Dynamical Symmetries and Sub-recoil Laser Cooling

W. F. Buell[†]

Department of Physics
State University of New York at Stony Brook
Stony Brook, NY 11794–3800

B. A. Shadwick[‡]

Physics Department
The University of California at Berkeley
Berkeley, CA 94720–7300

[†] Supported by National Science Foundation and the Office of Naval Research.

[‡] Supported by the U. S. DoE under contract No. DE-FG05-80ET-53088 and in part the U. S. DoE, High Energy Physics Division.

Abstract

We consider an n-level quantum system with Hamiltonian H and density matrix ρ where the dynamics are generated by the Liouville equation. The n^2-dimensional vector space $\mathcal{V}$ on which H and ρ act is spanned by a representation of the Lie algebra $u(n)$. Dynamical symmetries are manifest in a Hamiltonian living in a subspace of $\mathcal{V}$ which is spanned by a subalgebra of $u(n)$. This gives rise to certain constants of motion, the exact nature of which depends on the structure of the subalgebra. We have developed a procedure to automate the generation of maximal sub-algebra chains of $u(n)$ and the detection of dynamical symmetries and their corresponding constants of motion. By way of example, we consider several configurations for sub-recoil laser cooling by means of velocity selective coherent population trapping, (VSCPT) in which the states of the system are products of internal atomic states and external momentum states. In particular we

find that for certain values of the family momentum p, the Hamiltonian exhibits additional dynamical symmetry, leading to additional constants of motion, to which we attribute the coherent population trapping in the velocity selective dark state.

I. Bloch Sphere

The Bloch sphere is a well known way of representing the dynamics of a two level atom[1] with dipole moment $\boldsymbol{d}$ interacting with a near resonant electromagnetic field $\boldsymbol{E}$. The pseudospin vector $\boldsymbol{S} = (u, v, w)$ has components $S_i = \langle \sigma_i \rangle$, where the σ_i are the usual Pauli spin matrices:

$$
\begin{aligned}
\sigma_1 &= |1\rangle\langle 2| + |2\rangle\langle 1| \,, \\
\sigma_2 &= i\left(|1\rangle\langle 2| - |2\rangle\langle 1|\right) \,, \\
\sigma_3 &= -\left(|1\rangle\langle 1| - |2\rangle\langle 2|\right) \,,
\end{aligned}
\tag{1}
$$

and satisfy the commutaion relations

$$
\left[\sigma_i \,, \sigma_j\right] = 2i\,\epsilon_{ijk}\,\sigma_k \,. \tag{2}
$$

In the electric dipole and rotating wave approximations, with $\Delta \equiv \omega - \omega_0$ the detuning from exact resonance, and Rabi frequency $\Omega \equiv \boldsymbol{d} \cdot \boldsymbol{E}/2\hbar$, the dynamics of the pseudospin vector take the form[2] of precession about an effective "magnetic field," or torque vector $\boldsymbol{\Gamma} = (\Omega, 0, \Delta)$:

$$\boxed{\begin{aligned} i\hbar\dot{\boldsymbol{S}} &= [\boldsymbol{S}\,, H] \\ \dot{S}_i &= \epsilon_{ijk}\Gamma_j S_k\,. \end{aligned}} \tag{3}$$

It is in terms of the motion of this pseudospin vector that the well known phenomena of π & $\pi/2$ pulses, photon echo, Ramsey fringes, *etc.* are so easily visualized. (See figure 1.)

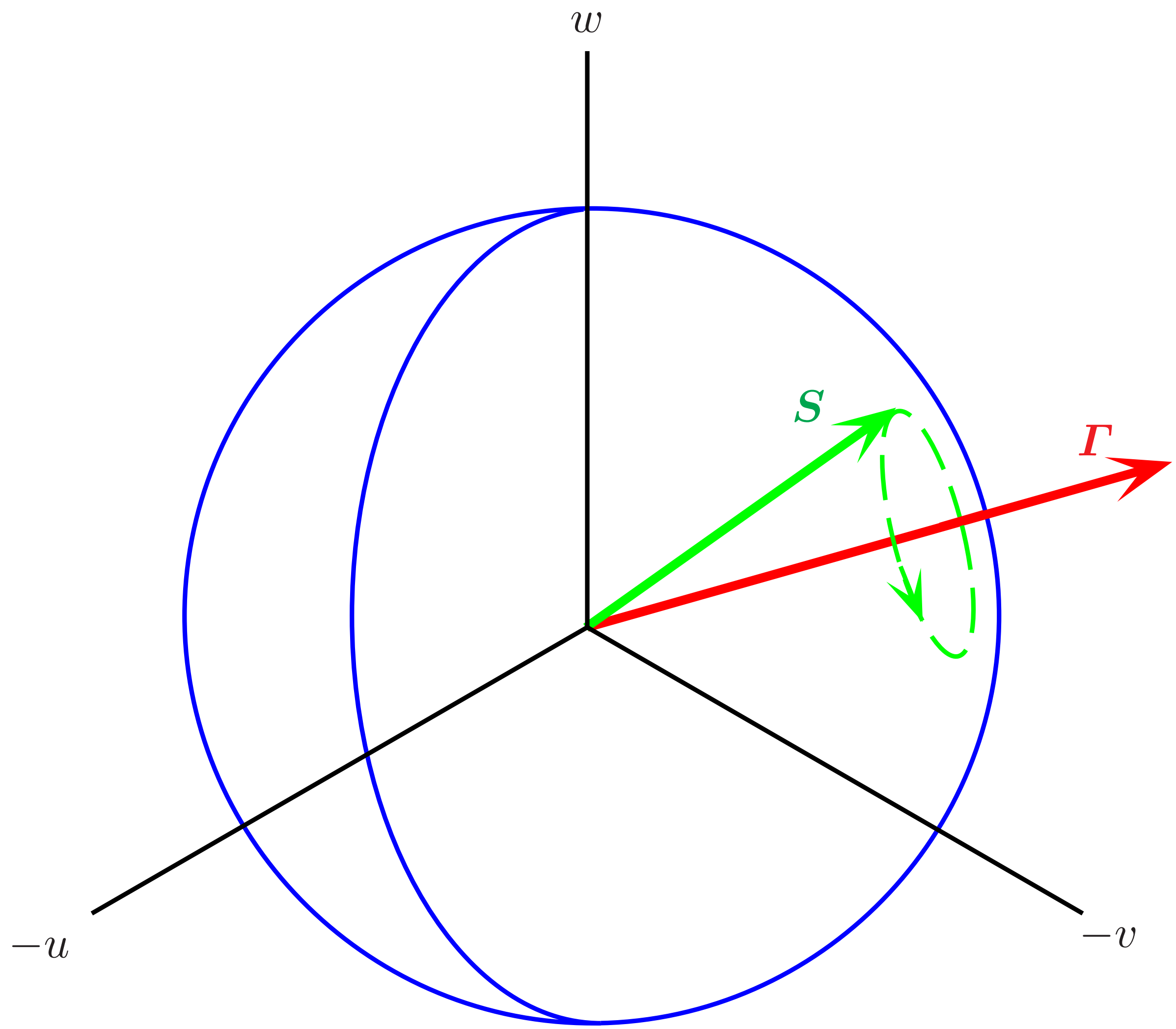


Figure 1: The two-level atom Bloch sphere. The axes u, v and w represent the expectation values $\langle\sigma_i\rangle$ of the Pauli spin matrices. The pseudospin vector $\boldsymbol{S}$ precesses about the torque vector $\boldsymbol{\Gamma}$, determined by the Hamiltonian of the system.

II. Introduction to Lie Algebra

Before discussing n-level systems, we review some of the basic concepts of the theory of Lie algebras. A Lie algebra $\mathfrak{g}$ is an $n(<\infty)$ dimensional complex linear vector space which is closed under the under the Lie product, $[\ ,\ ]$. This product satisfies the following

- Antisymmetry:

$$[x\,,\,y] = -[y\,,\,x]\,, \quad \forall\, x, y \in \mathfrak{g} \tag{4}$$

- Linearity:

$$[a\,x + b\,y\,,\,z] = a\,[x\,,\,z] + b\,[y\,,\,z]\,, \quad \forall\, a, b \in \mathbb{C}; \forall\, x, y, z \in \mathfrak{g} \tag{5}$$

- Jacobi Identity:

$$[x\,,\,[y\,,\,z]] + [y\,,\,[z\,,\,x]] + [z\,,\,[x\,,\,y]] = 0\,, \quad \forall\, x, y, z \in \mathfrak{g}\,. \tag{6}$$

Let $\{x_i\}$ be a basis for the vector space of $\mathfrak{g}$. Since $\mathfrak{g}$ is closed under the Lie product, we can write

$$[x_i\,,x_j] = \sum_{k=1}^{n} c_{ijk}\,x_k\,, \qquad i,j = 1,2,\ldots n\,, \tag{7}$$

where the constants c_{ijk} are called the structure constants for the algebra. These constants determine all of the properties of $\mathfrak{g}$.

An important concept for discussing dynamical symmetries is that of a subalgebra. A (proper) subalgebra, $\mathfrak{g}'$, of $\mathfrak{g}$ is a vector space of dimension less than that of $\mathfrak{g}$ that is closed under the Lie product. Thus, $\mathfrak{g}' \subset \mathfrak{g}$ if and only if

$$\boxed{[x\,,y] \in \mathfrak{g}'\,, \qquad \forall\, x,y \in \mathfrak{g}'\,.} \tag{8}$$

Of the many representations of abstract Lie algebras, the most relevant for studying the dynamics of the Liouville equation are the matrix representations. For every Lie algebra, $\mathfrak{g}$, there exists a one-to-one correspondence between the elements of $\mathfrak{g}$ and the elements of the set of $m \times m$ matrices where the Lie product is represented by the matrix commutator. Thus for $x, y \in \mathfrak{g}$

$$x \mapsto M_x, y \mapsto M_y \text{ and } [x, y] \mapsto M_x M_y - M_y M_x , \tag{9}$$

where M_x and M_y are $m \times m$ matrices. The Killing form provides a natural inner product on the matrix representation:

$$\boxed{(M_x\, , M_y) \equiv \frac{1}{2}\, \mathrm{tr}\, M_x M_y\, .} \tag{10}$$

Of particular interest is the algebra associated with the set of all traceless $n \times n$ Hermitian matrices Let $\mathcal{H}(n)$ denote this set. Notice that $\mathcal{H}(n)$ is a vector space, and it is easily seen that

$$[x\,,\,y] \subset \mathcal{H}(n)\,, \quad \forall x, y \in \mathcal{H}(n)\,. \tag{11}$$

Let $\{\lambda_i\}_{i=1}^{n}$ be a basis for $\mathcal{H}(n)$ then we can write

$$\boxed{[\lambda_i\,,\,\lambda_j] = 2i \sum_k f_{ijk}\,\lambda_k\,.} \tag{12}$$

This is the algebra $su(n)$.

We can choose this basis such that

$$\boxed{(\lambda_i\,,\,\lambda_j) = \delta_{ij}\,.} \tag{13}$$

Since $\{\lambda_i\}$ is a basis for $\mathcal{H}(n)$, any matrix $a \in \mathcal{H}(n)$ can be written as

$$a = \sum_{i=1}^{n} \alpha_i \lambda_i \,, \tag{14}$$

where

$$\alpha_i = \tfrac{1}{2} \operatorname{tr} a \lambda_i \,. \tag{15}$$

If we augment the set $\{\lambda_i\}$ with the identity matrix we obtain a basis for the space of all Hermitian $n \times n$ matrices. This corresponds to the algebra $u(n)$. Since the identity matrix commutes with all the other elements of the basis, the properties of $u(n)$ are only trivially different from those of $su(n)$.

III. Bloch Hypersphere for n-level systems

With the use of the concepts in section II, the Bloch sphere picture generalizes in a natural way to the case of multilevel systems interacting with polychromatic fields.[3] The density matrix ρ and Hamiltonian H (which we take to be traceless) are $n \times n$ Hermitian matrices which we can expand as

$$\boxed{\begin{aligned} \rho &= \frac{1}{n}\mathbf{I} + \frac{1}{2}S_j\,\lambda_j\,, \\ H &= \frac{\hbar}{2}\Gamma_j\,\lambda_j\,. \end{aligned}} \tag{16}$$

The λ_j are as in (12). The coefficients S_j and Γ_j in the expansions above are the n-level pseudospin and "torque" vectors, and are evidently

$$\boxed{\begin{aligned} S_j &= \mathrm{tr}(\rho\,\lambda_j) = 2(\rho\,,\lambda_j)\,, \\ \hbar\,\Gamma_j &= \mathrm{tr}(H\,\lambda_j) = 2(H\,,\lambda_j)\,, \end{aligned}} \tag{17}$$

That is,

$$\boldsymbol{S} = \Big(\{u_{jk}\},\{v_{jk}\},\{w_l\}\Big)\,, \tag{18}$$

where,

$$\begin{aligned} u_{jk} &= |j\rangle\langle k| + |k\rangle\langle j|\,, \\ v_{jk} &= i\,(|j\rangle\langle k| - |k\rangle\langle j|)\,, \\ w_l &= -\sqrt{\frac{2}{l(l+1)}}\left(\underbrace{|1\rangle\langle 1| + \cdots + |l\rangle\langle l| - l|l+1\rangle\langle l+1|}_{n^2-1}\right)\,, \end{aligned} \tag{19}$$

and

$$\boldsymbol{\Gamma} = \left(-\Omega_1, \ldots, -\Omega_{n-1}, 0, \ldots, 0, \Delta_1, \ldots, \Delta_{n-1}\right) . \tag{20}$$

The dynamics of the system are generated by the Liouville equation

$$\boxed{i\hbar\,\dot{\rho} = [H\,,\,\rho].} \tag{21}$$

From this and (16) it easy to show that the n-level pseudospin vector dynamics are described by a generalized precession equation:

$$\boxed{\dot{S}_i = f_{ijk}\,\Gamma_j\,S_k\,,} \tag{22}$$

in exactly the same manner as the two-level system.

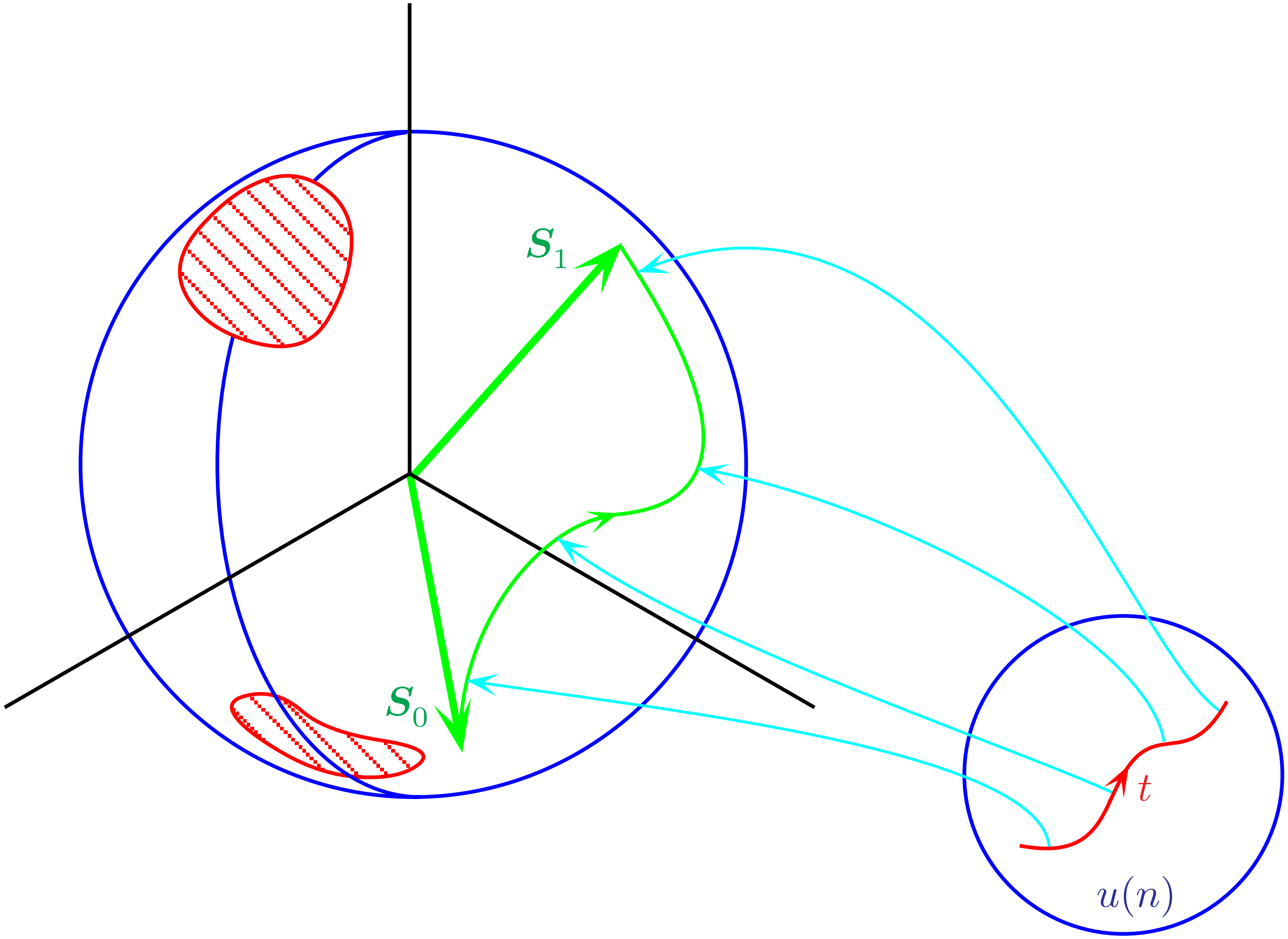


Figure 2: Dynamics on the n-level Bloch hypersphere. Hamiltonians generate dynamical trajectories in $u(n)$, which map onto paths on the $n^2 - 1$ dimensional hypersphere representing states of the system. There may exist patches on the sphere which are dynamically inaccessible from certain initial states.

IV. Trace Invariants and Dynamical Accessibility

In multilevel systems, there exist constants of motion in addition to the energy, $\langle H \rangle$, and the total population, $\mathrm{tr}\,\rho = 1$. Consider any scalar function of the density matrix $f(\rho)$. It follows directly from the Liouville equation and the cyclic properties of the trace that $c_f \equiv \mathrm{tr}\, f(\rho)$ is a constant of the motion:

$$\begin{aligned} \dot{c}_f &= \mathrm{tr}\left(f'(\rho)\dot{\rho}\right) , \\ &= \frac{1}{i\hbar}\,\mathrm{tr}\left(f' H\rho - f'\rho\, H\right) , \\ &= 0\,. \end{aligned} \tag{23}$$

It is then natural to ask how many of these constants are functionally independent. One way to answer this is to consider a representation where H is diagonal. In such a representation the diagonal elements of the density matrix do not evolve in time; thus there exist n such functionally independent constants of motion. One natural way[3] to enumerate these constants is as

$$\boxed{c_i = \operatorname{tr} \rho^i ; \qquad i = 1, \ldots n \, .} \tag{24}$$

(Since $\operatorname{tr} \rho$ does not evolve, we consider in what follows only the traceless part of the density matrix.) One consequence of the existence of such conserved quantities is the phenomenon of coherent population trapping. For example[3], one can show that for a three level (Λ) system it is not possible to transfer 100% of the population from an incoherent mixture of levels $|1\rangle$ and $|2\rangle$ to level $|3\rangle$ by Hamiltonian evolution. In this case such population trapping is a consequence of $\mathrm{tr}\,\rho^2 = \text{const}$. Such behavior is well known in classical mechanics, where it goes by the name of dynamical accessibility.[4] The importance of this behavior is that some states of the system may simply not be connected to other states by Hamiltonian evolution. (Since the density matrix for a pure state is a projection operator $\rho^2 = \rho$, any pure state is dynamically accessibly from any other pure state.) We are currently investigating the relevenace of such questions of dynamical accessibility to the fields of coherent control and quantum state engineering.

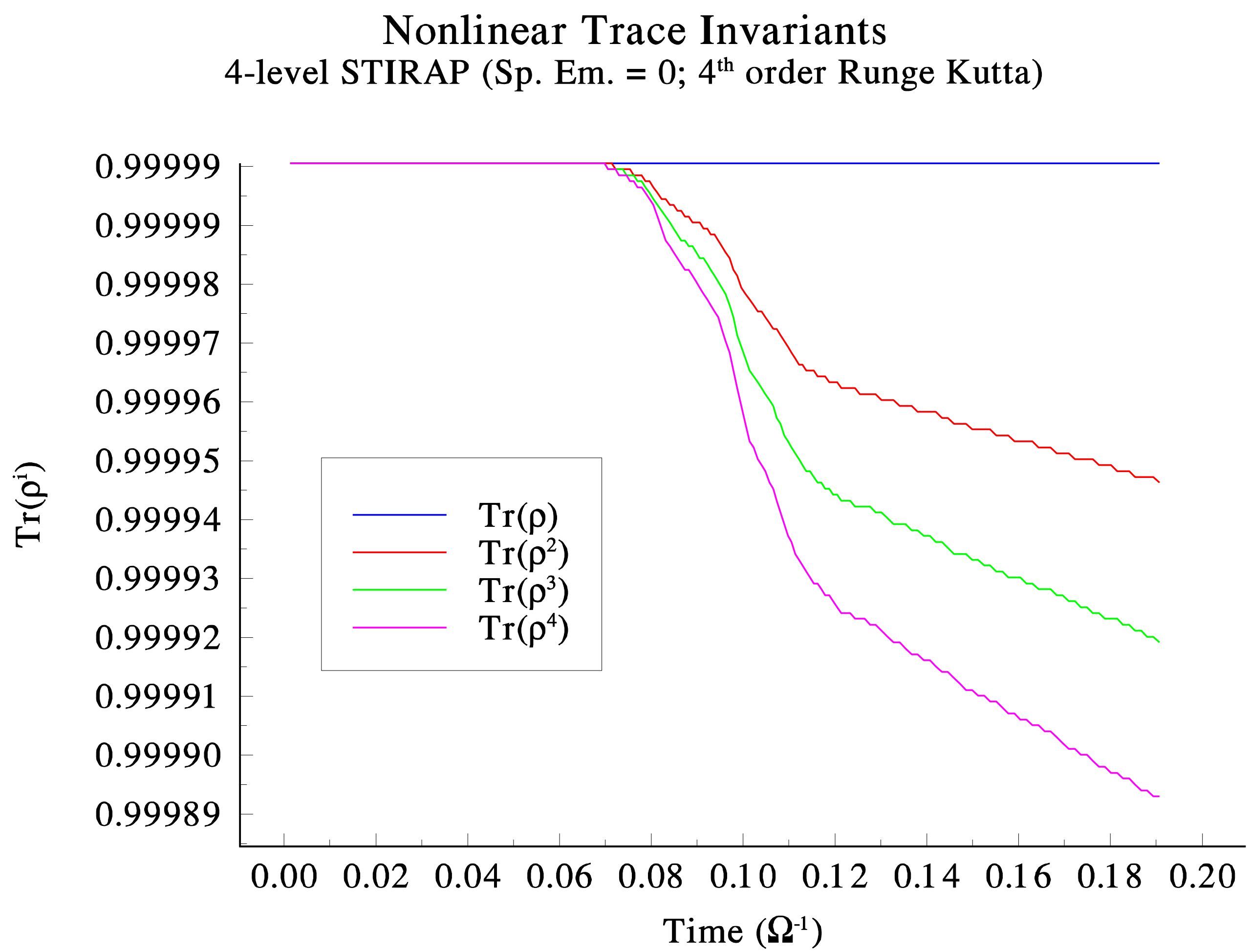


Figure 3: Example of nonconservation of $\mathrm{tr}\,\rho^i$ invariants with conventional numerical methods. Here we have solved the Optical Bloch Equations for a 4-level, 3-laser STIRAP-like excitation of Li Rydberg atoms.[5] Although $\mathrm{tr}\,\rho$ is conserved, the higher order invariants show some loss as the evolution progresses. We are investigating the application of exactly conservative integrator techniques to address this situation.

V. Dynamical Symmetries

Consider a general n-dimensional traceless Hamiltonian matrix H. Using the basis of $su(n)$ we can write

$$H = \sum_i h_i \lambda_i \,. \tag{25}$$

For a generic Hamiltonian, one would expect that $h_i \neq 0$, $\forall i$. Let $\mathfrak{g}'$ be a sub-algebra of $su(n)$. Denote the basis of $\mathfrak{g}'$ by $\{\lambda_a\}$ and consider a Hamiltonian that can be expressed using only those basis vectors belonging to $\mathfrak{g}'$; that is consider

$$H = \sum_a h_a \lambda_a \,. \tag{26}$$

To see how this structure affects the dynamics generated by the Liouville equation we decompose the traceless part of the density matrix as

$$\rho = \sum_a \rho_a \lambda_a + \sum_\mu \rho_\mu \lambda_\mu , \tag{27}$$

where $\{\lambda_\mu\}$ denotes those basis vectors of $su(n)$ not contained in $\mathfrak{g}'$. Now

$$\begin{aligned} \dot{\rho} &= i\hbar \sum_{a,b} \rho_a h_b [\lambda_a , \lambda_b] + i\hbar \sum_{\mu,b} \rho_\mu h_b [\lambda_\mu , \lambda_b] \\ &= \sum_i \dot{\rho}_i \lambda_i . \end{aligned} \tag{28}$$

Since $\mathfrak{g}'$ is a subalgebra, $[\lambda_a\,, \lambda_b] \in \mathfrak{g}'$ and we have

$$\boxed{\begin{aligned} \dot{\rho}_c &= 2\hbar \sum_{ab} \rho_a\, h_b\, f_{bac}\,, \\ \dot{\rho}_\nu &= 2\hbar \sum_{\mu b} \rho_\mu\, h_b\, f_{b\mu\nu}\,, \end{aligned}} \tag{29}$$

from which we see that the dynamics decouple. The projections of ρ onto $\{\lambda_a\}$ evolve independently of the projections of ρ onto $\{\lambda_\mu\}$. Notice that this decoupling depends critically on $\mathfrak{g}'$ being a subalgebra. The dynamics of ρ_a can be analysed without considering ρ_μ.

By a dynamical symmetry we mean a special structure ("symmetry") of the Hamiltonian that leads to this decoupling of the dynamics of ρ. To see how this leads to constants of motion, suppose that $\mathfrak{g}' = su(m)$, $m < n$.

Then in addition to the n constants $\operatorname{tr}\rho^i$, $i = 1 \ldots n$, there will be m constants $\operatorname{tr}\widehat{\rho}^i$, $i = 1 \ldots m$ where

$$\widehat{\rho} = \sum_a \rho_a \lambda_a \, . \tag{30}$$

In general these constants will be functionally independent of the trace invariants of the full system. The same general idea holds when $\mathfrak{g}'$ is some algebra other than $su(m)$.

VI. A Procedure for Determining Dynamical Symmetries

The problem of finding the dynamical symmetries of given is somewhat subtle; it is not enough to simply check for each subalgebra whether the Hamiltonian can be expanded using only the basis vectors of that subalgebra. This subtly is made clear by applying a unitary transformation to the basis vectors of $su(n)$. Let

$$\lambda_i' = U^\dagger \lambda_i U \,, \qquad i = 1, 2, \ldots, n^2 \,. \tag{31}$$

Now

$$[\lambda_i' \,, \lambda_j'] = U^\dagger [\lambda_i \,, \lambda_j] U \; = 2i \, f_{ijk} \lambda_k' \,, \tag{32}$$

and

$$h_i' \equiv (H \,, \lambda_i') = (U \, H \, U^\dagger \,, \lambda_i) = \beta_{ij} \, (H \,, \lambda_j) \,, \tag{33}$$

where

$$\beta_{ij} = (\lambda'_i\,, \lambda_j)\,. \tag{34}$$

Thus $\{\lambda'_j\}$ is also a representation of $su(n)$ but the expansion coefficients of H are different. Since both H and $U\,H\,U^\dagger$ generate the same dynamics, we must not only examine the given Hamiltonian but all unitarily equivalent Hamiltonians.

Fortunately there is a systematic method for performing an exhaustive search for dynamical symmetries of a given Hamiltonian H. One begins by constructing a list of maximal subalgebras of the dynamical algebra of the Hamiltonian. By examining only maximal subalgebras one can perform the full search in a recursive manner which tremendously reduces the actual number of subalgebras that need to be analyzed. (This is important as the total number of subalgebras of $su(n)$ grows better than exponentially in n.)

For each maximal subalgebra, the most general Hamiltonian, H_{SA}, is constructed using the basis of the subalgebra and its characteristic polynomial is computed. This polynomial is compared with the characteristic polynomial of H. If it is possible to make these polynomials agree for some choice of the expansion coefficients of H_{SA} then we know that H and H_{SA} are related by a unitary transformation and that H has a dynamical symmetry corresponding the maximal subalgebra used to construct H_{SA}. The process is continued by taking the subalgebra just found and computing all of its maximal subalgebras looking for additional symmetries. In this way one "descends" a chain of subalgebras until no further symmetry of the Hamiltonian is present.

In essence what we are doing is classifying all $n \times n$ (traceless) Hermitian matrices modulo their spectral characteristics. This is precisely what one would expect; the constants of motion that arise due to the dynamical symmetries are a consequence of various (potentially complex) relationships between the eigenvalues of the Hamiltonian. The most general n-dimensional Hamiltonian will have unrelated eigenvalues and will (obviously) admit no dynamical symmetry.

VII. Example

Consider a 3-level system with Hamiltonian

$$H = \begin{pmatrix} \frac{1}{3}\,d & -a & 0 \\ -a & -\frac{2}{3}\,d & -b \\ 0 & -b & \frac{1}{3}\,d \end{pmatrix} \tag{35}$$

The characteristic polynomial for H is

$$w^3 - \left(a^2 + b^2 + \frac{1}{3}\,d^2\right) w + \frac{1}{3}\,a^2 d \; + \frac{1}{3}\,b^2 d + \frac{2}{27}\,d^3\,. \tag{36}$$

The algebra $su(3)$ has a two maximal subalgebras, $u(2) = su(2) \oplus u(1)$ and $su(2)$. A basis for the maximal $su(2)$ subaglebra is

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \tag{37}$$

and the general form of the Hamiltonian is

$$H_{su(2)} = \begin{pmatrix} 0 & h_{12} & h_{13} \\ h_{12} & 0 & i\,h_{23} \\ h_{13} & -i\,h_{23} & 0 \end{pmatrix}, \tag{38}$$

where h_{12}, h_{13} and h_{23} are real. The characteristic polynomial of $H_{su(2)}$ is

$$w^2 - \left(h_{12}^2 + h_{13}^2 + h_{23}^2\right) w\,, \tag{39}$$

which implies that H exhibits the $su(2)$ symmetry only for $d = 0$.

We take as our basis vectors for $u(2)$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{40}$$

A generic Hamiltonian for this subalgebra is

$$H_{u(2)} = \begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{12} & h_{11} & 0 \\ 0 & 0 & 2h_{11} \end{pmatrix}, \tag{41}$$

where h_{11} and h_{12} are real. Equating the characteristic polynomial for H and $H_{u(2)}$:

$$w^3 - \left(a^2 + b^2 + \frac{1}{3}\,d^2\right) w + \frac{1}{3}\,a^2 d \; + \frac{1}{3}\,b^2 d + \frac{2}{27}\,d^3$$
$$= w^3 - \left(3h_{11}^2 + h_{12}^2\right) w + 2h_{11}^3 - 2h_{11}\,h_{12}^2\,, \quad (42)$$

we obtain equations for the free parameters h_{11} and h_{12}

This equation has many solutions, one solution is

$$h_{11} = -\frac{d}{6} \quad \text{and} \quad h_{12} = \frac{1}{2}\sqrt{4a^2 + 4b^2 + d^2} \qquad (43)$$

Hence there exists a unitary transformation U such that $U\,H\,U^\dagger$ has the form of $H_{u(2)}$.

Since $u(2) = su(2) \oplus u(1)$ is a direct sum subalgebra, the dynamics generated by $H_{u(2)}$ factor (trivially) once again. The $su(2)$ part is spanned by $\{\lambda_1, \lambda_2, \lambda_3\}$ while the $u(1)$ part is spanned by $\{\lambda_4\}$. The direct sum structure of $u(2)$ means that λ_4 commutes with all the other basis vectors. Thus

$$[\lambda_4\,, H_{u(2)}] = 0\,, \tag{44}$$

and we see that the projection of ρ into λ_4 is a constant of motion. The remaining conservation laws are given by

$$\mathrm{tr}\,\rho_s \qquad \text{and} \qquad \mathrm{tr}\,\rho_s^2 \tag{45}$$

where

$$\rho_s = \sum_{i=1}^{3} (\rho\,, \lambda_i)\,\lambda_i\,. \tag{46}$$

Since we have taken ρ to be traceless, only the second of these gives a nontrivial result. Explicitly we have

$$\boxed{\begin{aligned} c_1 &= (\rho\,,\lambda_4) = \frac{\sqrt{3}}{2}\left(\rho_{11}+\rho_{22}\right), \\ c_2 &= \operatorname{tr}\rho_s^2 = \frac{1}{2}\left(\rho_{11}^2+\rho_{22}^2\right) + 2\rho_{12}\,\rho_{21} - \rho_{11}\,\rho_{22}\,. \end{aligned}} \tag{47}$$

From these expressions we can extract simpler constants of motion:

$$\boxed{\begin{aligned} k_1 &= \rho_{11}+\rho_{22}, \\ k_2 &= \rho_{12}\,\rho_{21} - \rho_{11}\,\rho_{22}\,. \end{aligned}} \tag{48}$$

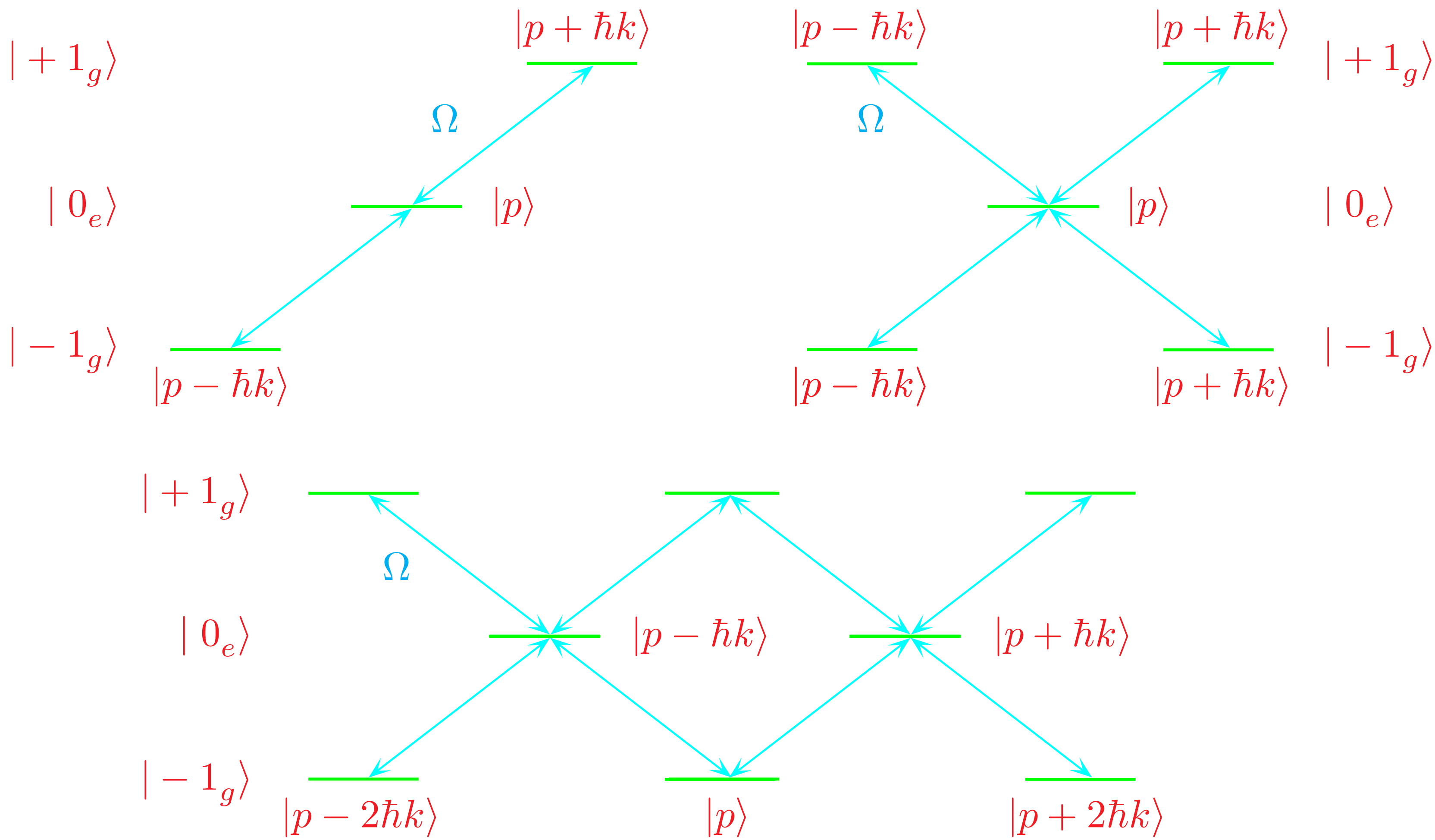
To obtain the form of the constants of motion in the basis of H we replace the elements of ρ in the above with the corresponding elements of $U\rho U^\dagger$.

Where U is the unitary matrix that brings H into the form of $H_{u(2)}$. There are many such matrices; for example

$$U = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & -\cos\theta \end{pmatrix}, \tag{49}$$

where $\tan\theta = a/b$. Using this transformation, our conservation laws become

$$\boxed{\begin{aligned} k_1 &= \left(a^2 - b^2\right)\rho_{11} + a^2\,\rho_{22} + a\,b\left(\rho_{13} + \rho_{31}\right), \\ k_2 &= a^2\left(\rho_{12}\,\rho_{21} - \rho_{11}\,\rho_{22}\right) + b^2\left(\rho_{11}\,\rho_{22} + \rho_{22}^2 + \rho_{23}\,\rho_{32}\right) \\ &\quad + a\,b\left(\rho_{13}\,\rho_{22} + \rho_{12}\,\rho_{23} - \rho_{22}\,\rho_{31} + \rho_{21}\,\rho_{32}\right). \end{aligned}} \tag{50}$$



$|p+\hbar k\rangle$
$|p-\hbar k\rangle$
$|p+\hbar k\rangle$
$|+1_g\rangle$
$|+1_g\rangle$
Ω
Ω
$|0_e\rangle$
$|p\rangle$
$|p\rangle$
$|0_e\rangle$
$|-1_g\rangle$
$|-1_g\rangle$
$|p-\hbar k\rangle$
$|p-\hbar k\rangle$
$|p+\hbar k\rangle$
$|+1_g\rangle$
Ω
$|0_e\rangle$
$|p-\hbar k\rangle$
$|p+\hbar k\rangle$
$|-1_g\rangle$
$|p-2\hbar k\rangle$
$|p\rangle$
$|p+2\hbar k\rangle$

Figure 4

VIII. Application to Sub-recoil Cooling

Much of laser cooling can be understood in terms of a semiclassical picture where point-like atoms undergo optical pumping processes in inhomogeneous light fields. The limiting factor in such cooling is the random momentum kicks from individual photon scattering events; the widths of the resulting momentum distributions are a few "recoil momenta": $p_r = \hbar k$. When atoms are cooled to near the recoil limit, the deBroglie wavelength of the atoms becomes comparable to the wavelength of the light being used to cool them, and a fully quantum mechanical treatment of the atomic momentum must be employed.

In order to breach the recoil limit, it is necessary to be able to "turn off" spontaneous emission, which we do by not making transitions to the excited state. One way of doing this is to make use of coherent population trapping to optically pump atoms into a velocity selective dark state by means of a random walk in momentum space. This is referred to as velocity selective coherent population trapping (VSCPT).[6] We now apply our formalism of dynamical symmetries to this means of subrecoil laser cooling, considering the cases where $p = 0$ and $p \neq 0$.

We consider several configurations for 1-D transverse VSCPT cooling of a $J = 1 \rightarrow 1$ system (appropriate to the $2\,^3S_1 \rightarrow 2\,^3P_1$ transition in metastable helium.) The simplest (see figure 4) is the $\sigma^+ - \sigma^-$ polarization configuration applied to the $J = 1 \rightarrow 1$ system. This results in a 3-level Λ-system with states $\{|0_e, p\rangle, |\pm 1_g, p \pm \hbar k\rangle\}$. The Hamiltonian is

$$H = \begin{pmatrix} \dfrac{(p-1)^2}{2m} & \frac{1}{2}\Omega & 0 \\ \frac{1}{2}\Omega & \delta + \dfrac{p^2}{2m} & \frac{1}{2}\Omega \\ 0 & \frac{1}{2}\Omega & \dfrac{(p+1)^2}{2m} \end{pmatrix};$$

this is the system treated in the preceding example. For $p = 0$, this Hamiltonain exhibits the $su(2)$ symmetry.

The next more complicated system is the 5-level system created in the lin$\perp$lin polarization configuration. Here the states of the system are $\{|0_e, p\rangle, |\pm 1_g, p \pm \hbar k\rangle, |\pm 1_g, p \mp \hbar k\rangle\}$. The Hamiltonian for this system is

$$H = \begin{pmatrix} \frac{(p-1)^2}{2m} & 0 & \frac{1}{2}\Omega & 0 & 0 \\ 0 & \frac{(p+1)^2}{2m} & \frac{1}{2}\Omega & 0 & 0 \\ \frac{1}{2}\Omega & \frac{1}{2}\Omega & \delta + \frac{p^2}{2m} & \frac{1}{2}\Omega & \frac{1}{2}\Omega \\ 0 & 0 & \frac{1}{2}\Omega & \frac{(p-1)^2}{2m} & 0 \\ 0 & 0 & \frac{1}{2}\Omega & o & \frac{(p+1)^2}{2m} \end{pmatrix}.$$

When the angle between the polarizations of the counterpropagating laser beams between 0 and $\pi/2$, the momentum families are no longer closed, and higher velocity VSCPT peaks can be observed[7]. The simplest subsystem describing such a situation is the 8-level system shown in the third part of figure 4. Here the higher order dark state is not an exact eigenstate of the total Hamiltonian, and the system exhibits only approximate dynamical symmetry for $p = 0$. We are currently working on incorporating such situations into our methods.

We can apply our method for finding dynamical symmetries to the Hamiltonian for the 5-level given above. The structure of this Hamiltonian changes radically when $p = 0$. For this system the dynamical algebra is $su(5)$. This algebra is more complicated than $su(3)$ having three maximal subalgebras $so(5) \sim sp(4)$, $su(3) \oplus su(2) \oplus u(1)$ and $su(4) \oplus u(1)$ of dimensions 10, 12 and 16 respectively. It turns out that in this representation $so(5)$ leads only to Hamiltonians having zero determinant and thus not of interest.

For $p \neq 0$ the relevant subalgebra is $su(3) \oplus su(2) \oplus u(1)$. This symmetry leads to four constants of motion; the first three of which are given below. The fourth constant is cubic in ρ and contains in excess of 220 terms and is omitted for brevity.

$$
\boxed{\begin{aligned}
k_1 &= \rho_{14} + \rho_{25} + \rho_{33} + \rho_{41} + \rho_{52}\,,\\
k_2 &= {\rho_{11}}^2 + \rho_{11}\,\rho_{14} + \rho_{12}\,\rho_{21} + \rho_{15}\,\rho_{21} + \rho_{12}\,\rho_{24} + \rho_{15}\,\rho_{24}\\
&\quad - \rho_{11}\,\rho_{25} - \rho_{14}\,\rho_{25} + 2\,\rho_{13}\,\rho_{31} + 2\,\rho_{23}\,\rho_{32} + \rho_{11}\,\rho_{33}\\
&\quad - \rho_{14}\,\rho_{33} - 2\,\rho_{25}\,\rho_{33} + 2\,{\rho_{33}}^2 + 2\,\rho_{13}\,\rho_{34} + 2\,\rho_{23}\,\rho_{35}\\
&\quad + \rho_{11}\,\rho_{41} - \rho_{25}\,\rho_{41} - \rho_{33}\,\rho_{41} + \rho_{21}\,\rho_{42} + \rho_{24}\,\rho_{42}\\
&\quad + 2\,\rho_{31}\,\rho_{43} + 2\,\rho_{34}\,\rho_{43} + 2\,\rho_{11}\,\rho_{44} + \rho_{14}\,\rho_{44} - \rho_{25}\,\rho_{44}\\
&\quad + \rho_{33}\,\rho_{44} + \rho_{41}\,\rho_{44} + {\rho_{44}}^2 + \rho_{21}\,\rho_{45} + \rho_{24}\,\rho_{45}\\
&\quad + \rho_{12}\,\rho_{51} + \rho_{15}\,\rho_{51} + \rho_{42}\,\rho_{51} + \rho_{45}\,\rho_{51} - \rho_{11}\,\rho_{52}\\
&\quad - \rho_{14}\,\rho_{52} - 2\,\rho_{33}\,\rho_{52} - \rho_{41}\,\rho_{52} - \rho_{44}\,\rho_{52} + 2\,\rho_{32}\,\rho_{53}\\
&\quad + 2\,\rho_{35}\,\rho_{53} + \rho_{12}\,\rho_{54} + \rho_{15}\,\rho_{54} + \rho_{42}\,\rho_{54} + \rho_{45}\,\rho_{54}\,,
\end{aligned}} \tag{51}
$$

$$
\boxed{
\begin{aligned}
k_3 = {} & {\rho_{11}}^2 - \rho_{11}\,\rho_{14} + \rho_{12}\,\rho_{21} - \rho_{15}\,\rho_{21} - \rho_{12}\,\rho_{24} + \rho_{15}\,\rho_{24} \\
& + \rho_{11}\,\rho_{25} - \rho_{14}\,\rho_{25} + \rho_{11}\,\rho_{33} - \rho_{14}\,\rho_{33} - \rho_{11}\,\rho_{41} - \rho_{25}\,\rho_{41} \\
& - \rho_{33}\,\rho_{41} - \rho_{21}\,\rho_{42} + \rho_{24}\,\rho_{42} + 2\,\rho_{11}\,\rho_{44} - \rho_{14}\,\rho_{44} \\
& + \rho_{25}\,\rho_{44} + \rho_{33}\,\rho_{44} - \rho_{41}\,\rho_{44} + {\rho_{44}}^2 + \rho_{21}\,\rho_{45} - \rho_{24}\,\rho_{45} \\
& - \rho_{12}\,\rho_{51} + \rho_{15}\,\rho_{51} + \rho_{42}\,\rho_{51} - \rho_{45}\,\rho_{51} + \rho_{11}\,\rho_{52} - \rho_{14}\,\rho_{52} \\
& - \rho_{41}\,\rho_{52} + \rho_{44}\,\rho_{52} + \rho_{12}\,\rho_{54} - \rho_{15}\,\rho_{54} - \rho_{42}\,\rho_{54} + \rho_{45}\,\rho_{54}\,, \\
k_4 = {} & {\rho_{11}}^2\,\rho_{14} + \rho_{11}\,{\rho_{14}}^2 + \rho_{12}\,\rho_{14}\,\rho_{21} + \cdots .
\end{aligned}
}
\tag{52}
$$

For $p = 0$ the Hamiltonian gains additional symmetry; the $su(3)$ piece decomposes into $su(2) \oplus u(1)$ and the projection of the Hamiltonian onto $u(2)$ vanishes leaving a $u(1) \oplus u(1)$ symmetry. The complete dynamical symmetry is $su(2) \oplus u(1) \oplus u(1) \oplus u(1)$. The procedure for computing the conservation laws is essentially the same as in the other cases. This symmetry leads to the 6 new constants of motion shown below. (We omit the full expression for k_2 for reasons of space.)

$$\boxed{\begin{aligned} k_1 &= 3\,\rho_{12} - \rho_{14} + 3\,\rho_{15} + 3\,\rho_{21} + 3\,\rho_{24} - \rho_{25} + 5\,\rho_{33}\,, \\ &\quad - \rho_{41} + 3\,\rho_{42} + 3\,\rho_{45} + 3\,\rho_{51} - \rho_{52} + 3\,\rho_{54}\,, \\ k_2 &= -{\rho_{12}}^2 + \rho_{12}\,\rho_{14} - 2\,\rho_{12}\,\rho_{15} + \rho_{14}\,\rho_{15} - {\rho_{15}}^2 + \cdots\,, \\ k_3 &= 2\,\rho_{11} - \rho_{14} + \rho_{25} + \rho_{33} - \rho_{41} + 2\,\rho_{44} + \rho_{52}\,, \\ k_4 &= \rho_{12} - \rho_{15} - \rho_{42} + \rho_{45}\,, \\ k_5 &= \rho_{21} - \rho_{24} - \rho_{51} + \rho_{54}\,, \\ k_6 &= \rho_{14} + \rho_{25} + \rho_{33} + \rho_{41} + \rho_{52}\,. \end{aligned}} \tag{53}$$

IX. Future Work

We plan to employ the conserved quantities determined by the methods described here in the solution of the optical Bloch equations using exactly conservative integration algorithms.[8] Such methods have the advantage of explicitly conserving all constants of motion during numerical integration of the dynamical equations. We also plan to investigate the role of dynamical accessibility in present proposals for coherent control and quantum state engineering.

The effects of non-Hamiltonian processes such as spontaneous emission have been ignored in this work except for the role they play in populating the trapping states. Future work will address these issues, as well as the investigation of approximate symmetries due to the fact that in configurations such as lin-angle-lin VSCPT, the momentum families are not closed.

References

[1] L. Allen and J.H. Eberly, *Optical Resonance and Two-Level Atoms* (Dover, New York, 1975).

[2] R.P. Feynman, F.L. Vernon, and R.W. Helwarth, "Geometrical Representation of the Schrödinger Equation for Solving Maser Problems," J. Appl. Phys. **28**, 49 (1957).

[3] F.T. Hioe and J.H. Eberly, "N-Level Coherence Vector and Higher Conservation Laws in Quantum Optics and Quantum Mechanics," Phys. Rev. Lett. **47**, 838 (1981).

[4] P.J. Morrison, "Hamiltonian Description of an Ideal Fluid," Institute for Fusion Studies Report **640-R**, (1994), Lectures delivered at Geophysical Fluid Dynamics Summer Program, Woods Hole, MA.

[5] G. Stevens, M. Widmer, F. Tudorica, C.-H. Iu and H. Metcalf, Coherent Excitation of Li Rydberg States and Application to Rydberg Atom Optics, DAMOP96 poster #FP.21.

[6] A. Aspect, E. Arimondo, R. Kaiser, N. Vansteenkiste, C. Cohen-Tannoudji, "Laser Cooling Below the One-Photon Recoil Energy by Velocity Selective Coherent Population Trapping," Phys. Rev. Lett. **61**, 826 (1988).

[7] M.T. Widmer, M.R. Doery, M.J. Bellanca, W.F. Buell, T. Bergeman and H.J. Metcalf, "High Velocity Dark States in Velocity Selective Coherent Population Trapping," Phys. Rev. A **53**, 946 (1996).

[8] B.A. Shadwick, J.C. Bowman and P.J. Morrison, "Exactly Conservative Integrators," Institute for Fusion Studies Report **698**, (1996), to appear in SIAM J. Appl. Math.

Re–Print Requests

Re–Print Requests