\sim}{u}\delta A_{\parallel g}/c\rangle_\ell\right)\cdot\underset{\sim}{\nabla}_X\ \ell n\ B\\ &+\langle(\underset{\sim}{\nabla}_U\ \tilde{\beta})\cdot\left[\underset{\sim}{\nabla}_X(\underset{\sim}{u}\cdot\delta\underset{\sim}{A}_g/c) - (\underset{\sim}{u}\cdot\underset{\sim}{\nabla}_X\delta\underset{\sim}{A}_g/c) - \gamma\underset{\sim}{\nabla}_X\delta\Phi_g\right]\rangle_\ell\\ &- u_\parallel\langle(\underset{\sim}{u}\cdot\underset{\sim}{\nabla}_X\ \underset{\sim}{e}_\parallel)\cdot\delta\underset{\sim}{A}_g/c\rangle - \langle\delta A_{\parallel g}(\underset{\sim}{u}\cdot\underset{\sim}{\nabla}_X)\underset{\sim}{e}_\parallel\cdot\underset{\sim}{u}/c\rangle_\ell\\ &+ (q/m_0)\left(\underset{\sim}{E}_0\cdot\langle\underset{\sim}{u}\delta\Phi_g\rangle_\ell/c^2 - \gamma E_{0\parallel}\langle\delta A_{\parallel g}\rangle_\ell/c\right) + i\ell\omega_\alpha\langle\delta\psi_g\rangle_\ell\\ &+ \Omega\left[\langle(\tilde{\beta}\delta\Phi_g)'_\alpha\rangle_\ell(\partial/\partial\varepsilon + \gamma\partial/B\partial\mu)\right.\\ &\left.-\langle(\tilde{\beta}u_\parallel\delta A_{\parallel g}/c)'_\alpha\rangle_\ell\ (\partial/B\partial\mu)\right]\}\ (\partial F_{g0}/B\partial\mu)\ ,\end{aligned} \tag{39}$$

and $\delta_{\ell,0}$ is the Kronecker delta. In deriving Eqs. (37) to (39), we have noted that only the $\ell = 0$ component of $\langle S_2\rangle_\ell$ is important; that is, for $\ell \neq 0$, $|\langle S_2\rangle_\ell|/|\langle S_1\rangle_\ell| \sim O(\lambda)$, and we have kept only the leading-order terms. Note also that the last two terms in $\langle S_2\rangle_\ell$ vanish for $\ell = 0$.

Equations (21), (32), (36), and (37) then determine the perturbed distribution function δF_g and, hence, δF. Knowing δF, we can calculate the linear density and current perturbations, which, when coupled with Maxwell's equations, provide a complete description of the linear wave dynamics.

3. RESULTS WITH THE EIKONAL ANSATZ

Since in many practical applications one is interested in the regime where the perpendicular wave length is much shorter than L_0, let us consider the results obtained by adopting the following WKB description for the perturbed quantities:

$$\delta F(\underset{\sim}{x},\underset{\sim}{u}) = \delta\overline{F}(\underset{\sim}{x},\underset{\sim}{u}) \exp\left(i \int^{\underset{\sim}{x}_\perp} \underset{\sim}{k}_\perp \cdot d\underset{\sim}{x}_\perp\right) = \delta\overline{F}(\underset{\sim}{X},\underset{\sim}{U}) \exp\left(i \int^{\underset{\sim}{X}_\perp} \underset{\sim}{k}_\perp \cdot d\underset{\sim}{X}_\perp - i\theta(\underset{\sim}{k}_\perp)\right) , \tag{40}$$

where $\theta(\underset{\sim}{k}_\perp) = \underset{\sim}{k}_\perp \cdot \underset{\sim}{u} \times \underset{\sim}{e}_\parallel/\Omega$, $|k_\perp L_0| >> 1$, and where $\delta\overline{F}$ as well as $\underset{\sim}{k}_\perp$ contain slow spatial variations in $\underset{\sim}{X}_\perp$. Meanwhile, we also have

$$x dF_g(\underset{\sim}{X},\underset{\sim}{U}) = \delta\overline{F}_g(\underset{\sim}{X},\underset{\sim}{U}) \exp\left(i \int^{\underset{\sim}{X}_\perp} \underset{\sim}{k}_\perp \cdot d\underset{\sim}{X}_\perp\right) , \tag{41}$$

and, hence,

$$\delta\overline{F}_g = \delta\overline{F} \exp\left[-i\theta(\underset{\sim}{k}_\perp)\right] . \tag{42}$$

Equation (37) then reduces to

$$\langle L_g\rangle_\ell \langle \delta H_g\rangle_\ell = \left(\hat{u}_\parallel \underset{\sim}{e}_\parallel \cdot \underset{\sim}{\nabla}_X - iQ_\ell\right)\langle \delta\overline{H}_g\rangle_\ell = i(q/m_0)P_\ell(F_{g0})\langle \delta\overline{\psi}_g\rangle_\ell , \tag{43}$$

where

$$Q_\ell = \gamma\omega - \underset{\sim}{k}_\perp \cdot \underset{\sim}{u}_D - \ell(\Omega - \omega_\alpha) , \tag{44}$$

$$P_\ell(F_{g0}) = \left[\omega\partial/\partial\varepsilon + \ell\Omega\ \partial/B\partial\mu + (\underset{\sim}{k}_\perp \times \underset{\sim}{e}_\parallel/\Omega)\cdot\underset{\sim}{\nabla}_X\right]F_{g0} , \tag{45}$$

$$\langle \delta\overline{\psi}_g\rangle_\ell = J_\ell(z)\left[(\gamma - \ell\omega\Omega/k_\perp^2 c^2)\delta\overline{\Phi} - u_\parallel \delta\overline{A}_\parallel/c \right.$$
$$\left. - i(\ell\Omega/k_\perp^2 c)(\underset{\sim}{B}\cdot\underset{\sim}{\nabla}_x)(\delta\overline{A}_\parallel/B)\right] - u_\perp J_\ell'(z)\delta\overline{B}_\parallel/k_\perp c , \tag{46}$$

J_ℓ is the Bessel function, $z = |k_\perp u_\perp/\Omega|$, $J'_\ell = dJ_\ell(z)/dz$, and $\langle \overline{S}_2\rangle_\ell$ is of higher order and ignorable here. Furthermore, the Lorentz gauge, $\underset{\sim}{\nabla}_x \cdot \delta\underset{\sim}{A} + \frac{1}{c}\frac{\partial}{\partial t}\delta\Phi = 0$ is adopted here. From Eq. (43), $\langle \delta\overline{H}_g\rangle_\ell$ may be obtained by integrating along the magnetic field line and will be formally written as

$$\langle \delta\overline{H}_g\rangle_\ell = i(q/m_0)P_\ell(F_{g0})\langle \overline{L}_g\rangle^{-1}_\ell \langle \delta\overline{\psi}_g\rangle_\ell \quad . \tag{47}$$

The perturbed distribution function in $(\underset{\sim}{X},\underset{\sim}{U})$ phase space, $\delta\overline{F}$, is then given by

$$\delta\overline{F} = \frac{q}{m_0}\Big[\delta\overline{\Phi}\,\frac{\partial}{\partial\varepsilon} + \Big(\gamma\delta\overline{\Phi} - \frac{u_\parallel \delta\overline{A}_\parallel}{c}\Big)\frac{1}{B}\,\frac{\partial}{\partial\mu}\Big]F_0 + \sum_\ell \langle \delta\overline{G}_g\rangle_\ell \exp[i\theta(\underset{\sim}{k}_\perp) - i\ell\alpha] \ , \tag{48}$$

and

$$\langle \delta\overline{G}\rangle_\ell = -(q/m_0)\langle \delta\overline{\psi}_g\rangle_\ell(\partial F_{g0}/B\partial\mu) + \langle \delta\overline{H}_g\rangle_\ell \quad . \tag{49}$$

Here, in Eq. (48), we have neglected the $(\delta\underset{\sim}{A} \times \underset{\sim}{e}_\parallel/c\Omega)\cdot\underset{\sim}{\nabla}_x F_0$ term [c.f. Eq. (22)] which can be shown to be generally of higher order. With $\delta\overline{F}$ given by Eq. (48), the corresponding Maxwell's equations can then be expressed as

$$\Big(k^2 - \frac{\omega^2}{c^2}\Big)\delta\overline{\Phi} = 8\pi^2\sum_j q \int \frac{\gamma B d\mu d\varepsilon}{|u_\parallel|} \Big\{\frac{q}{m_0}\Big[\delta\overline{\Phi}\,\frac{\partial F_0}{\partial\varepsilon} + \Big(\gamma\delta\overline{\Phi} - \frac{u_\parallel \delta\overline{A}_\parallel}{c}\Big)\frac{\partial F_0}{B\partial\mu}\Big]$$

$$+ \sum_\ell \langle \delta\overline{G}_g\rangle_\ell J_\ell(z)\Big\} \ , \tag{50}$$

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\delta\bar{A}_\parallel = \frac{8\pi^2}{c} \sum_j q \int \frac{Bd\mu d\varepsilon}{|u_\parallel|} u_\parallel \left[\frac{q}{m_0}\left(\gamma\delta\bar{\Phi} - \frac{u_\parallel \delta\bar{A}_\parallel}{c}\right)\frac{\partial F_0}{B\partial\mu} + \sum_\ell \langle\delta\bar{G}_g\rangle_\ell \, J_\ell(z)\right] , \quad (51)$$

and

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\delta\bar{B}_\parallel = \frac{8\pi^2}{c} k_\perp \sum_j q \int \frac{Bd\mu d\varepsilon}{|u_\parallel|} u_\perp \sum_\ell \langle\delta\bar{G}_g\rangle_\ell \, J'_\ell(z) , \quad (52)$$

where

$$k^2 \equiv k_\perp^2 - (\underset{\sim}{B}\cdot\underset{\sim}{\nabla}_x)(B^{-1}\, \underset{\sim}{e}_\parallel \cdot \underset{\sim}{\nabla}_x) , \quad (53)$$

j is the species index, and a sum over $\pm u_\parallel$ is understood in the velocity integration. Equations (47) and (49)-(52) thus provide a complete description of the linear wave properties within the framework of the eikonal ansatz. There are new channels for wave-particle resonances due to the velocity-dependent coefficient, γ, in $\langle\bar{L}_g\rangle_\ell$ in addition to those mentioned in Papers I and II.

We can also evaluate the $\langle\delta\bar{H}_g\rangle_\ell$ for both trapped and untrapped particles in axisymmetric tokamaks with long parallel wave-lengths $[k_\parallel/k_\perp \lesssim O(\lambda)]$ and the ballooning-mode representation (Connor, Hastie and Taylor, 1980; Frieman et al., 1980; Pegoraro and Schep, 1981). Results similar to those presented in Section 4 of Paper II are then obtained, except with the following essential modifications: (i) change all $\underset{\sim}{v}$'s into $\underset{\sim}{u}$'s, e.g. $v_\parallel \to u_\parallel$, $v_\perp \to u_\perp$, and $\underset{\sim}{v}_D \to \underset{\sim}{u}_D$; and (ii) redefine

$$(I_\ell)_a^b = \int_a^b d\eta (JB/|u_\parallel|)(\gamma\omega - \ell\Omega + \ell\omega_\alpha - \underset{\sim}{k}_\perp \cdot \underset{\sim}{u}_D) \quad , \tag{54}$$

$$\langle \delta\overline{\psi}_g' \rangle_\ell = \left[(\gamma - \ell\omega\Omega/k_\perp^2 c^2)\delta\overline{\Phi} - u_\parallel \delta\overline{A}_\parallel / c\right] J_\ell(z') - u_\perp J_\ell'(z') \delta\overline{B}_\parallel / k_\perp c \quad , \tag{55}$$

$$\langle \delta\overline{\psi}_g' \rangle_{\ell 1} = (\gamma - \ell\omega\Omega/k_\perp^2 c^2 J_\ell(z') \delta\overline{\Phi} - u_\perp J_\ell'(z') \delta\overline{B}_\parallel / k_\perp c \quad , \tag{56}$$

$$\langle \delta\overline{\psi}_g' \rangle_{\ell 2} = J_\ell(z') |u_\parallel| \delta\overline{A}_\parallel / c \quad . \tag{57}$$

4. Conclusion and Discussion

We have generalized the systematic gyrokinetic formalism developed in Papers I and II to include full relativistic effects. The theory is valid for arbitrary frequencies and magnetic field configurations. Both the plasma and magnetic field inhomogeneities and finite Larmor radii effects are retained. The equation is greatly simplified when the WKB eikonal ansatz is imposed. The perturbed distributions for the circulating, as well as trapped, particles in the axisymmetric tokamaks are obtained with the ballooning mode representation. In addition to the proper gyrokinetic equation now being expressed in terms of the momentum $\underset{\sim}{p} = m_0 \underset{\sim}{u}$ instead of the velocity $\underset{\sim}{v}$, the other necessary changes in the formalism were also generated by using the relativistic Vlasov equation and the Lorentz gauge. This new relativistic gyrokinetic equation has already found application in the theoretical interpretation of stability results observed in a recent relativistic electromagnetic particle simulation of the Elmo Bumpy Torus (Ohsawa et al., 1983).

During the final writing of this paper, it brought to our attention that recently Dr. Littlejohn (1983) has also worked on the relativistic gyrokinetic equation. However, he is only interested in low-frequency perturbations, using Hamiltonian formalism and imposing the WKB eikonal ansatz from the beginning.

Acknowledgment

This work was supported by the United States Department of Energy under Contract DE-FG05-80ET-53088 and Contract DE-AC02-76-CH03073.

REFERENCES

Berk H. L. , Van Dam J. W. , Rosenbluth M. N. , and Spong D. A. (1983) Physics of Fluids 26, 201.

Chen L. and Tsai S. T. (1983a) Physics of Fluids 26, 141; (1983b) Plasma Physics 25, 347.

Connor J. W. , Hastie R. J. , and Taylor J. B. (1980) Plasma Physics 22, 757.

Frieman E. A. , Rewoldt G. , Tang W. M. , and Glasser A. H. (1980) Physics of Fluids 23, 1750.

Littlejohn R. G. (1983) University of California at Los Angeles Report #709.

Ohsawa Y., Dawson J. N., and Van Dam J. W. (1983), University of California at Los Angeles Report #712, submitted to Physics of Fluids.

Pegoraro F. and Schep J. J. (1981) Physics of Fluids 24, 478.

Quon B. H. , Dandl R. A. , DiVergilio W. F. , Guest G. E. , Lao L. L., Lazar A. H., Samec T. K., and Wuerker R. F. (1982) TRW Report #82-STM-100 (to be published in Physics of Fluids).

Rosenbluth M. N. , Tsai S. T., Van Dam J. W. , and Engquist M. G. (1983), Institute for Fusion Studies Report #98 (submitted to Phys. Rev. Lett.).

Rutherford P. H. and Frieman E. A. (1968) Physics of Fluids 11, 569.

Taylor J. B. and Hastie R. J. (1968) Plasma Physics 10, 479.