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Some aspects of the Geometry of Poisson Dynamical Systems

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Some aspects of the Geometry of Poisson Dynamical Systems

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To my friends and family

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Some aspects of the Geometry of Poisson Dynamical Systems

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The purpose of this thesis is to examine various characterizations of nonmaximal rank in dynamical systems for which a generalization of the Poisson bracket generates the dynamics. In particular, for systems with symmetry, there is a connection between Goldstone modes of spontaneous symmetry breaking induced by any G-invariant polynomial potential (G being the symmetry group of the system), and subcasimirs that define nonmaximal symplectic foliations. The connection is made explicit using ideas from invariant theory, orbit space geometry, Poisson geometry, and work already done in the area of patterns of symmetry breaking. Examples illustrating some of these ideas are given in both finite and infinite dimensions. The moment algebra of an ideal fluid in $2+1$ dimensions is an example where some ideas from invariant theory can be profitably used. Finally, it is indicated how a unified bundle description might be given of all these phenomena using the notion of principal bundle reduction, and of submaximal distributions and their integrability.

The nature of the infinite dimensional analog of the above characterizations is also indicated in the conclusion.

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Chapter 1

Introduction

1.1 Symmetry in Mathematics and Physics

The role of symmetry in physics has traditionally been that of a device to simplify the problem at hand. Using hints from the symmetry constraints it is often possible to guess at the correct solution from amongst the infinitely many plausible solutions, without actually solving the problem. It may thus be regarded as a tool to study the behavior of a system, be it composed of a collection of differential equations, or a space of solutions from extremizing an action; or even to probe the geometry of a manifold on which the symmetry acts. In physics, the role of symmetry has been central to the formulation of quantum mechanics, as pioneered by Wigner (who introduced the term 'little group', which plays a central role in this thesis), and of gauge theories, which form the backbone of the so called 'Standard Model' and its generalizations (like supersymmetry and string theory) to include gravitation.

The pioneers of the subject of classical and celestial mechanics were quick to appreciate the virtues of symmetry in solving complicated systems of differential equations that arose in their work, but it took a long time for the methods of modern Lie group theory to become established within the

subject of mechanics (but see [1] for an early pioneering work). The historical developments [2] dealt with symmetries in the context of the mechanical systems being studied, and their evolution dynamics. It must be pointed out that the vision of symmetry that S. Lie had was much more general than the sense in which the concept is commonly used today. Group theory as used in most of contemporary physics is the theory of finite groups, or Lie groups, and their generalizations to infinite dimensions. Lie first discovered differentiable groups in a more general setting, as transformations induced simultaneously on the dependent and the independent variables of an infinite dimensional (continuum) system. The nugget of Lie group theory was then isolated from its rich context by later workers such as F. Klein, who had elementary geometric considerations to motivate him, rather than the complex situations that Lie had in mind. Thus, classical and celestial mechanics proceeded relatively old-fashionedly, while the new stylistic developments in formal Lie group theory and representation theory were intensely worked out; it is this latter theory that physicists find fit to apply to their work today.

Motivated by the success of Lie group theory in other branches of physics, a revival was made during the latter half of the 20th century to apply them to classical (Hamiltonian and Lagrangean) mechanics. The technique of restricting to the isotropy subgroup (or 'little group') gave rise to the important procedure of *reduction*, or reducing the number of degrees of freedom in a system with symmetry. In essence, this was well known to the workers of yore (such as Lie, Cartan and Poincare), when they used reduction to reduce

the order of the differential equations that governed the evolution of a physical system. Reduction (in the modern sense) first required stringent assumptions on the type of action (the associated moment map had to be weakly regular), which was later relaxed to include singular variants. Now, the methods of Lie groups and representation theory are well embedded within the mechanics community; the next step, of incorporating the full vision of Lie in modern terms, has yet to be completed [3].

Most of modern mechanics is modeled on a phase space of physical variables, such as positions and momenta, as an even dimensional symplectic manifold. A dual point of view, which regards the algebra of functions as fundamental, is found to be a generalization of phase space. The advantage of this view is the possibility of having odd dimension, as is often required when the symplectic manifold is acted upon by an odd-dimensional group, or if the restriction to a nondegenerate symplectic form is far too restrictive. Thus, we take our dynamical systems to be modeled on phase spaces that are Poisson manifolds, and at times assume that the phase space arises as a quotient space of a symplectic manifold by a group action (by a procedure known as Poisson reduction).

A Poisson manifold is endowed with an (in general) degenerate Poisson structure (or Poisson bracket), and this plays an important role when analyzing the equilibria of a system. Equilibria can be found, on the one hand, by extremizing the action, which gives equations of motion whose time evolution is set to zero; on the other hand, they could also be found by simply setting the

gradient of a suitable governing Hamiltonian to be equal to zero. The problem with degenerate (or noncanonical) Poisson brackets is that these two methods do not commute: that is to say, more equilibria can result from the action principle because of the extra submanifolds along which the Poisson structure could itself vanish. Similar considerations apply in attempting to ascertain the nature of stability of the equilibria. The usual methods developed (such as the 'energy-Casimir', or Arnol'd method [1]) work only for *regular* points, where the rank of the Poisson tensor is maximal. At singular points, where the rank is nonmaximal, additional Casimir functions (called *subcasimirs*) arise, and these have to be taken properly into account to study stability. The problems in the infinite dimensional (or functional) setting are even more complex, as function spaces are more complicated than finite dimensional manifolds. (We shall, aside from illustrating with some examples, only touch upon the functional issues in the final (concluding) chapter.)

Symmetry enters in the following way in the study of the geometry of Poisson manifolds. Given a C^∞ manifold M, the largest group that can act on it is the infinite dimensional group of diffeomorphisms, $\mathit{Diff}(M)$, of all differentiable maps that take M to itself. This group is too vast for many purposes, so we restrict to subgroups of the full diffeomorphism group that preserve certain additional structures on M. In this way [5], we arrive at Lie group actions on the manifold, $\Theta : G \times M \longrightarrow M$, where $G \subset \mathit{Diff}(M)$. With additional restrictions on the type of action (proper, for example), orbit

[1] Similar stability methods had been developed earlier, see for instance [4].

spaces with Hausdorff quotient topology can be defined. We can therefore use subgroups of the diffeomorphism group to probe the structural details of M. In particular, if M happens to be a Poisson manifold, one of these is the Poisson structure. (The action is then assumed to preserve the Poisson structure.)

In the second appendix to Chapter 2, it will be shown how the orbits of the diffeomorphism group and subgroups thereof define generalized foliations on M, which for a Poisson manifold are generated by the *characteristic distribution* that its Poisson structure defines. In this view, any vestige of symmetry left over from restricting to the various orbits reflects in an obvious way the geometry of the orbit. One of the aims in this thesis will be to give a dynamical interpretation for these orbits, both regular and singular, using the tools of symmetry breaking and invariant theory. The extrema of potentials that are defined as G-invariant functions (or polynomials) on M are intimately related to the orbit geometry, which in turn is governed by the characteristic distribution. In this manner, we find a connection between the extrema of arbitrary potentials, and the subcasimirs that pin down a given orbit.

An infinite dimensional counterpart of this problem would involve modeling an infinite dimensional phase space, which is usually taken to be a Banach manifold, and the Lie groups acting on the phase space could be defined on a finite dimensional subspace (local regularity assumption) in much the same way as the finite dimensional case. However, essential functional analytic aspects of the problem arise, which we do not encounter in the finite dimensional case. The actions and functions we consider are C^∞, or those functions that can

be extended from closed sets to C^∞ functions on the whole domain, and this rules out potentially physical phenomena such as leaves of a foliation that are singular at arbitrarily many points, and are supported by singular integrals. In the final chapter we shall return to these issues, and point out generalizations of the thesis that could be made with reasonable straightforwardness to the functional setting. A few ideas such as anti-selfadjoint index theory could be applied to detect rank change of the infinite dimensional Poisson structure, which in the functional case is a cosymplectic *operator* and not a tensor as it is in the finite dimensional case. A proper functional setting for noncanonical Hamiltonian dynamics (whose phase space would be modeled by the infinite dimensional Poisson manifold) would first need to be laid [6] before investigating these questions with the aim of explaining phenomena already seen, and predicting those that haven't yet been seen. Although every Banach manifold is locally diffeomorphic to a Banach vector space, the calculus of functionals is obviously more varied and complicated than the ordinary calculus of invariant polynomials and functions that are used in the thesis.

The roles played by *invariants* in physics and mathematics are too myriad to even state here. In general, in both mathematics and physics, one is interested in objects that are left invariant with respect to a certain operation, or that characterize a situation globally. For example, in classical Hamiltonian dynamics, the invariants of a symmetry are guaranteed by Noether's theorem, and by Lie's theorem for the case of a system governed by differential equations [3]. In the geometry of the associated symplectic and Poisson manifolds,

the various integral invariants of a system of vector fields or differential forms give rise to the theory of foliations (c.f. Appendix 2 to Chapter 2), and of differential forms [5, 7]. Cartan exploited the role of invariants to set up various *equivalence problems*, which characterize the general properties of the manifold in terms of its differential germs. Invariant theory, both in its classical and geometric forms, seeks to display explicitly elements of tensor spaces that are invariant under group actions, and then interprets their geometry as being "universal" on the orbits of the group action. Later, we shall use these results applied to the problem of spontaneous symmetry breaking, which is also similar in spirit to their use for characterizing bifurcations in *equivariant* bifurcation theory [8]. Finally, we note that invariant polynomials play a big role in defining characteristic classes of a principal fiber bundle; indeed several results used here in the context of semisimple algebras and their invariants find their generalizations in the guise of the properties of the Weil algebra and Weil homomorphisms in equivariant cohomology theory. (We shall return to this in the appendix to Chapter 4, and in the concluding chapter, where a fiber bundle setting is proposed.)

1.2 Main ideas and results

Firstly, we note the important *caveat*: All orbits, both regular and singular, are assumed to be closed. This means that group actions that are designed to produce a dense orbit (by the expedient of using an irrational parameter on a torus component, for example) are purposely excluded. Oth-

erwise, it can be shown that even for proper actions of a compact group on a compact manifold, the Casimirs (and hence subcasimirs) do not determine the orbits [9]. For the ideas used and developed in this thesis, it will prove imperative that Casimirs and subcasimirs *do* determine the orbits; if there is a way of getting around this caveat, then it is one that the author is unaware of. This, of course, poses a stringent restriction on the types of Poisson structure we shall be able to tie up with a dynamical interpretation.

The main ideas and results of the thesis are the following:

1. Given a Poisson manifold, linearization of the Poisson structure makes it possible to look at a Lie-Poisson foliation. Then, using the embedding theorems of Whitney, it is sufficient to consider a linear (hence orthogonal) action on a finite-dimensional vector space.

2. The leaves of the characteristic distribution defined by the Poisson structure become orbits that are fully transitive with respect to the orbit group action, with different isotropy subgroups characterizing orbits of various dimensions (always even, since they are symplectic).

3. Assuming these orbits are (locally) closed, invariant theory may be used to characterize the Casimirs and subcasimirs that appear as functions whose level sets define the orbits. This phenomenon is shown to be general in a series of examples from finite dimensions to infinite dimensions.

4. The passage from a larger isotropy group (associated with a thinner orbit, or a more degenerate leaf) to a smaller one (associated with a

fatter orbit, or a less degenerate leaf) is related to the appearance of extra *Goldstone modes*, that in turn become leaf coordinates of the bigger orbit. The reverse process, called *symmetry restoration*, maps a suitable functional combination of these Goldstone modes into the (differentials of) subcasimirs that pin down the thinner orbit [2].

5. Since the breaking and restoration of symmetry requires a Hamiltonian (or a potential) which may depend on control parameters, by varying the controls, it is possible to move from generic to nongeneric orbits. At the same time, by looking at the invariant theory of the situation, it becomes possible to see new subcasimirs come and go in the process.

The main task of the thesis is to outline a synthesis of the above ideas and results, using notions already developed in the literature of Poisson geometry, group actions on a manifold, invariant theory, geometric mechanics, and patterns in hadronic symmetry breaking. These ideas are then applied to characterize rank-changing events in Poisson dynamical systems.

1.3 Organization of the thesis

In the next chapter, we begin with a short introduction to the local structure of Poisson manifolds. Symmetry enters in when we consider the orbits of the characteristic distribution to coincide with those of a certain Lie

[2]Weinstein [10] calls the generic orbits symplectic leaves, and the thinner orbits 'symplectic bones'.

group action. The physical interpretation of Casimirs, functions that specify these orbits, is next outlined with examples. The two appendices are concerned with the generalization of the classical integrability result of Frobenius, to distributions of nonconstant rank. At the end, we indicate how these ideas find an application in the global foliation of a Poisson manifold.

Next, in Chapter 3, we consider examples in finite and infinite dimensions. We begin with the standard example of a 3-dimensional rigid body (the simplest nontrivial example of a noncanonical system) and its generalization to n dimensions. The latter provides an opportunity to work in some standard facts about semisimple Lie algebras, which we duly record. Semi-direct products (and their special case, direct products) are considered next, first in general, and then in the guise of two examples, one from finite dimensions (rigid body under the influence of a constant gravitational field) and the other from infinite dimensions (a 2+1 field theory of fluids, involving the algebra of moments). In the appendix, we outline a brief description of the classical invariant theory procedure that is utilized for the ideal fluid example.

Chapter 4 is concerned with actions of groups on manifolds, and the important constructs that appear therein. Some of it is standard textbook fare, save for an occassional example or generalization that hearkens to the purpose at hand. Among results of this type, mention may be made of the *singular* reduction procedure, which generalizes the *clean* reduction theorem of Marsden and Weinstein, and the result of Whitney, which permits any action to be embedded as a linear action (or representation) on a vector space. It is

the latter result that we will invoke when we restrict our attentions to a local neighborhood of a point which sports the Weinstein linearization outlined in the Chapter 2. Next, we consider the various orbit types and their associated strata. In Section 4.3, the methods of invariant theory are used to connect the geometry of orbit strata with the algebra of invariant polynomials and invariant functions. This is followed by an appendix which describes two equivalent approaches to the characterization of regular and singular leaves using some cohomology theory.

In Chapter 5, all the threads of ideas in the previous chapters are now collided with the notion of spontaneous breaking of symmetry, as used in bosonic field theory. The work done in the late 1980s and early 1990s on patterns of symmetric breaking (that is, variation of the isotropy group of solution space with variation in *control* parameters) is first outlined, and later, used in the Poisson context to make a statement about connecting certain types of Goldstone bosons with the functions (Casimirs and subcasimirs) that determine the level sets of the orbits of the characteristic distribution.

We conclude the thesis with an overview of topics that have not been treated at all, like the fiber bundle setting, or infinite dimensional Poisson manifolds, and conjecture how some of these results might go over to those cases. At this point, we review what the thesis was about, and also indicate some future work in progress.

A Bibliography is set up with references to authors of books and articles and private communications, in the same order that they were invoked in the

text of the thesis (read linearly from start to finish).

Finally, here are some general notations assumed throughout:

- $\mathfrak{X}(M)$: the space of all vector fields on a differentiable manifold M.
- $\Lambda^*(M)$: subalgebra of forms (antisymmetric covariant tensors) of arbitrary degree in the tensor algebra of M. For instance, $\Lambda^0(M)$ are the functions $C^\infty(M)$, $\Lambda^1(M)$ are 1-forms, etc.
- $\Lambda^{-*}(M)$: subalgebra of antisymmetric contravariant tensors of arbitrary degree in the tensor algebra of M. For example, $\Lambda^{-1}(M) \equiv \mathfrak{X}(M)$, $\Lambda^{-2}(M)$ are 2-covectors, and so on.
- G is a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual, or Lie coalgebra.
- We adopt the following convention in dealing with manifold and vector space *dimensions*: with the first mention of an m-dimensional manifold M, its dimension will be indicated, as in M^m, but after that will mostly be referred to as M, as in T_xM, or M/G. In case we do not wish to emphasize the dimensionality until later, there shall be no dimension superscripts at first mention. (The obvious exception to this rule is $\mathbb{R}^n$, $\mathbb{C}^n$, etc.)
- We *do not* adopt any consistent convention with regard to upper and lower indices, whether the summation convention (summing over repeated indices) is invoked or not. In this regard, we also note that

the structure constants of a semisimple Lie algebra are assumed to be antisymmetric in the indices chosen to represent the Poisson structure, regardless of the positions of those indices; eg., C^i_{jk}, C^{ij}_k etc. have the same skew-symmetry properties.

- If $\mathbb{K}$ denotes a field (for example, $\mathbb{K} = \mathbb{R}$), then $\mathbb{K}^*$ is the field with zero element (the origin) excluded. ($\mathbb{K}$ shall always stand for a field of zero characteristic.)

- $\forall$ = 'for all'; $\exists$ = 'there exists'; $\exists!$ = 'there exists (a) unique'; dim = dimension; s.t. = such that.

Chapter 2

Poisson Geometry

2.1 Symplectic and Poisson manifolds

2.1.1 Preliminaries

It is a theorem of Darboux that any symplectic manifold can be provided with local coordinates about any point that reduces the symplectic form to a canonical one. A symplectic manifold is an even dimensional differentiable manifold M^{2n} with a closed 2-form $\omega \in \Lambda^2(M)$ such that $d\omega = 0$, called the symplectic form, that is nondegenerate (i.e., $\omega(X, Y) = 0$ for all $X \in \mathfrak{X}(M) \Rightarrow Y = 0$). Let local coordinates guaranteed by Darboux' theorem about a point $x_0 \in M$ be given by $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. Then the symplectic form has the following *canonical form*:

$$\omega(x)|_{x_0} = dq_i \wedge dp_i$$

(Here, and throughout, the summation convention — summing repeated indices over the appropriate index range — is in force, unless otherwise noted.) The symplectic form $\omega(x)$ $(= \omega_{ij}(x) dx^i \wedge dx^j)$ can be represented as a $2n$ by $2n$ matrix

$$(\omega_{ij})|_{x_0} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix}.$$

Since the above matrix is nondegenerate, it has a well defined inverse, denoted by J^{ij} such that $\omega_{ik}J^{kj} = \delta_i^j$. In a neighborhood of x_0, the covariant antisymmetric bilinear tensor that this defines has the canonical form

$$J(x)|_{x_0} = \frac{\partial}{\partial q_i} \bigwedge \frac{\partial}{\partial p_i}. \tag{2.1}$$

The tensor J can equivalently and completely characterize the symplectic manifold. If $f \in C^\infty(M)$, then the *Hamiltonian vector field* corresponding to it is given by $df = \omega(X_f, \cdot)$, or by $J(df, \cdot) = X_f$.

For a pair of C^∞ functions f, g we therefore have the relation

$$J(df, dg) = \omega(X_f, X_g). \tag{2.2}$$

This isomorphism between the symplectic form and the J tensor (also called *cosymplectic form*) breaks down in the context of a general Poisson manifold, P. A Poisson structure is specified by a Lie algebra structure on $C^\infty(P)$ by means of an antisymmetric algebra operation called the Poisson bracket [11]

$$\{\cdot, \cdot\} : C^\infty(P) \times C^\infty(P) \longrightarrow \mathbb{R}.$$

It satisfies two properties, aside from antisymmetry,

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \tag{2.3}$$

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \tag{2.4}$$

which are called Leibniz's rule and Jacobi identity, respectively.

In a coordinate neighborhood $\{x^i\}$, and for $f, g \in C^\infty(P)$ the above properties imply the following form for the bracket:

$$\{f, g\} = \{x^i, x^j\} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}.$$

Setting

$$\{x^i, x^j\} := J^{ij},$$

we see that the Poisson bracket of coordinate functions completely determines the cosymplectic form,

$$J(x) = J^{ij}(x) \frac{\partial}{\partial x^i} \bigwedge \frac{\partial}{\partial x^j}. \tag{2.5}$$

We shall call it 'Poisson tensor' or 'Poisson structure' in what follows.

By definition the Poisson bracket induces a Lie algebra structure on the space of Hamiltonian vector fields. Since $J(df, \cdot) = \{f, \cdot\} = X_f$, the Leibnitz rule is just the derivation property of vector fields on the function algebra, and the Jacobi identity expresses the anti-homomorphism between the Lie algebras $(C^\infty(P), \{\cdot, \cdot\})$ and $(\mathfrak{X}(P), [\cdot, \cdot])$,

$$X_{\{f,g\}} = -[X_f, X_g]. \tag{2.6}$$

A Poisson manifold with its Poisson structure is a generalization of a symplectic manifold equipped with a symplectic structure.[1] For one, the

[1] Another way to generalize a symplectic manifold is by a *presymplectic structure*, where the manifold is even dimensional with a closed 2-form ω, which is now allowed to be degenerate.

former *is* defined on an odd dimensional differentiable manifold. Secondly, while a symplectic form implies (by nondegeneracy) a Poisson structure, the converse is not generally true. It was known to Lie (for instance) that a modified version of Darboux' theorem held when the Poisson structure matrix, (J^{ij}) had constant rank (the *regular* case).

If $\dim(P) = m$ and the rank of the matrix is l, then firstly $l = 2n$, and moreover local coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n, c_1, \ldots, c_k)$ exist about any point $x_0 \in P$ such that the Poisson structure has the *noncanonical form*

$$(J^{ij})|_{x_0} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & 0 \\ -\mathbf{1} & \mathbf{0} & 0 \\ 0 & 0 & \mathbf{0_k} \end{pmatrix}, \tag{2.7}$$

which would be a canonical form except for the $\mathbf{0_k}$. Here, $m = 2n + k$, and zeros appearing in the matrix have the appropriate dimensions.

(In terms of the Poisson brackets, we have the following commutation relations:

$$\{q_i, p_j\} = \delta_{ij}, \quad \{q_i, q_j\} = \{p_i, p_j\} = 0,$$

and c_k's commute with everything.)

Clearly, $\det(J^{ij}) = 0$, and so this is not the inverse of any symplectic structure. However, by the Frobenius' theorem, it is possible to find *leaves* of a regular foliation generated by a system of vector fields of rank $2n$, which are symplectic with the symplectic form coming from (roughly speaking) the part of the matrix (J^{ij}) that is an invertible $2n$ by $2n$ submatrix. The remaining k dimensions are transversal to the leaves and comprise the *transverse* structure

to the foliation. For a regular foliation, the transverse structure is trivial, and is simply generated by the *complementary* dimension subspace of 1-form (Pfaffian) fields of rank k, from amongst the space of all 1-form fields, $\Lambda^1(P)$. (Precise statements are given in Appendix 2.) The Figure 2.1 below indicates the geometry of the regular case.

The leaves are regular submanifolds of P that are locally just level sets of the k functions in a neighborhood with generalized Darboux coordinates, $\{c_k = const.\}$. These are called *Casimirs* (or Casimir invariants), and play an important role in both the physics and geometry of Poisson dynamical systems. The Casimirs form the center of the Poisson function algebra, that is, $\{c_k \in C^\infty(P) | \{c_k, f\} = 0 \quad \forall f \in C^\infty(P)\}$. Geometrically, as indicated above, the differentials, $\{dc_k\}$ generate the transverse foliation of complementary dimension to the symplectic leaves. Physically, these are precisely the leaves where dynamics, specified by a Hamiltonian, would be constrained to lie within.

2.1.2 Nonconstant rank

Let $\rho(J)$ denote the rank of the Poisson structure. If $\rho(J)(x)$ is not constant as $x \in P$ varies, then the distribution of vector fields generated by the invertible part of J^{ij} is integrable in the generalized sense, with leaves of foliation of varying (even) dimension. (In the second appendix to this chapter, some facts about generalized distributions and their foliations are included.) The Figure 2.2 below shows a variable rank Poisson structure, with maximal

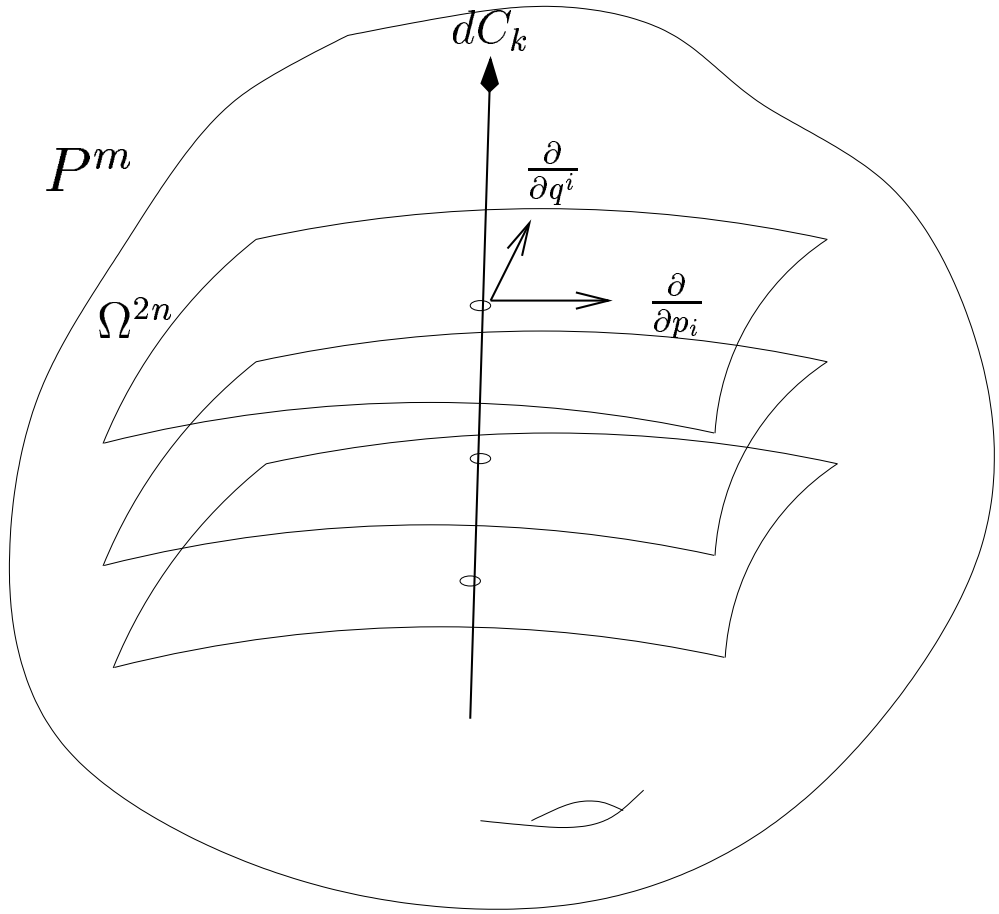

Figure 2.1: Regular Poisson manifold showing foliation by symplectic leaves (here i runs from $1, \ldots, n$).

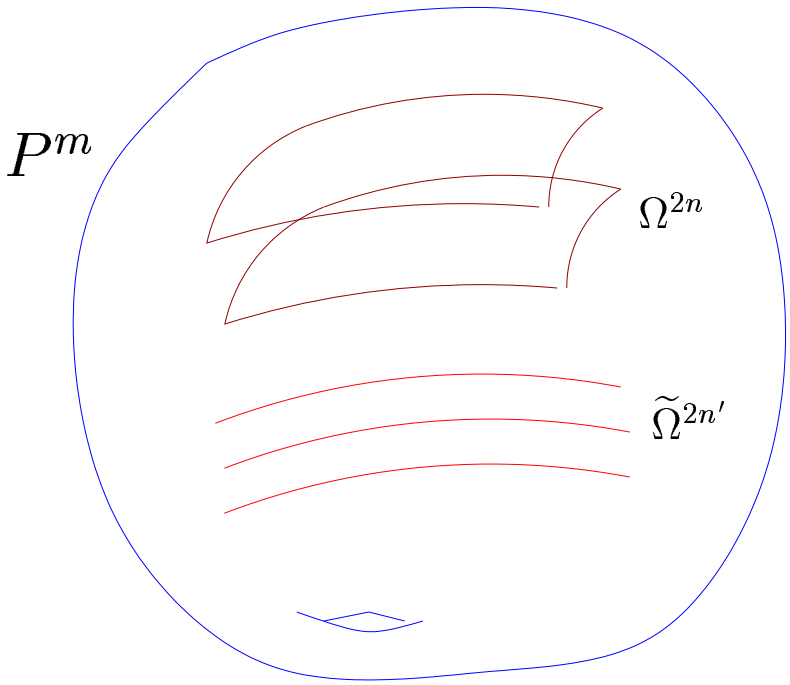

Figure 2.2: Singular Poisson manifold showing foliation by symplectic leaves of various dimensions ($n > n'$).

dimension leaves labeled Ω and the nonmaximal ones by $\widetilde{\Omega}$.

If the generalized distribution is differentiable, then $\rho(J)(x)$ is lower semicontinuous (cf. Appendix 2, Section 2.4), so its rank is constant in an open neighborhood of $x \in P$, and in the generalized sense, the distribution of symplectic leaves is integrable (the leaves are always even dimensional since J is skew symmetric), both for maximal and nonmaximal leaves. However, the *corank* is upper semicontinuous, and so the *complementary* Pfaffian distribution fails to be integrable in the generalized sense. Thus it becomes important to study the transverse structure where rank changes occur. (Semicontinuity

is reviewed in Appendix 1, Section 2.3.) Since J is skew-symmetric, it follows that its rank is even, so the leaves of a generalized foliation drop in dimension by even integers, when rank of J decreases.

2.1.3 Local structure of Poisson manifolds

The structure of a Poisson manifold in the local neighborhood of a point $x_0 \in P$ was studied in a paper by Weinstein [12, 13]. Essentially, the Poisson structure decomposes into a symplectic part and a singular part for which various germs can be postulated. The one that shall be used in this thesis is the 'linear structure', which entails zero rank at the origin. Other germs, such as quadratic, are used in quantization and computation of L-P (Lichnerowicz-Poisson) cohomology, and we shall not touch upon these.

The *generic* points are those for which the symplectic part of the Poisson structure has maximum dimension. Points other than generic are called *singular* — at these points, the symplectic part falls in dimension, and the singular germ is one that incorporates the coordinates that have disappeared.

In terms of the structure matrix, we have in a neighborhood of x_0,

$$(J^{ij})|_{x_0} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & 0 \\ -\mathbf{1} & \mathbf{0} & 0 \\ 0 & 0 & \pi^{ij}(y) \end{pmatrix} \tag{2.8}$$

where $(q_1, \ldots, q_n, p_1, \ldots, p_n, y_1, \ldots, y_k)$ are coordinates of P^{2n+k} in the above neighborhood, and

$$\pi^{ij}(y) = \{y^i, y^j\}, \quad \pi^{ij}(0) = 0$$

is the transverse Poisson structure. In tensor notation, we have:

$$J(x)|_{x_0} = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \bigwedge \frac{\partial}{\partial p_i} + \sum_{j,l=1}^{k} \pi^{jl}(y) \frac{\partial}{\partial y_j} \bigwedge \frac{\partial}{\partial y_l}.$$

The transverse structure is governed by the π^{kl} part. There are some theorems and conjectures as to what form it could assume. If

$$J^{ij}(x) = C_l^{ij} x^l,$$

the Poisson structure is said to be a *linear* one. Weinstein conjectured that the germ of the transverse structure about the origin is *linearizable*, that is, coordinate transformations may be made whereby the Poisson structure becomes a linear one. (This is akin to Morse theory [14] where a Morse function in the neighborhood of a nondegenerate critical point, can be given a quadratic germ.) With some restrictions that were noted later, a version of the conjecture is true. Without getting into the details, here is a short sketch of the theory ([13]; see also [15]).

A. Splitting theorem

In the neighborhood of any point of $x_0 \in P$, the Poisson manifold P^m is a product of a symplectic part and a degenerate Poisson part (in local coordinates, the expressions given above). The regular part of P is an open dense subset of P, and here the symplectic factor arranges in leaves of locally constant dimension. The degenerate factor or transverse structure is defined

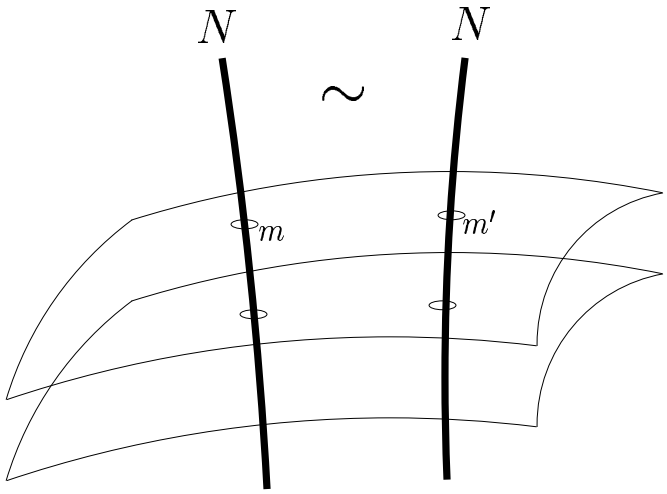

Figure 2.3: The transverse structure to a leaf at two points.

up to its *isomorphism class*, which is the same for all points along the leaf, see Figure 2.3.

B. Linearization

In a neighborhood of the point where rank is zero, the Poisson structure has a Taylor series expansion of the form

$$J(x)|_0 = \sum_{i,j,l=1}^{m} C_l^{ij} x^l \frac{\partial}{\partial x^i} \bigwedge \frac{\partial}{\partial x^j} + o(x^2)$$

where C_l^{ij} are structure constants of a Lie algebra $\mathfrak{g}$ of dimension m and rank k (where k is the corank of the generic foliation by symplectic leaves). If terms of $o(x^2)$ vanish, we could locally take P^m to be a vector space V^m such that its dual, V^* of linear functions on V is the Lie algebra $\mathfrak{g}$, which means, locally,

$V \cong \mathfrak{g}^*$. Thus, linearization happens when the degenerate Poisson structure at the point of total degeneracy is isomorphic to a *Lie-Poisson structure.*

If $(\mu^1, \ldots, \mu^m)$ are coordinates of $\mathfrak{g}^*$, $(\frac{\delta}{\delta\mu^1}, \ldots, \frac{\delta}{\delta\mu^m})$ the corresponding coordinate basis of $\mathfrak{g}$, and $\langle\ ,\ \rangle$ the $\mathfrak{g}-\mathfrak{g}^*$ pairing, then for any pair of functions $f, g \in C^\infty(\mathfrak{g}^*)$, the *Lie-Poisson bracket* is defined by [2]

$$\{f,g\}(\mu) = \left\langle \mu, \left[\frac{\delta f}{\delta\mu}, \frac{\delta g}{\delta\mu}\right]\right\rangle = C_l^{ij}\mu^l \frac{\delta f}{\delta\mu^i}\frac{\delta g}{\delta\mu^j}, \tag{2.9}$$

where $[\ ,\ ]$ is the Lie bracket on $\mathfrak{g}$, and C_k^{ij} the structure constants of $\mathfrak{g}^*$.

C. Conditions for linear structure

When can a Poisson structure be made linear about the point of zero rank? Here we give some results gleaned from the literature on this topic [13] [15].

1. A given Poisson structure is *formally* linearizable at a point if the associated tangent Lie algebra (defined as the algebra of differentials of functions in the cotangent space at that point) is *semisimple.*

2. A C^∞ structure is C^∞ linearizable if its tangent Lie algebra is of compact type. (A semisimple, compact, Lie algebra $\mathfrak{g}$ is said to be of compact type if there exists an involutive automorphism s (i.e., $s \neq e$, $s^2 = e$) whose fixed point set is compactly imbedded as a subalgebra of $\mathfrak{g}$.)

[2] Throughout the thesis, we ignore the $(\pm)$ convention for the Lie-Poisson bracket. The difference arises in using left-invariant $(-)$ or right-invariant $(+)$ vector fields for the isomorphism between T_eG and $\mathfrak{g}$.

3. Let $P = \mathfrak{g}^*$. Then the transverse Lie algebra to $\mu \in \mathfrak{g}^*$ is $\mathfrak{g}_\mu$, the stabilizer of μ in $\mathfrak{g}$ with respect to the coadjoint action. (These terms are explained in Chapter 4, Subsections 4.1.1 and 4.1.4). Essentially, the orbits of the dual to any Lie algebra are coadjoint orbits, whose symmetry group is $\mathfrak{g}_\mu$. On the tangent space to the orbits, the vectors are spanned by elements of $\mathfrak{g}$, since we could identify $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$. Then the Lie algebra itself splits into two parts: the tangent to the coadjoint orbit, spanned by $\mathfrak{m}$, and the transverse to it, spanned by $\mathfrak{g}_\mu$.

   Linearizable transverse structure is possible only if the complement to the stabilizer is an ideal of $\mathfrak{m}$ in $\mathfrak{g}$, that is

$$\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{m},$$

$$[\mathfrak{g}_\mu, \mathfrak{m}] \subseteq \mathfrak{m}.$$

   Several counterexamples for which these conditions are not met are given in the above cited references. In an interesting paper by Oh [16], it is shown that the isomorphism class of the transverse structure in some of these counterexamples admits a quadratic structure. (Quadratic transverse structures come into play during quantization, see for example [17].)

2.2 Kinematics and Dynamics

Casimirs, being functions that commute with *every function*, form the center of the Poisson algebra, $(C^\infty(P), \{\ ,\})$. In the case $P = \mathfrak{g}^*$, they are

closely related [18] to the Casimir operators in the universal enveloping algebra (where monomials in algebra elements form an associative algebra), $\mathfrak{S}(\mathfrak{g})$ [19]. It was in this context that the term 'Casimir' was first applied, in representation theory.

For a regular Poisson manifold, the leaves of the symplectic foliation are level sets of the Casimir functions (in suitable local coordinates). For the example of $P = \mathfrak{so}(3)^*$, the Casimir in coordinates (μ^1, μ^2, μ^3) is simply equal to $c = (\mu^1)^2 + (\mu^2)^2 + (\mu^3)^2$, and the level sets are 2-spheres about the origin. (See Figure 2.4 below.)

The symplectic form on the leaves can be found, see the first example of the next chapter for details. (In Figure 2.4, the coordinates to the tangent plane m_1 and m_2 stand for the canonical leaf coordinates, cf. Subsection 3.1.1.)

Away from the origin, the leaves are 2-dimensional, and the rank of J is two. At the origin, the singular leaf is a point, and the rank of J there is zero. At that point, any set of functions of the three coordinates that intersect in the solution set of a point can be used as good Casimirs. Such *extra* functions that pop up whenever rank change occurs are called *subcasimirs*. Part of the thesis is concerned with characterising them, and pointing out a dynamical interpretation of these functions.

Subcasimirs, by definition, are functions that separate singular leaves, where the rank of J is less than maximal. By the theory of generalized distributions (Appendix 2, Section 2.4), it is clear that these cannot have a nice

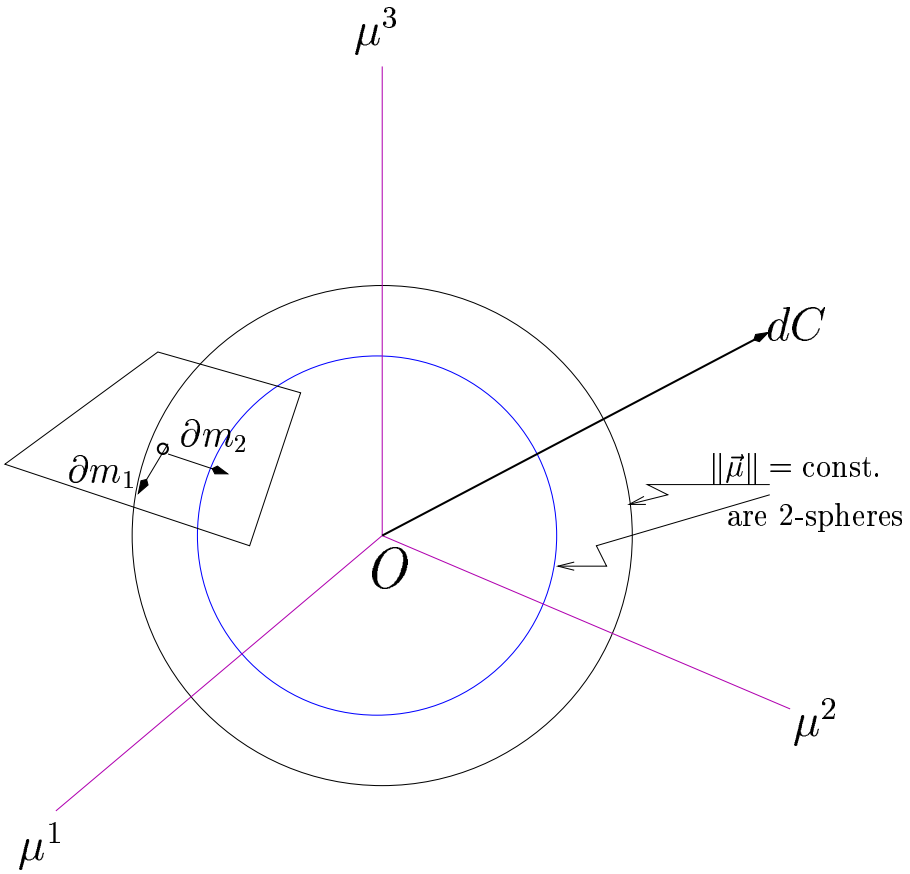

Figure 2.4: Rigid body in 3-D as a Poisson manifold

differential geometric interpretation. In the regular case, the gradients of the Casimirs simply generate the transversal distribution to the symplectic distribution, and the geometry is clear: the gradients fill out the remaining directions. At the singular points, there is no neighborhood about them where rank (being *lower semicontinuous*) stays constant, and hence no means of associating gradients to the corresponding point. (In the example of $P = \mathfrak{so}(3)^*$, this reduces to the observation of there being no well defined notion of tangency and hence gradient at the singular origin.)

We take a cue from the fact that in the case of $P = \mathfrak{g}^*$, the Casimirs of the transverse structure are related to the *algebraic geometry* of the coadjoint orbits. By imposing some symmetry group G acting on the manifold (which we take to be a linear vector space), and a G-invariant potential on it, the 'gradients' at various points, both regular and singular, can be defined in the sense of algebraic varieties.

Since a Poisson structure is purely *kinematical* while a choice of Hamiltonian is *dynamical* (for example, for the Lie coalgebra to $SO(3)$, the rigid body Hamiltionian is only *one* of infinitely many possible, but no matter what the choice, the norm of angular momentum is always conserved), thus through the algebraic geometry of orbits, it is possible to make a connection between kinematics and dynamics.

2.3 Appendix 1: Upper and Lower semicontinuity

Let S be a topological space and f a real valued function $f : S \longrightarrow \mathbb{R}$. The notions of upper semicontinuity (u.s.c) and lower semicontinuity (l.s.c) are less restrictive than continuity, and are useful in several situations in analysis, such as maxima and minima problems or variational principles. Here, we define them, note a couple of results of importance, and give two simple examples [20] [21].

f is u.s.c if for all $\alpha \in \mathbb{R}$, $f^{-1}(-\infty, \alpha)$ is open in S.

f is l.s.c if for all $\alpha \in \mathbb{R}$, $f^{-1}(\alpha, \infty)$ is open in S.

Equivalently, f is u.s.c if the set $\{x|f(x) < \alpha\}$ is open in S for all $\alpha \in \mathbb{R}$ (or $\{x|f(x) \leq \alpha\}$ is closed in S).

f is l.s.c if the set $\{x|f(x) \leq \alpha\}$ is closed in S for all $\alpha \in \mathbb{R}$ (or $\{x|f(x) > \alpha\}$ is open in S).

Some properties

1. If S is compact, then a u.s.c function is bounded above and achieves its maximum, while a l.s.c function is bounded below and assumes its minimum.

2. A real valued function on S is l.s.c iff its *negative* is u.s.c.

3. A function is *continuous* iff it is both u.s.c and l.s.c. On a compact set such a function attains its extremum (could be max. or min.).

Examples

1. Let $E \subset S$ be measurable. Then its *characteristic function* is defined by

$$\chi_E(x) = \begin{cases} 1 & if \quad x \in E \\ 0 & otherwise \end{cases}$$

From the figures 2.5 below it is easily seen that $\chi_E(x)$ is l.s.c on an open set and u.s.c on a closed set.

2. Let P^m be an m-dimensional Poisson manifold, $\rho(J)(x)$ the rank of the corresponding Poisson structure, and $\hat{\rho}(J)(x) = m - \rho(J)(x)$ the *corank*. Then from Property 2 above, it is clear that the former is l.s.c and the latter is u.s.c on P. Since the foliation is by symplectic leaves, jumps down (or up) in rank occur by even integers. Hence the rank may be taken as a function from P onto the set of even integers, $2\mathbb{Z}$, $\rho(J) : P \longrightarrow 2\mathbb{Z}$.

2.4 Appendix 2: Generalized Distributions and their Foliations

The classical theorem of Frobenius states that a distribution is integrable iff it is involutive. As stated and proven in most treatments, the distribution is assumed to be of constant rank. Here, the corresponding generalization of the theorem to the case of variable rank distributions is briefly

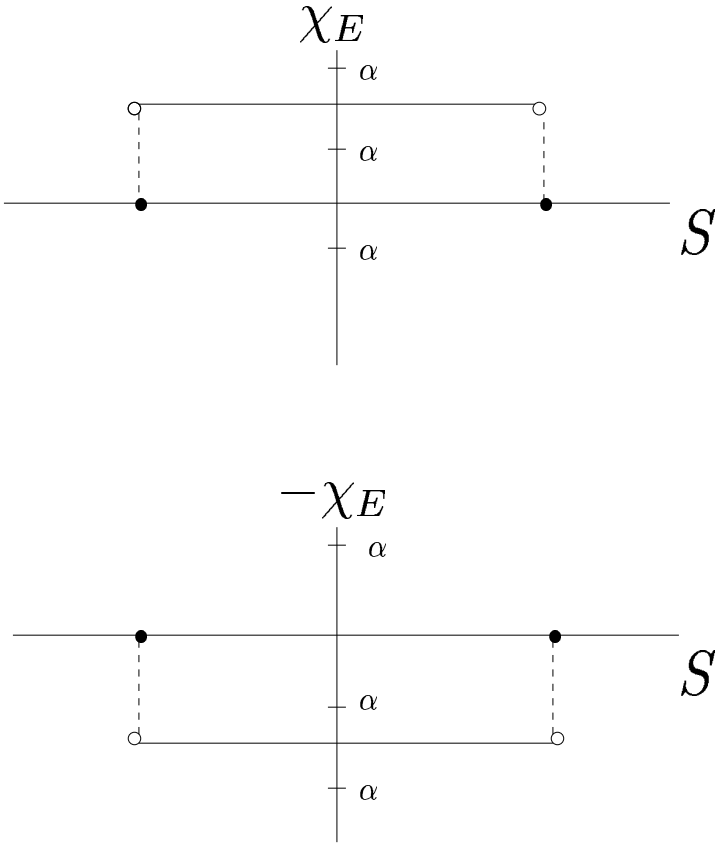

Figure 2.5: Upper and lower semicontinuity

indicated. These were first sought in the context of Riemannian Foliations, and the so called *reachability problem* in control theory [22]. The general theory was worked out in a series of papers by Sussmann, Stefan, Hermann, Molino and others [23] [24] [25] [26]. Subsection 2.4.1 deals with the topological aspects (global), and Subsection 2.4.2 treats the differential aspects (local). Subsection 2.4.3 applies the theory to the case of Poisson manifolds, which was cursorily indicated in Section 2.1 (see also [15], [27]). There is a different, algebraic approach to these issues, pioneered by E. Cartan, for the case that M is a cotangent bundle of some configuration space. (We refer to [28] for this development.)

2.4.1 Completely integrable distributions: global theory

Let M be a differentiable manifold. A *generalized distribution* (hereafter simply, distribution) on M is a subset $\Delta \subset TM$ such that $\forall x \in M$, $\Delta(x) = \Delta \cap T_xM$ is a vector subspace of T_xM. Dim $\Delta(x)$ is called the *rank* of Δ at x. In other words, a distribution is a map that assigns to each point a linear subspace of the tangent space to M at that point.

A *differentiable section* of Δ is a vector field $X \in \mathfrak{X}(M)$ on a neighborhood $U_x \subset M$, such that for $y \in U_x$, $X(y)$ lies in the fiber $\Delta(y)$.

The distribution Δ is said to be *differentiable* if $\forall x \in M$ and $v \in \Delta(x)$ there exists a differentiable section X of Δ defined on a open neighborhood U_x such that $X(x) = v$.

Let Δ be a distribution on M. A *Pfaffian system* Δ^0 is canonically

associated to it. The fiber $\Delta^0(x)$ is the annihilator of the fiber $\Delta(x)$ of Δ at x:

$$\Delta^0(x) = \{\alpha \in T_x^*(M) |\ \langle \alpha, v \rangle = 0 \ \ \forall \ \ v \in \Delta(x)\}$$

We have the relation rank (Δ) + rank (Δ^0) = dim (M).

The rank of a distribution Δ is a lower semicontinuous function (see Section 2.3) on M, so the rank of its annihilator Δ^0 is a upper semicontinuous function. Thus the latter is not a *differentiable* distribution according to the above definition unless its rank is constant in a neighborhood and hence constant everywhere.

An *integral* of a distribution Δ on a manifold M is a pair (S, i) of a connected manifold S and an immersion $i : S \longrightarrow M$ such that

$$\mathrm{d}i_x(T_xS) \subseteq \Delta(i(x)).$$

If equality holds, the integral is said to be of *maximal* dimension at that point $x \in S$.

An *integral manifold* of Δ is an immersion by canonical injection of an integral of Δ. The integral manifolds are in general not embedded submanifolds, so the subspace topology will not usually apply.

A differentiable distribution Δ is called *completely integrable* iff for every $x \in M$ there exists a maximal integral manifold of Δ which contains x.

Theorem (*generalized foliations*)

Let Δ be a completely integrable distribution on a differentiable manifold M that is differentiable in the sense defined above. Then at every point $x \in M$ there is a unique integral manifold S of Δ which is maximal (i.e., has maximal dimension everywhere and contains no other maximal ones). The maximal integral manifolds of Δ form a partition of M into disjoint *leaves* and form the generalized foliation of the given distribution. Further, if (S, i) is an integral of Δ of maximal dimension, then i is a local diffeomorphism from S onto an open subset $i(S)$ of a leaf of the generalized foliation defined by Δ.

Corollary (*constant rank case*)

For constant rank Δ, if (S', i') is an integral manifold of Δ (not necessarily of maximal dimension[3]) then i' is an *immersion* of S' in a leaf of Δ on M.

2.4.2 Local conditions for complete integrability

Now, let us consider the manifold M acted upon by the group of local diffeomorphisms $G = \mathit{Diff}(M)$. Closure and other group properties shall then imply infinitesimal conditions on the vector fields that generate the group along the leaves of the generalized foliations. This leads to a natural extension of Frobenius' criterion for integrability of a distibution to the case of generalized distributions.

[3]For example, a *contact manifold* is $2n+1$ dimensional with leaves of constant dimension $\leq n$.

Let $D \subset \mathfrak{X}(M)$. If G_X is the one parameter group of local diffeomorphisms generated by $X \in D$, then there is the smallest group G_D that contains the union of $\{G_X\}$ $\forall X \in D$. This is the group of local diffeomorphisms generated by D (a subgroup of G).

G_D-equivalence defines *orbits*. The orbits have the following topology:

Let $T = (t_1, \ldots, t_n)$, $t_i \in \mathbb{R}$, $x \in M$, $X \in D^n$ and $\Phi_{X,x}$ the map $T \longrightarrow X_T(x)$ (the flow map giving flow curves), with some $\Omega_{X,x} \subseteq \mathbb{R}^n$ as domain for the flow times t_i. Then the orbit of D through x, S, has the topology that makes the flow map $\Phi_{X,x}$ continuous.

Let Δ be a distribution on M. Let $x \longmapsto g \cdot x$ denote the action of the group of local diffeomorphisms G on M. Then Δ is said to be *G-invariant* if $dg : \Delta(x) \longmapsto \Delta(g \cdot x)$, on the same orbit. Dim $(\Delta(x))$ is constant along a G-orbit.

The following constructs aid in distinguishing the variable rank property of Δ.

1. Let $D \subseteq \mathfrak{X}(M)$. Then there exists a distribution Δ which is spanned by D (that is, any section X of Δ can be written as a linear combination of vector fields from D), call it Δ_D. A distribution that is of the form Δ_D for some everywhere defined subset $D \subseteq \chi(M)$ is called a C^∞ distribution.

2. The smallest distribution that is G_D-invariant and contains Δ_D will be denoted by P_D. Dim $(P_D(x))$ is called the *rank* of the orbit S.

3. $D \subseteq \mathfrak{X}(M)$ is involutive if whenever $X \in D, Y \in D$, then $[X, Y] \in D$.

   Let D^* be the smallest involutive set containing some (in general non-involutive) D. Then the following inclusions hold:

$$\Delta_D \subseteq \Delta_{D^*} \subseteq P_D. \tag{2.10}$$

4. Recall from Subsection 2.4.1 that an integral manifold $S \subset M$ for Δ has its tangent space at any $x \in S$ evaluated by the distribution, that is, $T_x S = \Delta(x)$. A C^∞ distribution Δ that is integrable (has integral manifolds through every point) must be involutive. (The converse is not true; see example below.) Also recall that a *completely* integrable distribution has integral manifolds of maximal dimension.

5. The set P_D has more "directions" so that flows can move away from and come back to the orbit in question, upon the action of the group of diffeomorphisms. Thus we can expect it to characterize variable rank distributions.

1. **Theorem I**

   Let M be a differentiable manifold, D a set of C^∞ vector fields.

   (a) If S is a G_D-orbit of D, then S admits a unique differentiable structure such that $S \subset M$ is an immersed submanifold. The dimension of S equals its rank, and its topology is the orbit topology (defined above).

(b) Every orbit of D is a maximal integral submanifold of P_D.

(c) P_D is involutive.

2. **Theorem II**

Let Δ be a C^∞ distribution on M, D a set of C^∞ vector fields which spans Δ. The following are equivalent:

(a) Δ is integrable.

(b) Δ is completely integrable.

(c) Δ is G_D-invariant.

(d) For every $X \in D$, $t \in \mathbb{R}$, $x \in M$, dX_t maps $\Delta(x)$ to $\Delta(X_t \cdot x)$.

(e) For $x \in M$, $\exists\ X^1, \ldots, X^k$ from D such that $\Delta(x)$ is the linear hull of $X^1(x), \ldots, X^k(x)$. Further, if $X \in D$, there is an $\epsilon > 0$ and C^∞ functions f_j^i $(1 \leq i, j \leq k)$ may be found for $t \in (-\epsilon, \epsilon)$ s.t.

$$[X, X^i](X_t(x)) = \sum_{j=1}^{k} f_j^i(t) X^j(X_t(x)), \quad i = 1, \ldots, k.$$

(f) $\Delta = P_D$.

3. **Theorem III**

Let D be a set of C^∞ vector fileds that satisfy the property of local finiteness: for $x \in M$, $\exists\ X^1, \ldots, X^k \in D$ such that

(a) $X^1(x), \ldots, X^k(x)$ span $\Delta_D(x)$

(b) $X \in D \Longrightarrow \exists\ U_x$ and C^∞ functions f_j^i satisfying

$$[X, X^i](x') = \sum_{j=1}^{k} f_j^i(x') X^j(x'), \quad i = 1, \ldots, k, \quad x' \in U_x.$$

If D is of local finite type, then the distribution Δ_D is completely integrable. If dimension k of $\Delta(x)$ does not change with x, then the distribution is completely integrable iff it is involutive.

An Example

This example is taken from Sussmann [23]. Let $M = \mathbb{R}^2$, $(x, y) \in M$, $D = \{X, Y\}$, $X = \frac{\partial}{\partial x}$, $Y = \phi(x)\frac{\partial}{\partial y}$, $\phi(x) = 0$, $x \leq 0$ and $\phi(x) > 0$, $x > 0$.

$P_D(X, Y)$ has dim 2, $x > 0$. Every $(x, y) \in M$ can be joined by a piecewise integral curve of D. So P_D has dim 2 everywhere. But Δ_{D^*} has dim 1 when $x \leq 0$.

Also, note that the above distribution is involutive:

$$[X, Y](x, y) = \frac{d\phi(x)}{dx}\frac{\partial}{\partial y} = 0,$$

since $\frac{d\phi(x)}{dx} = 0$, $x \leq 0$. This implies that D satisfies involutivity, although it is not integrable. Thus, an involutive generalized distribution violates the classical Frobenius' theorem, and is not necessarily integrable.

Constant rank distributions (*Frobenius' theorem*) [27]

Let M^m be an m-dimensional differentiable manifold, and Δ a differentiable distribution of constant rank $2n$. Then Δ^0, the annihilator of the distribution Δ is a regular Pfaffian system of rank $k = m - 2n$. Clearly, Δ is a differentiable vector subbundle of TM, while Δ^0 is a differentiable vector subbundle of T^*M.

The following are equivalent:

1. Δ is completely integrable.

2. If X, Y are differentiable sections of Δ, $[X, Y]$ is also (that is, Δ is involutive).

3. For every differentiable section $\eta \in \Delta^0$, the 2-form $d\eta$ belongs to the ideal of the algebra of exterior forms on an open set U_x generated by the differentiable sections of Δ^0. That is, $\exists\ (\alpha_1, \ldots, \alpha_{2n})$ sections of Δ^0, and $(\beta_1, \ldots, \beta_{2n})$ Pfaffians defined on the open set U_x s.t.

$$d\eta = \sum_{i=1}^{2n} \alpha_i \wedge \beta_i.$$

Geometrically, about every point $x \in M$ there is a "box" neighborhood U_x with chart coordinates $(x^1, \ldots, x^m)$ s.t. Δ and Δ^0 in U_x are generated by $2n$ vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{2n}}$ and $m - 2n = k$ Pfaffian forms $dx^{2n+1}, \ldots, dx^m$ respectively. The vector fields generate maximal integral manifolds for Δ, and the exact 1-forms generate the complementary dimension maximal integral manifolds for Δ^0.

2.4.3 Application to Poisson manifolds

Let $(P^m,\ J)$ where $m = 2n + k$, be a Poisson manifold with Poisson structure J. Using the notion of a Hamiltonian vector field, we can define the following generalized distribution for J called the *characteristic distribution* [15]. It is defined by setting the spanning vector fields to be precisely the Hamiltonian vector fields,

$$\Delta^c_{P,J}(x) = \{v \in T_xP |\ \exists\ \ f \in C^\infty(P)\ \ s.t.\ \ X_f(x) = v\}, \tag{2.11}$$

$\forall x \in P$.

The main result is that $\Delta^c_{P,J}$ is completely integrable in the generalized sense. The proof of this is simple: from the anti-homomorphism between the Poisson algebra and the algebra of derivations of Hamiltonian vector fields (2.6) it follows that the distribution is involutive

$$X_{\{f,g\}} = -[X_f, X_g],$$

$\forall f, g, h \in C^\infty(P)$, and the X_f's generate the orbits of infinitesimal Poisson automorphisms (which fact follows from $\mathfrak{L}_{X_h}(J(df, dg)) = 0$, by using the elementary properties of the Lie derivative & Leibniz's rule for J, cf. (2.3), (2.4)).

Hence the hypotheses of the previous subsections apply, and the characteristic distribution is completely integrable.

Furthermore, noting that dim $(\Delta^c_{P,J}(x))$ = rank $(J(x))$ = an even integer, we see that symplectic structures which are nondegenerate can be given on these leaves of invariance of the distribution under Poisson automorphisms.

(This can be formally done by restricting the Poisson bracket to be that of coordinate functions defined on the leaf locally, and then extended to a local neighborhood in P.)

Conversely, given a differentiable manifold and a generalized foliation defined by $\Delta^c_{P,J}$ such that each leaf is symplectic, the Hamiltonian vector field tangent to each leaf can be extended (by extending the Hamiltonians), to globally defined vector fields on P, which can then be used to assemble a Poisson bracket out of all the leaves:

$$\{f, g\} := X_f g.$$

This is the Poisson structure (independent of the extension) that contains the symplectic foliations of varying dimension.

Chapter 3

Examples in finite and infinite dimensions

In this chapter, we shall consider some examples of dynamical systems where rank change of the Poisson tensor J manifests in physically important contexts. The first example is that of a rigid body, which has already been mentioned in the previous chapter. In many ways, this example is the prototype for the orbit geometry we need later; in some ways it is not, for example it has no partial degeneracy (where rank of J is less than full but at the same time not zero). The next example generalizes the first one to the case of an arbitrary semisimple Lie algebra. We mention the obvious direct product with the product Poisson structure, as well as the *semidirect product* with the semidirect product Poisson structure, before launching on to the next set of examples both from finite dimensions and a two degree of freedom infinite dimensional system of stratified flow in an ideal fluid, and its resulting moment algebra. A brief appendix outlines the rules involved in computing the Casimirs using methods of invariant theory. This is one indication that the methods of invariant theory and orbit geometry shall play a role in characterizing rank-changing events, which was one of the results announced in the Introduction (see Section 1.2).

3.1 Rigid bodies in n dimensions

3.1.1 3-D rigid body

In 3 dimensions, a rigid body is pictured as rotating freely in space about its center of mass, under no external force. The symmetry group is $SO(3)$, acting on configuration space of three positions (the three Euler angles, for instance) or a phase space of six dimensions, with three positions and three conjugate momenta (defined to be the product of angular velocity and moment of inertia). The group action is pictured as the coadjoint action of $SO(3)$ on the space of angular momenta, $\mu^i \in \mathfrak{so}(3)$, for $i = 1, 2, 3$. The equations of motion are the standard three Euler equations (see, for instance [29], [30]). These equations of motion follow most expeditiously by considering a *noncanonical* formulation involving the momenta alone, which comprise a phase space that is now a Poisson manifold.

We use the linear Poisson structure J suggested by the structure constants ϵ_{ijk} of the Lie algebra of $SO(3)$. As was indicated in Section 2.1.3, this is called the 'Lie-Poisson' structure. For two functions of the angular momenta, $f, g \in C^\infty(\mathfrak{g}^*)$, it is given by

$$\{f, g\}(\mu) := \epsilon_{ijk}\mu^k \frac{\partial f}{\partial \mu^i}\frac{\partial g}{\partial \mu^j} \equiv J(df, dg),$$

a construct that is quite *independent* of the actual form of the Hamiltonian. To recover Euler's equations, the Hamiltonian of choice would be

$$H = \frac{1}{2}\sum_{i=1}^{3} \frac{(\mu^i)^2}{I_i}, \tag{3.1}$$

with I_1, I_2, and I_3 the principal moments of inertia of the body. Using Hamilton's equations, the three Eulerian equations are given by:

$$\begin{aligned} \dot{\mu}^i &= \{\mu^i, H\} \\ &= \epsilon_{ijk}\mu^k I_j^{-1}\mu^j, \text{ for } i = 1, \ldots, 3. \end{aligned} \tag{3.2}$$

We notice that the norm of the angular momentum (norm evaluated with respect to the inner product of the vector space $\mathfrak{g}^*$) is a conserved quantity; that is to say,

$$C = \frac{1}{2}\sum_{i=1}^{3}(\mu^i)^2 \tag{3.3}$$

has null time evolution,

$$\dot{C} = \{C, H\} \equiv 0,$$

and is called a *Casimir* for the Lie-Poisson bracket. Actually, more is true: From the properties of a Poisson bracket (2.3) and (2.4), it follows that

$$\{C, f\} \equiv 0$$

for *any* function $f \in C^\infty(\mathfrak{g}^*)$. Thus, Casimirs are constants along the orbits of the Poisson structure itself, without any reference to the chosen Hamiltonian. (In other words, Casimirs are kinematical, while the Hamiltonian equations (3.2) are dynamical.)

As shall be seen in Subsection 4.1.4, these orbits coincide with those of the coadjoint action of a general Lie group G on its Lie algebra dual $\mathfrak{g}^*$. The

coadjoint orbits (cf. Figure 2.4) in this case are 2-spheres of constant angular momenta, spreading out from the origin $\mu^i = 0\ \forall i$, and the dynamics governed by (3.2) intersect these spheres in ellipsoids of inertia. The six points where ellipsoids of various energies might be tangent with a given angular momentum sphere, form the equilibria of the system.

Writing out the Poisson structure of the rigid body, we find a 3 by 3 antisymmetric matrix

$$(J^{ij}) = \begin{pmatrix} 0 & \mu^3 & -\mu^2 \\ -\mu^3 & 0 & \mu^1 \\ \mu^2 & -\mu^1 & 0 \end{pmatrix}. \tag{3.4}$$

We observe that away from the origin $\mu^i = 0$, the rank of the above matrix is 2. (It has to be an even integer, and less than 3, because an odd antisymmetric matrix has zero determinant.) At the origin, the rank is 0. The equation for null eigenvalues (away from the origin) can be solved by setting

$$J \cdot v = 0,$$

which results in a system of three linear equations. Now, we assume that the null eigenvector $v_i = \partial_i C$, so that the system of equations can be written in terms of their partial characteristics [31] and integrated to give the Casimir (3.3) (up to a numerical proportionality, which is chosen by convention to be 1/2).

At the origin, we could no longer possibly assume that $v_i = \partial_i C$, since the gradient of a function is well defined only in a neighborhood of a point,

whereas the rank, being lower semicontinuous, has no neighborhood of the origin of rank zero. So in this *totally* degenerate case, any function of the three momenta whose graphs have in common the origin would qualify as Casimirs. (For this reason, in most of what follows in this chapter, the totally degenerate point will be excluded from consideration.)

It is also easily seen that the leaf coordinates, that lead to a canonical symplectic structure away from the degenerate origin (the so-called Kirillov Kostant form, cf. Subsection 4.1.4), are given by [30] the (non unique) pair $(m_1 = \mu^3, m_2 = \tan^{-1}(\frac{\mu^1}{\mu^2}))$.

Then the leaf symplectic form (which is easily verified to be closed) is given by

$$\omega(m_1, m_2) = dm_1 \wedge dm_2.$$

(We note that the Jacobi identity reduces to the well known tensor identitiy for the ϵ_{ijk} symbol, $\epsilon_m^{ij}\epsilon_j^{kl} + \epsilon_m^{kj}\epsilon_j^{li} + \epsilon_m^{lj}\epsilon_j^{ik} \equiv 0$.)

These coordinates also effect the transformation that is guaranteed by the generalization of Darboux' theorem (2.7). That is, the (noncanonical) coordinate transformation

$$(\mu^1, \mu^2, \mu^3) \longmapsto (m_1, m_2, C)$$

allows the Poisson structure (3.4) to be written in the (nearly canonical) form

$$(J^{ij}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As a prelude to what is to come in the later chapters, we note that the two dimensional leaf tangents coincide with the two roots of the Lie algebra with respect to the adjoint system, while the Casimir is an invariant — that is, it forms the central element (one that commutes with all other elements) of the enveloping algebra to the Lie algebra. The value for the Casimir invariant is in fact simply equal to $\frac{1}{4}\mathrm{tr}J^2$, where the $1/4$ normalization is purely conventional. (Also, we note that the first invariant, $\mathrm{tr}J = 0$, by construction (likewise for all odd invariants), while the even dimensional invariants are simply a function of the quadratic Casimir.)

From the symmetry viewpoint, we note that each point on the leaf is invariant not under the full $SO(3)$, but rather only a $U(1)$ factor. This makes physical sense, as all $SO(2)$ rotations about an axis through the origin leave the point (and its antipodal point: which reflects the fact that there is also a Weyl symmetry, $S_2 \simeq \mathbb{Z}_2$ in this case) invariant. Also, any two points on the same sphere are related by conjugate rotations, with the conjugate element coming from $SO(3)$.

Stability

Adding a term linear in the momentum to the rigid body Hamiltonian (3.1), allows us to see the relevance of rank change to stability of the solution [32]. Let us assume the rigid body is charged and placed in a uniform magnetic

field. This gives a modified Hamiltonian:

$$\widetilde{H} = \sum_{i=1}^{3} \left(\frac{(\mu^i)^2}{I_i} + B^i \mu^i \right), \tag{3.5}$$

(with B^i the 3 components of the magnetic field). We choose the Poisson structure to be the same [1] as the one before (3.4), so that there is one Casimir invariant, given by (3.3). The equations of motion (3.2) now take the modified form:

$$\dot{\mu}^i = \{\mu^i, \widetilde{H}\} = \epsilon_{ijk} \mu^k (I_j^{-1} \mu^j + B^j). \tag{3.6}$$

To determine equilibria, we set the time derivatives of the dynamical variable in the above equation (3.6) to zero. One such equilibrium is the stationary one, $\mu^i = 0$ $i = 1, \ldots, 3$. It can be shown [29] that the correct function to extremize in the presence of a Casimir invariant (that is, for a noncanonical system) is the *free energy*, defined to be $F = \widetilde{H} + \lambda C$, with the Casimir C given above (3.3), and λ is an arbitrary Lagrange multiplier. Extremizing this function would yield all the equilibria of the system, as it takes into account the constraint imposed by the Casimir. Clearly, thanks to the additional term λC, variation with respect to the momenta μ^i do not yield, for any value of λ, the stationary equilibrium.

[1] While the choice of a different Hamiltonian may change the dynamics, the *kinematics* depends on only the Poisson structure chosen. For example, the norm of the angular momentum is known to be conserved, and the above choice correctly reflects this fact. (One of the aims in Chapter 5 will be to relate kinematics and dynamics using the notions of patterns in symmetry breaking.)

This inequivalence between the equilibria implied by setting the time evolution equations of motion to zero, and those obtained by setting the free energy variation to zero illustrates why rank change is important. The stationary equilibrium corresponds precisely to the point where the rank of the Poisson structure falls from two to zero. This information is captured by the equations of motion, but not by the extremization of free energy. Thus, the set of equilibria obtained by considering constrained variations through varying F is a subset of the full set of equilibria. (This is a general symptom of the rank change phenomenon.) We also note that the new equilibria obtained by extremizing F alone are not to be found via the equations of motion (3.6) with $\widetilde{H}$ in the evolution Poisson bracket, but can be obtained by substituting F for $\widetilde{H}$.

Similar considerations apply when considering stability of the solutions. The second derivative of the free energy (which arises from a variation of the equilibria) is the quantity considered to ascertain whether the equlibrium point is a stable one. Again, conditions for stability that result from adding the appropriate Casimir invariant break down at the origin, where rank change of the Poisson structure occurs. This is because, in order to correctly guage stability at these points, the subcasimirs that arise upon rank change have also got to be taken into account.

3.1.2 n-D rigid body generalization

Now the group is taken to be $SO(n)$, or as a generalization, any semisimple Lie group, G (in which event the term 'rigid body' ought not to be taken literally). As in the 3-dimensional case, we write the Lie-Poisson bracket (2.9) to be the one suggested by the structure constants of the Lie algebra $\mathfrak{g}$; for any functions $f, g \in C^\infty(\mathfrak{g}^*)$ we set

$$\{f,g\}(\mu) := \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] \right\rangle = \mu^i \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right]_i , \tag{3.7}$$

where $\frac{\delta f}{\delta \mu} = (\frac{\delta f}{\delta \mu})^l e_{(l)}$, $\frac{\delta g}{\delta \mu} = (\frac{\delta f}{\delta \mu})^m e_{(m)}$, where $\{e_{(i)}\}$ is a basis for $\mathfrak{g}^*$, and the inner bracket $[\ ,\ ]$ stands for the Lie algebra bracket,

$$[e_{(l)}, e_{(m)}] = C_{nlm} e_{(n)}$$

and C_{nlm} the structure constants of the Lie algebra. The Poisson tensor now becomes

$$J^{lm} = C_n^{lm} \mu^n. \tag{3.8}$$

(Although not explicitly stated, use has been made above of the natural identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ using the (nondegenerate) Killing form.)

We note that the origin is once again a point of total degeneracy, where rank of the Poisson tensor is zero. Elsewhere, it is equal to the *corank* of the Lie algebra, which is also the codimension of its Cartan subalgebra. In fact, the Cartan torus is generated by the *duals* of the differentials of the Casimirs, which are each responsible for one $U(1) \simeq SO(2)$ factor. These various $U(1)$ factors

commute, as they must for a torus, and this fact is captured in one of the orbit theorems that distinguish between regular and singular points. (When rank of J is less than full, the differentials of the Casimirs of the reduced dimensional orbit (which now include new functions called subcasimirs) no longer commute, and hence form a nonabelian subalgebra of $\mathfrak{g}$. These considerations are echoed in Chapter 4, where orbit geometry is used to relate the invariants of the orbits to the Casimirs and subcasimirs of the Poisson structure.)

Elsewhere, away from the origin, the leaves have the dimensionality given by the difference between dimension d of $\mathfrak{g}$ and its rank r, which is always an *even* integer, as it also equals the number of roots (which are none other than weights of the adjoint representation). So the orbits are always even dimensional, and there is in fact a nondegenerate symplectic form (4.16) that inverts (3.7) away from the origin [2].

The symplectic structure on the coadjoint orbits of a Lie coalgebra to a Lie group can also be shown to arise from a symplectic reduction of the cotangent bundle $M = T^*G$ by G (further modded out by the isotropic subgroup G_μ at $\mu \in \mathfrak{g}^*$). That is, $\Omega(\mu) = M/G/G_\mu := M//G$. The symplectic structure on the leaf $\Omega(\mu)$ is none other than the Kirillov Kostant form, while the Poisson structure of $\mathfrak{g}^* = \boldsymbol{\rho}^{-1}(\mu)$, with $\boldsymbol{\rho} : M \longrightarrow \mathfrak{g}^*$ the moment map, is

[2]We emphasize the distinct usage of the word 'rank' for *both* the Lie algebra and the Lie-Poisson structure on its dual: the former refers to the number of Casimirs, while the latter refers to the coadjoint orbit dimension. We continue using the commonly used term 'rank' for either case, taking care to note that rank of a Lie algebra is numerically equal to the corank of the Lie-Poisson structure on its dual.

the Lie-Poisson structure (3.7) above. (The Lie-Poisson structure itself can be shown to arise from a single Poisson reduction of T^*G by G. For more details on reduction in general, see Subsection 4.1.6.)

For examples, let us consider a few semisimple Lie algebras. $G = SU(n)$ has dim $G = d = n^2 - 1$, rank $G = r = n - 1$, so $d - r = n(n-1)$, which is always even. SO(n) has $d = n(n-1)/2$ and $r = (n-1)/2$ (n odd), or $r = n/2$ n even). In either event, $d - r = (n-1)^2/2$ (n odd), or $d - r = n^2/4$ (n even). Below is a short list for the various classical and exceptional Lie algebras, of dimension n and rank r, illustrating the above general fact:

$$\begin{aligned}
A_r = SU(r+1):\ n - r &= (r+1)^2 - 1 - r \\
&= r(r+1) \\
B_r = SO(2r+1):\ n - r &= r(2r+1) - r \\
&= 2r^2 \\
C_r = Sp(2r):\ &\text{—same as for } B_r\text{—} \\
D_r = SO(2r),\ r \geq 3:\ n - r &= r(2r-1) - r \\
&= 2r(r-1)
\end{aligned}$$

$$
\begin{aligned}
G_2: \ n-r &= 14-2 \\
F_4: \qquad &= 52-4 \\
E_6: \qquad &= 78-6 \\
E_7: \qquad &= 133-7 \\
E_8: \qquad &= 248-8.
\end{aligned}
$$

All semisimple Lie algebras in the adjoint representation have the Casimirs determined by the adjoint action [33]. There are r of them, omitting the first order one, which is always zero. Let $\mathfrak{g}$ be a semisimple Lie algebra, and ϑ any representation operator. Then for $\xi_1, \ldots, \xi_n \in \mathfrak{g}$, let $[\xi_i, \xi_j] = C_{ij}^k \xi_k$. The Killing form (which is always nondegenerate for a semisimple $\mathfrak{g}$), is defined by $B_{mm'} = B(\xi_m, \xi_{m'}) = C_{mn}^r C_{m'r}^n$. Transform to a new basis,

$$\eta_i = (B^{-1})_{ji} \xi_j.$$

Then the Casimir operators are given by the relation

$$C_j = \sum_{p_1,q_1,p_2,q_2,\ldots=1}^{n} C_{p_1 q_1}^{q_2} C_{p_2 q_2}^{q_3} \ldots C_{p_j q_j}^{q_1} \, \vartheta(\eta_{p_1}) \, \vartheta(\eta_{p_2}) \ldots \vartheta(\eta_{p_j})$$

or, since $C_{pq}^r = (ad\xi_p)_{rq}$,

$$C_j = \sum_{p_1,q_1,\ldots=1}^{n} tr\{ad(\xi_{p_1}) \ldots ad(\xi_{p_j})\} \, \vartheta(\eta_{p_1}) \, \vartheta(\eta_{p_2}) \ldots \vartheta(\eta_{p_j}).$$

where the index on the Casimir runs from 1 to rank $\mathfrak{g}$.

Let us now consider what happens when one of the Casimirs becomes zero. This is the same as having one dot on the Dynkin diagram erased [34],

and we have a so called S-subalgebra $\mathfrak{g}' \subset \mathfrak{g}$. The rank drops by 1, and the leaf dimension is now reduced by an even integer. For example, for A_{r-1}, the difference $d - r = r^2 - r$, so the change in leaf dimension is by $2r$. For B_r and C_r, the change is $2(2r-1)$, for $D_{r\geq 3}$ it is $4(r-1)$. (For exceptionals, Dynkin's list [35] must be consulted.)

In Chapter 5, some of these considerations will be used in the notion of *symmetry breaking*. When symmetry breaking occurs from a group G to a subgroup H, then there is the question of whether the rank changes [36], and if so how it does. In general, from a purely dynamic viewpoint, it is unclear if a breaking of symmetry always entails rank change and if so, by how much. But in the case of the Poisson manifold foliation, we see that rank always changes by an even integer; so symmetry breaking patterns modeled on a Poisson manifold are always required to be rank changing. (In practice, this would require a suitable choice of potential and control parameters.)

Finally, there is the question of parametrizing the various singular orbits (defined as those whose dimension is less than what is generically the case) that arise as orbits of the coadjoint action of Lie groups on the corresponding Lie coalgebras. Again, by methods close to invariant theory, several of these have been classified, and orbits that are representable as homogeneous spaces have a natural parametrization in terms of algebraic geometry, as the intersection of certain algebraic curves. For details of this type of work we refer to [37], [38], [39], [40]. For more details on this aspect, see also Subsection 4.1.4.

3.2 Semidirect product case

One source of examples for subcasimirs are semidirect products, and their special case, direct products. We first define the bracket in general, and give a simple example of a rigid body in a gravitational field, followed by an example taken from [41] with more degrees of freedom.

3.2.1 Definitions and Properties

Let G and H be two Lie groups with their product operations denoted by $\circ$ and $\star$, respectively. Let $\vartheta : G \times H \longrightarrow H$ be a C^∞ action, so that for $g \in G$ fixed, we have $\vartheta_g : H \longrightarrow H \in Aut\ (H) \simeq GL(H)$. By the hypotheses of an action (cf. Subsection 4.1.1),

$$\vartheta_g(h \star h') = \vartheta_g(h) \star \vartheta_g(h') \tag{3.9}$$

for $h, h' \in H$.

Define a composition rule on $G \times H$ by

$$(g, h) \cdot (g', h') = (g \circ g', h \star \vartheta_g(h')) \tag{3.10}$$

for $g, g' \in G$ and $h, h' \in H$.

It can be easily seen, using the properties of the action ϑ in (3.9) that the product '$\cdot$' defined above (3.10), is a legitimate group operation on $G \times H$. That is, it is

1. Associative:

$$\begin{aligned}
(g,h)\cdot[(g',h')\cdot(g'',h'')] &= (g,h)\cdot(g'\circ g'', h'\star\vartheta_{g'}(h'')) \\
&= (g\circ g'\circ g'', h\star\vartheta_g(h'\star\vartheta_{g'}(h''))) \\
&= (g\circ g'\circ g'', h\star\vartheta_g(h')\star\vartheta_g(\vartheta_{g'}(h''))) \\
&= (g\circ g'\circ g'', h\star\vartheta_g(h')\star\vartheta_{g\circ g'}(h'')) \\
&= (g\circ g', h\star\vartheta_g(h'))\cdot(g'',h'') \\
&= [(g,h)\cdot(g',h')]\cdot(g'',h'').
\end{aligned}$$

   for $g, g', g'' \in G$ and $h, h', h'' \in H$.

2. Unit is (e_G, e_H):

   Since $\vartheta_g(h\star e_H) = \vartheta_g(h) = \vartheta_g(h)\star\vartheta_g(e_H)$, we have $\vartheta_g(e_H) = e_H$. Using this we verify:

$$\begin{aligned}
(g,h)\cdot(e_G,e_H) &= (g\circ e_G, h\star\vartheta_g(e_H) \\
&= (g,h) \\
&= (e_G,e_H)\cdot(g,h).
\end{aligned}$$

3. Inverse exists and is unique:

$$\begin{aligned}
(g,h)\cdot(g,h)^{-1} &= (e_G,e_H) \\
&= (g,h)\cdot(g',h'), \text{ say,} \\
&= (g\circ g', h\star\vartheta_g(h')) = (e_G,e_H)
\end{aligned}$$

$\Longrightarrow g' = g^{-1}$ and $h' = \vartheta_g^{-1}(h^{-1})$. Since $\vartheta_g \star \vartheta_{g^{-1}} = \vartheta_{e_G} \Longrightarrow \vartheta_g^{-1} = \vartheta_{g^{-1}}$, hence

$$\begin{aligned}(g,h)^{-1} &= (g^{-1}, \vartheta_g^{-1}(h^{-1}) \\ &= (g^{-1}, \vartheta_{g^{-1}}(h^{-1})).\end{aligned}$$

Uniqueness follows easily.

$G \times H$ with the above product structure (3.10) is called semidirect product, and the homomorphism ϑ from G to $GL(H)$ is denoted $G \ltimes_\vartheta H$.

For the special case that ϑ is *trivial*, that is, $\vartheta_g = Id_H \ \forall g \in G$, we get

$$(g,h) \cdot (g',h') = (g \circ g', h \star h')$$

or, a *direct product* bracket, $G \ltimes_{Id_H} H \simeq G \times H$.

As an example of the semidirect product, let G be a matrix group and H a vector space (with $\star \equiv +$ as the Abelian group operation), and ϑ a (linear) representation of G on $H = V$.

The product (3.10) then becomes

$$(g,v) \cdot (g',v') = (g \circ g', v + \vartheta_g(v')) \tag{3.11}$$

where $g, g' \in G$, $v, v' \in V$. If G is a subgroup of the orthogonal group (and hence norm preserving), this is the *Euclidean* group of motions.

3.2.2 Infinitesimal version of Semidirect Product

Let G and H be Lie groups, with $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras, isomorphic to their respective tangent spaces at their identities, e_G and e_H.

For every action $\vartheta_g : H \longrightarrow H \in GL(H)$ we define the corresponding tangent map,

$$d\,\vartheta_g : T_{e_H}H \longrightarrow T_{e_H}H \in GL(\mathfrak{h})$$

by writing, for a 1-parameter family curve $g(t) = \exp\, tX$, with $g(1) \equiv g$,

$$\vartheta'_X(Y) \equiv d\,\vartheta_g(Y) = \left.\frac{d}{dt}\right|_{t=0} \vartheta_{\exp\, tX}(Y), \tag{3.12}$$

for $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$.

ϑ' is a homomorphism from $\mathfrak{g}$ into the derivations of $\mathfrak{h}$, namely $Aut\ (\mathfrak{h})$, i.e., it satisfies the following two properties:

$$\begin{aligned} \vartheta'_{[X_1,X_2]_\mathfrak{g}} &= \vartheta'_{X_1} \star \vartheta'_{X_2} - \vartheta'_{X_2} \star \vartheta'_{X_1}, \\ \vartheta'_X([Y_1,Y_2]_\mathfrak{h}) &= [\vartheta'_X(Y_1), Y_2]_\mathfrak{h} + [Y_1, \vartheta'_X(Y_2)]_\mathfrak{h}. \end{aligned}$$

Define the semidirect product of Lie algebras $\mathfrak{g} \ni X_{1,2}$ and $\mathfrak{h} \ni Y_{1,2}$ by

$$[(X_1,Y_1),(X_2,Y_2)] = ([X_1,X_2]_\mathfrak{g}, [Y_1,Y_2]_\mathfrak{h} + \vartheta'_{X_1}(Y_2) - \vartheta'_{X_2}(Y_1)). \tag{3.13}$$

It can be shown that the semidirect product defined above satisfies the usual properties of a Lie algebra, namely skew-symmetry, bilinearity and Jacobi's identity; further, using the defining relation (3.12), that the product, denoted by $\mathfrak{g} \ltimes_{\vartheta'} \mathfrak{h}$ is the Lie algebra of $G \ltimes_\vartheta H$. Further, the *dual* to the semidirect product algebra, denoted by $\mathfrak{g}^* \ltimes_{\vartheta^*} \mathfrak{h}^*$, has the structure of a (degenerate)

Poisson manifold. The semidirect *representation* then occurs via the coadjoint action of the first Lie algebra upon the second (see, for instance, [42]):

$$\{F,G\}(\mu,a) = \{F,G\}_{L-P} + \left\langle a, \vartheta'\left(\frac{\delta F}{\delta \mu}\right)\cdot\frac{\delta G}{\delta a}\right\rangle - \left\langle a, \vartheta'\left(\frac{\delta G}{\delta \mu}\right)\cdot\frac{\delta F}{\delta a}\right\rangle, \tag{3.14}$$

where $F, G : \mathfrak{g}^* \ltimes_{\vartheta^*} \mathfrak{h}^* \longrightarrow \mathbb{R}$, and $F_\mu \in \mathfrak{g}$, $F_a \in \mathfrak{h}$ by identification with the dual. The term subscripted 'L-P' stands for the direct sum of the Lie-Poisson structures on $\mathfrak{g}$ and $\mathfrak{h}$.

Examples of such semidirect Lie-Poisson structures will occur through the rest of this chapter. Many other examples of semidirect products and their extensions may be found in [43].

3.2.3 Examples of rank change

The finite dimensional examples below illustrate how subcasimirs result by restricting oneself to lower dimensional orbits from the generic orbits. In the spirit of the main idea, this process may be imagined to occur via certain control parameters present in a potential or Hamiltonian.

Rigid body under gravity

Departing from the notation of the rigid body a bit, we denote by $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$, the angular momentum of the body, and by $\vec{z} = (z_1, z_2, z_3)$, the (constant) gravitational vector from the center of mass. The configuration space is the 6-dimensional vector $(\vec{\mu}, \vec{z})$. $SO(3)$ acts according to the semidirect product bracket (3.11), so we write $W = (SO(3) \ltimes_\vartheta \mathbb{R}^3)$, where ϑ is the Adjoint

representation (cf. Subsection 4.1.4), and define the product rule at the group level

$$(R, v) \cdot (R', v') = (RR', Rv' + v)$$

with $R, R' \in SO(3)$, $v, v' \in \mathbb{R}^3$.

The dual of the Lie algebra of the semidirect product group is $\mathfrak{w}^* = \mathfrak{so}(3)^* \ltimes_{\vartheta^*} \mathbb{R}^3$, where ϑ^* is the coadjoint representation, with the semidirect product Poisson structure given by

$$J(\vec{\mu}, \vec{z}) = \left(\begin{array}{ccc:ccc} 0 & \mu_3 & \mu_2 & 0 & z_3 & z_2 \\ -\mu_3 & 0 & -\mu_1 & -z_3 & 0 & -z_1 \\ -\mu_2 & \mu_1 & 0 & -z_2 & z_1 & 0 \\ \hdashline 0 & z_3 & z_2 & 0 & 0 & 0 \\ -z_3 & 0 & -z_1 & 0 & 0 & 0 \\ -z_2 & z_1 & 0 & 0 & 0 & 0 \end{array} \right). \tag{3.15}$$

The matrix can be abbreviated for convenience, by

$$J(\vec{\mu}, \vec{z}) = \left(\begin{array}{cc} \hat{\mu} & \hat{z} \\ \hat{z} & 0 \end{array} \right), \tag{3.16}$$

where the hatted entries correspond to the subblocks in the above matrix (or in other words, $\hat{\mu}$ could be viewed as the isomorphism between a cross product and its antisymmetric matrix representation in the semidirect algebra).

The rank of the matrix (3.15) is easily seen to be 4. There are two Casimirs, $C_1 = \vec{\mu} \cdot \vec{z}$ and $C_2 = \|\vec{z}\|^2$ generically, which means that in Darboux' coordinates, the maximal symplectic leaves have dimension 4. When $z_i \equiv$

$0\ \forall i$, then the matrix reduces to the one examined in the previous subsection, namely that of a free rigid body. A new subcasimir function that arises is $C_3 = \|\vec{\mu}\|^2$, in addition to a trivial subcasimir that says that z coordinate dependence does not exist, $C_4 = f(z) \equiv 0$. The leaves are 2-spheres, as in the rigid body case, with a totally degenerate origin when all subcasimirs collapse to arbitrary functions of μ and z which intersect at zero.

Underwater vehicle dynamics

This example is taken from [41]. It is the extension of the preceding example to include one more force that operates on an underwater vehicle, namely buoyancy, so there are more rank changes here. Denote the buoyancy vector by $\vec{b} = (b_1, b_2, b_3)$, the phase space vector by $(\vec{\mu}, \vec{z}, \vec{b})$, the semidirect product group by $W = SO(3) \ltimes (\mathbb{R}^3, \mathbb{R}^3) \simeq (SO(3) \ltimes \mathbb{R}^3) \ltimes \mathbb{R}^3$, (for simplicity, we supress the adjoint homomorphism) with the group product operation defined as:

$$(R, v, w) \cdot (R', v', w') = (RR', Rv' + v, Rw' + w).$$

The coadjoint space is given by $\mathfrak{w}^* = \mathfrak{so}(3) \ltimes \mathbb{R}^3 \ltimes \mathbb{R}^3$, with the Lie-Poisson structure (in the condensed notation of (3.16))

$$J(\vec{\mu}, \vec{z}, \vec{b}) = \begin{pmatrix} \hat{\mu} & \hat{z} & \hat{b} \\ \hat{z} & 0 & 0 \\ \hat{b} & 0 & 0 \end{pmatrix},$$

which is now a 9 by 9 matrix. At a generic point the coadjoint orbit has dimension 6, as there are 3 Casimirs $C_1 = \vec{b} \cdot \vec{z}$, $C_2 = \|\vec{z}\|^2$, $C_3 = \|\vec{b}\|^2$. Nongeneric orbits come about in two stages:

1. 4-dimensional orbits that occur when $\vec{z} \,||\, \vec{b}$, which leads to two subcasimirs, $C_4 = \vec{\mu} \cdot \vec{z}$, $C_5 = \vec{\mu} \cdot \vec{b}$.

2. 2-dimensional orbits (akin to free rigid body) that arise when $\vec{z} = \vec{b} = 0$, with the nontrivial subcasimir $C_6 = \|\vec{\mu}\|^2$. As before, the origin in all 9 coordinates is a totally degenerate point.

3.3 Moment algebra for $(2+1)$-D Ideal Fluid

This example [44] is in essence an infinite dimensional one [3], but proceeding by stages in the order of the moments, it is possible to examine the semidirect Poisson bracket like in the previous examples. The details of this example might appear in some future publication with the persons referenced above; in the interests of brevity, we shall state the bare minimum to illustrate the ideas of the thesis.

A stratified fluid is a modification of an incompressible ideal fluid, so we consider the latter first. It is by now well known that the ideal fluid flow has a noncanonical Hamiltonian structure in function space [29]. For the purpose at hand, we consider the dynamics of the scalar *vorticity*, $\omega(x, y, t) = \hat{z} \cdot \nabla \times v$, where v is the Eulerian (i.e., spatial and not co-moving, or material) velocity, assumed to be divergence free, $\nabla \cdot v = 0$. The latter condition permits introduction of a streamfunction $\psi(x, y, t))$ such that its skew-gradient is the

[3] I thank Tom Yudichak for many useful discussions on the invariant theory aspect of this example, as well as help with Maple and Mathematica.

velocity:

$$(v_x, v_y) = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \right).$$

The Hamiltonian is specified over a domain $D \subseteq \mathbb{R}^2$, as the kinetic energy of the vortex (finite if the vorticity is chosen to vanish at infinity sufficiently rapidly),

$$\begin{aligned} H[\omega] &= \frac{1}{2} \int_D v^2 dx dy \\ &= \frac{1}{2} \int_D \omega \psi dx dy. \end{aligned} \tag{3.17}$$

In infinite dimensions, the Poisson bracket is defined in a functional setting. If $F, G \in C^\infty(\mathfrak{B})$, with $\mathfrak{B}$ some Banach space, are functionals of variables $\{\omega^i\}$, $i = 1, \ldots, n$, which are real-valued functions, then

$$\{F, G\} = \int_D \frac{\delta F}{\delta \omega^i} \mathfrak{J}^{ij} \frac{\delta G}{\delta \omega^j} d\mu = \left\langle \frac{\delta F}{\delta \omega}, \mathfrak{J} \frac{\delta G}{\delta \omega^j} \right\rangle$$

is the Poisson structure. (It satisfies the Leibnitz rule (2.3) and Jacobi identities (2.4), besides — owing to integration by parts — being manifestly skew-symmetric.) For the scalar vortex, there is a Poisson bracket (cf. [45, 46], [47]) given by

$$\{F, G\} = \int_D \omega \left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega} \right] dx dy = \left\langle \omega, \left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega} \right] \right\rangle, \tag{3.18}$$

which has a form that resembles the Lie-Poisson bracket (3.7), with the inner Lie bracket being a function space tangent algebra bracket. Any arbitrary functional of the form $\int_D C(\omega) dx dy$ is a Casimir for the above bracket (which

is related to the fact that all dynamics are constrained to rearrangements, cf. [48], [49], [50]).

The equation of motion for the vortex is governed by Hamilton's principle, $\delta H[\omega] \equiv 0$, which yields, upon use of the above bracket,

$$\frac{\partial \omega}{\partial t} = \{\omega, H\} = [\omega, \psi].$$

One of the interesting facts about the scalar vortex is that the evolution dynamics can be studied entirely in terms of its projection down to a finite dimensional space of *moments* [51]. This process is akin to the reduction in variables of a phase space due to the presence of a symmetry. (In this case the symmetry group is that of all volume preserving diffeomorphisms.) The reduced dynamics can be studied in its own right and projected back onto the original (infinite dimensional) space to recover the full dynamics. One of the main problems of plasma physics is to see if reduction by moments could obtain for a large class of theories that are describable through a Lagrangian or an Hamiltonian.

We supress the integration measure and domain of integration in what follows. The moment description involves a space of moments, (a_1, a_2, a_3), where

$$\begin{aligned} a_1 &= \int \frac{1}{2} x^2 \omega, \\ a_2 &= \int xy\omega, \\ a_3 &= \int \frac{1}{2} y^2 \omega. \end{aligned} \tag{3.19}$$

The moments could easily be shown to satisfy the following commutation relations, where we use the Lie-Poisson structure (3.18) to evaluate them:

$$\begin{aligned} \{a_1, a_2\} &= 2a_1, \\ \{a_2, a_3\} &= 2a_3, \\ \{a_3, a_1\} &= -a_2, \end{aligned} \tag{3.20}$$

which is the standard commutation algebra of the Lie algebra of $SO(2,1) \simeq SL(2,\mathbb{R})$. The Casimir for this Lie algebra (where the metric induced by the Killing form has signature (1, 1, -1)), is $C = a_1^2 + a_2^2 - a_3^2$ (whose level sets are now noncompact hyperboloids, and so the symplectic leaves they define have mild orbifold singularities). In physics, when a field theory system is fully describable by a finite set of moments, it is said to have the property of *closure*. This moment algebra reduces the infinite dimensional scalar vortex dynamics to that of a three parameter vortex patch, known as a Kida vortex [51, 29].

Now, let us consider adding a term proportional to the density, $\rho(x, y)$ to the Hamiltonian (3.17). This procedure is called *stratification*, in the event that the density varies in the vertical direction alone. (In experimental situations [52] the flow is in the $x - z$ plane, with the y direction being considered 'infinite'). Adding a term proportional to ρy would amount to adding gravitational potential energy to the pure kinetic model of (3.17). (Other terms could be added with no loss of generality to the considerations to follow, but their physical meaning is obscure.) We consider a moment expansion for the

density profile [4],

$$\rho(x,y) \sim \sum b_{ij},$$

with b_{ij} representing the moments for density:

$$b_{ij} = \int x^i y^j \rho, \tag{3.21}$$

where $i, j = 0, 1, \ldots$.

We note that b_{ij} are essentially monomials in x and y, so we form a series of vector spaces, $\{\mathbb{V}^k\}$ consisting of all moments of homogeneous degree k. (For example, $\mathbb{V}^2 = (b_{20}, b_{11}, b_{02})$ for quadratic order. In general, $\mathbb{V}^k$ has $k+1$ elements.)

The algebra of homogeneous polynomials under the Poisson bracket like (3.18) would display the following derivation rule,

$$\{\ ,\ \} : \mathbb{V}^m \times \mathbb{V}^n \longrightarrow \mathbb{V}^{(n+m-2)}$$

so that only the tensor product of all these spaces $\mathbb{V}^1 \otimes \mathbb{V}^2 \otimes \cdots$ is closed under the bracket (except for $n = m = 3$). The situation however changes when we *define* the following semidirect generalization of the Lie-Poisson bracket (3.18)

$$\{F, G\} = \int \omega[F_\omega, G_\omega] + \rho([F_\omega, G_\rho] - [G_\omega, F_\rho]), \tag{3.22}$$

with F, G arbitrary functionals of both the vorticity and the density, and the subscripts on them with respect to these arguments stands for the operation of functional derivative.

[4] A similar moment expansion obtains for the vorticity, but we restrict the vorticity moments to be the same three quadratic ones as before.

It turns out that [44] the moment algebra of the stratified fluid with the above semidirect algebra, is actually closed to any order of truncation of the density moments (3.21). That is, the space of monomials given by $\{a_1, a_2, a_3\} \cup \{\mathbb{V}^k\}$ for any k, is closed with respect to the above bracket. Hence, the dynamics of the stratified fluid may be projected down to its space of moments order by order, to any desired level of approximation. (It is as yet unknown if the evolution of the moments with time according to the above semidirect Poisson bracket algebra faithfully renders, at any level of truncation, the physical dynamics got from the equations of motion directly.)

We note that the above bracket (3.22) represents the *dual* to the semidirect product of $\mathfrak{sl}(2)$ with $\mathbb{V}^k$ via the adjoint action. That is, we collect together the moment algebra and take their Poisson brackets mutually to get a structure matrix representing the semidirect product of the dual algebra. For the case of $k = 1$, we would get a 5 by 5 matrix of quadratic a_i moments, and linear b_{01}, b_{10} moments. There is one Casimir. For the case of $k = 2$, we would end up with a 6 by 6 matrix that is virtually identical to the semidirect product of the rigid body in gravity (3.15) (except for scaling, signature, and a linear transformation to diagonal coordinates [51]), and is given below:

$$J_6 = \left(\begin{array}{cccccccc} 0 & 2a_1 & a_2 & \vdots & 0 & b_{20} & 2b_{11} \\ -2a_1 & 0 & 2a_3 & \vdots & -2b_{20} & 0 & 2b_{02} \\ -a_2 & -2a_3 & 0 & \vdots & -2b_{11} & -b_{02} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 2b_{20} & 2b_{11} & \vdots & 0 & 0 & 0 \\ -b_{20} & 0 & b_{02} & \vdots & 0 & 0 & 0 \\ -2b_{11} & -2b_{02} & 0 & \vdots & 0 & 0 & 0 \end{array} \right). \qquad (3.23)$$

Here, we have used the cross-commutation relations:

$$\begin{aligned} \{a_1, b_{ij}\} &= j b_{(i+1)(j-1)} \\ \{a_2, b_{ij}\} &= (j-i) b_{ij} \\ \{a_3, b_{ij}\} &= -i b_{(i-1)(j+1)}. \end{aligned}$$

There are two quadratic Casimirs, just as for (3.15) (except with one minus sign in each), and similar comments regarding rank change and sub-casimirs apply (although now, with the moments as vectors in a vector space $\mathbb{V}^6 = (a_1, a_2, a_3, b_{20}, b_{11}, b_{20})$ there is no physical meaning to the interpretation of parallel vectors, as with the rigid body in gravity). The Casimirs are $C_6^1 = b_{11}^2 - b_{20}b_{02}$ and $C_6^2 = a_1 b_{02} - a_2 b_{11} + a_3 b_{20}$. (Compare with $z_1^2 + z_2^2 + z_3^2$ and $\mu_1 z_1 + \mu_2 z_2 + \mu_3 z_3$ for the case of the rigid body in gravity.)

Things get more interesting in dimensions greater than 6, or $k > 3$. It turns out that the spaces $\mathbb{V}^7$, $\mathbb{V}^8$, and so on [5], actually all have as many

[5] In a Mathematica computation, we have gone up to dimension 12, but saw no reason to push it further.

Casimirs as are needed to carve out a 6 dimensional manifold. What's more, while the Casimirs of J_6 and less are determined by integrating the null covectors, those of J_7 and higher, can be obtained from the methods of *invariant theory*. This is signalled by the fact that the Casimirs for J_7 and higher are all (1) determinants, so that the sum of coefficients of the terms in the Casimir expression is identically zero; (2) the Casimirs do not depend on the vorticity moments (a_1, a_2, a_3) at all; (3) the invariants of the representation of $SL(2)$ on the vector space $\mathbb{V}^k$ for $k > 3$ are quadratic, cubic, quartic, and so on, without repetitions. (Note: One Casimir, namely the quadratic in b's, also arises in the 6-D case by invariant theory methods outlined in the appendix below.)

For instance, the Casimir for J_7 on $\mathbb{V}^7$ is a quartic, given by $C_7 = 4b_{12}^3 b_{30} + 4b_{21}^3 b_{03} - 6b_{03}b_{12}b_{21}b_{30} + b_{03}^2 b_{30}^2 - 3b_{12}^2 b_{21}^2$. The two Casimirs of J_8 are a quadratic and a cubic, given by $C_8^1 = b_{04}b_{40} - 4b_{13}b_{31} + 3b_{22}^2$ and $C_8^2 = -b_{22}^3 + 2b_{13}b_{22}b_{31} - b_{04}b_{31}^2 - b_{13}^2 b_{40} + b_{04}b_{22}b_{40}$. (In general, the *number* of Casimirs that appears in any dimension can be easily found by determining the number of null covectors of the given J_{k+3} matrix, using Mathematica or Maple; integrating them is however cumbersome. It would seem that once it is known how many null covectors exist, invariant theory methods — see appendix below — would provide a combinatorial algorithm for computing the Casimirs.)

The Casimirs at any given moment truncation will bring the system down to a certain 6 dimensional manifold. Many interesting questions arise: Is this the same manifold for every choice of $k > 3$? Would a subsequent rank change from the class of 6 manifolds give a four manifold that would

resemble the generic leaf of the 6-D case (carved out by the level sets of a pair of homogeneous polynomials C_6^1 and C_6^2 in $\mathbb{R}^6$)? (For $k = 1$, a similar question arises.) As of this writing, these questions are yet under investigation. The author hopes to return to the investigation of these and other related matters with the initiators of this particular approach [44].

Finally, we mention that the functional setting for the pure ideal (i.e., without the density term, cf. (3.17)) fluid requires the phase space to be the dual of all divergence free vector fields (or the Lie coalgebra of all volume preserving diffeomorphisms), where point vortices, vortex filaments, patches, etc. comprise the singular orbits, and their individual vortex strengths are the subcasimirs [53].

3.4 Appendix: Some Invariant theory

In the computation of the Casimir invariants of the compressible fluid semidirect product bracket, use was essentially made of the combinatorial rules for determining the tensor invariants of the representation of $SL(2)$ on vector spaces of arbitrary dimension. Here, by means of a simple example, we indicate the procedure for a low dimensional case. The higher order moments need more and more tensor powers to be considered, a combinatorial task that can prove tedious. The main reference for this book-keeping is the paper by Mihailovs [54].

Consider the action of $SL(2)$ on its defining vector space V, a 2-D space, according to fractional linear transformations: $v = (x, y) \in V$ by $x \longmapsto ax + b$

and $y \longmapsto cy + d$, where the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$, (where $ad - bc = 1$). The work of finding a basis for the subspace of $SL(2)$ invariants of $V^{\otimes m} = V \otimes \cdots \otimes V$ was done by numerous authors ([54] and references therein); it is a basic problem of invariant theory to do this for any classical Lie group (cf. Subsection 4.3.1).

The action extends naturally to tensor products. All classes of irreducible representations are parametrized by nonnegative integers $k = 0, 1, \ldots,$ and can be represented by the actions on the *symmetric* powers of V, $S^k V$. Let ϑ_k denote the representation on the $(k+1)$-dimensional linear space of homogeneous polynomials in the two coordinates of V, with basis formed by the monomials (the b_{ij} moments in our case), $(x^k, x^{k-1}y, \ldots, y^k)$.

Using various combinatorial techniques, it is possible to determine tensor invariants for arbitrary tensor products of the symmetric powers of V. For example, in the case of J_8 above, we would have $k = 4$, and a basis of 5 monomials, $(b_{40}, b_{13}, b_{22}, b_{31}, b_{04})$. The quadratic Casimir comes from considering $S^a V \otimes S^a V$, which has exactly one invariant, given by

$$(x \wedge y)^a = \sum_{i=0}^{a} (-1)^i \begin{pmatrix} a \\ i \end{pmatrix} x^{a-i} y^i \otimes x^i y^{a-i},$$

and so for $a = 4$, we get the terms

$$\begin{aligned}(x \wedge y)^4 \;=\; & +1x^4 \otimes y^4 \\ & -4x^3y \otimes xy^3 \\ & +6x^2y^2 \otimes x^2y^2 \\ & -4xy^3 \otimes x^3y \\ & +1y^4 \otimes x_4,\end{aligned}$$

which, upon symmetrizing (a process called restitution (see, for instance [55], [56]), or getting to polynomial invariants from tensor invariants) and collecting together the terms according to the definition of the moments available, yields the quadratic Casimir, C_8^1. The cubic casimir is found by considering the cubic tensor product $S^4\,\mathbb{V} \otimes S^4\,\mathbb{V} \otimes S^4\,\mathbb{V}$, which again has exactly one invariant under the action ϑ_4, which is given by $((x \wedge y)^2 \otimes 1)(x \otimes 1 \otimes y - y \otimes 1 \otimes x)^2(1 \otimes (x \wedge y)^2)$. A straightforward if tedious computation yields C_8^2.

Two points are worthy of note. First, since any such tensor product of symmetric powers has only one invariant, we would need to consider as many tensor products for a given moment truncation as there are Casimirs. But, if there is no independent way to ascertain where to stop, one could go on considering higher tensor products, only to realize that the new invariants are functionally dependent on the previous ones. So, while the methods of invariant theory are useful for determining Casimirs directly, without having to perform integration by the method of characteristics (see, however [57], where a method based on Pfaffian distributions is used) they are most effective

when one knows in advance how many Casimirs are expected. Secondly, from the general statement of the First Fundamental Theorem of invariant theory (invariant theory is covered in more detail in Section 4.3), it is clear that these polynomial invariants are determinants. That they are is seen by the fact that the coefficients in the Casimirs add up to zero.

Chapter 4

Orbit Geometry and Invariant theory

Manifolds with symmetry are one way to obtain Poisson manifolds with the linear Poisson structure at the origin. This is done by considering the level sets of a *moment map* for the group action. Some general constructions involving the moment map are given in this chapter. The Poisson manifold is next assumed to be a linear vector space $\mathbb{V}$ acted upon by a symmetry group G, which is assumed to be a Lie group. Using Whitney's embedding theorem for actions on arbitrary manifolds, this can always be done. For proper actions, we can define the notion of *orbit space*, which is to say the quotient $\mathbb{V}/G$; the rest of the chapter gives the *invariant theory* description of the strata (defined below) of both $\mathbb{V}$ and the orbit space $\mathbb{V}/G$. The results needed for later use in Chapter 5, involve what is called "classical invariant theory". Here a connection is made between orbits of a group action and the invariants that parametrize the orbit, which are related, via the notion of symmetry breaking (cf. Chapter 5), to the Casimir and subcasimir invariants. The leaves of a Poisson manifold P are orbits of the *characteristic* distribution defined using its Poisson structure, J. There is a way to characterize the structure of the leaf using a bigger cohomology (called L-P cohomology) of it which encompasses both the leaf and its surroundings (the transversal to the

leaf), and which depends on the J chosen. This, and the related notion of equivariant cohomology, shall form the subject of the appendix. In particular, for the case of $P = \mathfrak{g}^*$, there is a simple relation between the Lie algebra cohomology, the Casimir algebra, and the L-P cohomology.

4.1 Manifolds with Symmetry

In this section, we let M^n be a differentiable manifold, and specialize to the case of symplectic and Poisson manifolds by adding or relaxing additional structures. There is a wide variety of reduction procedures available in the literature, so only a small selection of them is presented here. The main idea is the following: if a manifold is acted upon by a group of transformations preserving some structure (as stipulated), then it is possible to go to a lower dimensional manifold, where the group action is not visible. In physics terms, the group action reduces the number of degrees of freedom by providing a priori *conserved quantities* which reduce the original system to a smaller system, one that is left invariant under the symmetry.

4.1.1 Basic notions

The references for this section are the excellent texts by Bredon and Duistermaat & Kolk ([58], [59]; see also [60], Appendix B).

Let G act on M. That is, there is a C^∞ map, called the (left) action

map,

$$\Theta : G \times M \longrightarrow M \tag{4.1}$$

satisfying the following properties,

$$\Theta(e, x) = x \tag{4.2}$$

$$\Theta(g_1 g_2, x) = \Theta(g_1, \Theta(g_2, x)) \tag{4.3}$$

$\forall g_1, g_2, e$ (the identity element) $\in G$, and $x \in M$. The action above may be viewed as a transformation group taking the manifold to itself,

$$\Theta_g : M \longrightarrow M. \tag{4.4}$$

Then the properties above can be written $\Theta_e = Id_M$, and $\Theta_{g_1 g_2} = \Theta_{g_1} \cdot \Theta_{g_2}$. The map Θ_g is said to be a *realization* of $g \in G$ on the space M. If the action is linear, and M a vector space, then it is called a *representation* of G on M. (The action then becomes that of the group of automorphisms of M.)

[Every left action determines a right action $x \longmapsto x \cdot g$ and vice versa. Set $\widetilde{\Theta}_g = \Theta_{g^{-1}}$. Then $\widetilde{\Theta}_g$ satisfies the hypotheses of a *right* action, namely, $\widetilde{\Theta}_{g_1} \cdot \widetilde{\Theta}_{g_2} = \widetilde{\Theta}_{g_2 g_1}$.]

An action is *effective* if $\Theta(g, x) = x \ \forall x \in M \Rightarrow g = e$. That is, an effective action moves at least *some* points on the manifold.

The action is said to be *free* if it is free of fixed points, i.e., $g \neq e \Rightarrow \Theta(g, x) \neq x$.

If any two points of M can be connected by the action, then it is said to be *transitive*. For $x, y \in M$, $\exists \;\; g \in G$ so that $\Theta(g, x) = y$.

The *orbit* through a point $x \in M$ is the set of points in M which are in the image of Θ_g, $\forall g \in G$. We denote the left action $x \longmapsto g \cdot x$, and the orbit through x by $\Omega(x) \equiv G \cdot x = \{\Theta(g, x) | \;\; g \in G\}$.

[Thus, the action is transitive if the only orbit is M itself. In this case, M is called a *homogeneous space*.]

4.1.2 Invariant points and subspaces

The most interesting actions are those that are effective and not free. In this event, there are subgroups of G that leave fixed certain points and subsets of M. The significance of fixed points is that in a neighborhood of a fixed point $x \in M$, the action Θ_g can be *linearized* for $g \in G_x$.

1. The set of elements of G that leave a point $x \in M$ fixed forms a closed subgroup, and hence a Lie subgroup of G, and is called variously the *stabilizer, isotropy subgroup, 'little group', or residual symmetry* of the point x. We shall use these terms interchangably in what follows. Formally,

$$G_x = \{g \in G | \;\; \Theta(g, x) = x\} \tag{4.5}$$

   Since $ghg^{-1} \cdot (g \cdot x) = gh \cdot x = g \cdot (h \cdot x) = g \cdot x$, for $h \in G_x$, it follows that

$$G_{g \cdot x} = g G_x g^{-1} \tag{4.6}$$

$\forall x \in M$, or stabilizers of points in the same G-orbit are related by conjugacy. The conjugacy class of stabilizers of a given orbit is said to constitute its *type*, denoted by (G_x). For the example of the rigid body in Subsection 3.1.1, at generic points the orbits of the coadjoint action of $SO(3)$ on $\mathfrak{so}(3)^*$ have residual symmetry $SO(2)$, and those of two distinct points on the same orbit (a 2-sphere) are related by a conjugate rotation, as is easily seen (cf. also Figure 2.4).

Conversely, if $H \subset G$ is a stabilizer of a point $x \in M$, then so are all its conjugate subgroups.

2. A subset $N \subset M$ is *G-invariant* if $g \cdot y \in N$, for any $y \in N$, $g \in G$, and $\Theta_{|G \times N}$ is an action on N. A function $f : M \longrightarrow \mathbb{R}$ is G-invariant if $f(g \cdot x) = f(x)$ $\forall g \in G$ and $x \in M$. If M and M' are two G-manifolds, with G-actions Θ and Θ' respectively, and $F : M \longrightarrow M'$ a differentiable map, then F is called *G-equivariant* if

$$F(\Theta(g, x)) = \Theta'(g, F(x)).$$

$\forall x \in M$ and $g \in G$.

Finding an explicit description in terms of invariant polynomials, of G-invariant objects is one of the main functions of classical invariant theory. Both invariant functions and equivariant maps turn out to be expressible in terms of a minimal number of polynomial invariants; for the case of a linear action, this is in turn expressible in terms of the Casimirs of the associated enveloping algebra of the Lie algebra $\mathfrak{g}$ of G.

3. Define

$$G_M = \bigcap_{x \in M} G_x.$$

It is a closed, normal subgroup of G consisting of all the elements of the group that fix all the points of M. If this is a finite dimensional Lie subgroup H, then the quotient $\widetilde{G} = G/H$ is a Lie subgroup such that $\widetilde{G}_M$ is a *discrete* subgroup of G. The action is called *almost effective* if G_M is a discrete subgroup $\Gamma \subset G$ different from the identity, and effective if it is exactly the identity.

Theorem

The orbit through $x \in M$, $\Omega(x) \equiv G \cdot x$ is a smooth submanifold of M, and $G \longmapsto G \cdot x$ by $g \longmapsto \Theta(g, x)$ $\forall g \in G$ is a smooth submersion. Further, there is a *homeomorphism* of differentiable manifolds,

$$\Omega(x) \equiv G \cdot x \simeq G/G_x. \tag{4.7}$$

Thus, orbits through $x \in M$ may be regarded as images, via the action map, of the homogeneous space got from modding the group by the stabilizer of x.

4.1.3 Infinitesimal actions, flows

The action of a group on a manifold constitutes a higher dimensional generalization of a flow, where the various flows in general do not commute with each other (they would, if the group were abelian).

To every action $\Theta : G \times M \longrightarrow M$, and for every Lie algebra element $\xi \in \mathfrak{g}$, the flow generator corresponding to the 1-parameter group $g(t) = \exp\, t\xi$ is given by

$$\xi_M(x) = \left.\frac{d}{dt}\right|_{t=0} \Theta(\exp\, t\xi, x) \in \quad \mathfrak{X}(M). \tag{4.8}$$

These vector fields generate the orbits of the action. In other words, the action $\Theta : G \times M \longrightarrow M$ induces a linear *anti-homomorphism*,

$$\Theta_*([\xi, \eta]_\mathfrak{g}) = -[\Theta_*\xi, \Theta_*\eta]_{\mathfrak{X}(M)} = -[\xi_M, \eta_M], \tag{4.9}$$

where $\xi_M = \Theta_*\xi$, etc., so that $\Theta_*(\mathfrak{g})$ is a finite dimensional subspace in the space of all vector fields (an infinite dimensional Lie algebra) on M, $\mathfrak{X}(M)$. Note:

1. The Lie algebra of G_x, the isotropy subalgebra, is simply the kernel of this derived action, that is, $\mathfrak{g}_x = \ker\, \Theta_*(x) \quad \forall x \in M$.

2. The *minus* sign in the homomorphism between the Lie algebras of the group and the flows on M occurs because conventionally, left-invariant vector fields on G are chosen to be isomorphic to the Lie algebra $\mathfrak{g}$. Were *right*-invariant vector fields chosen instead, this minus sign would not appear. As such, it has nothing to do with the minus sign in the homomorphism between the Poisson algebra and the algebra of vector fields, which comes because of the Jacobi identity. (The similar minus sign that is conventionally added for the same reason to the Lie-Poisson bracket (cf. (2.9), Subsection 2.1.3) shall be ignored in this thesis.)

4.1.4 Ad, ad, Ad*, ad*

The important case when $M = G$ itself, and the action that of conjugation, gives rise to the most studied Poisson structures, namely the one that gives rise to the leaves of the foliation by *coadjoint orbits*. The adjoint action is primarily an organizing tool in representation theory, to keep track of the non-commutative (in general) nature of G. We recall the definitions first, and follow with some facts about the orbit structure of the adjoint representation and its contragredient cousins. As seen in Subsection 3.1.2 the structure theory of Lie groups and Lie algebras are intimately related to the orbit structure of $\mathfrak{g}^*$. Also, the image of the moment map of a group G acting on a manifold M is seen to be the union of coadjoint orbits. The Casimirs of the Lie-Poisson structure of the coadjoint space are related to the Casimirs of the orbit space P/G of a Poisson manifold P (cf. Subsection 4.1.5).

The Ad, ad, (and dual, Ad*, ad*) actions are defined in two stages of linearization, one on the Lie group and another on the Lie algebra levels.

Let G be a Lie group. Define the *conjugation* operator, $C_g \in Aut\ (G)$ by $C_g : G \longmapsto G$ such that

$$C_g(h) = ghg^{-1} = L_g \circ R_{g^{-1}} \cdot h \tag{4.10}$$

$\forall h \in G$, and where L_g and R_g denote the left and right translation operations, $L_g \cdot h = gh$, $R_g \cdot h = hg$.

The *Adjoint* representation $Ad : G \longmapsto Aut\ (\mathfrak{g}) \simeq GL(\mathfrak{g})$ by $Ad_g : \mathfrak{g} \longmapsto \mathfrak{g}$ is

defined by the relation

$$Ad_g\xi = C'_g \cdot \xi = (L_g \circ R_{g^{-1}})'|_e \cdot \xi. \tag{4.11}$$

$\forall \xi \in \mathfrak{g}$ and $g \in G$.

[For *matrix* groups, this is $g\xi g^{-1}$ itself, with $g \in GL(n,\mathbb{R})$ and $\xi \in \mathfrak{gl}(n,\mathbb{R})$.]

The *adjoint* action, $ad : \mathfrak{g} \longmapsto End\ (\mathfrak{g}) \simeq \mathfrak{gl}(\mathfrak{g})$ by $ad_\eta : \mathfrak{g} \longmapsto \mathfrak{g}$ is given by differentiating the Ad operator at the identity of a 1-parameter curve $g(t) = \exp t\eta$,

$$ad_\eta\xi = \left.\frac{d}{dt}\right|_{t=0} Ad_{g(t)}\xi = [\eta,\xi]_{\mathfrak{g}}. \tag{4.12}$$

The *Coadjoint* representation $Ad^* : G \longmapsto GL(\mathfrak{g}^*)$ by $Ad^*_g : \mathfrak{g}^* \longmapsto \mathfrak{g}^*$ is contragredient to the Adjoint one,

$$\langle Ad^*_g(\mu), \xi\rangle = \langle \mu, Ad_{g^{-1}}(\xi)\rangle, \tag{4.13}$$

with $\mu \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$, $g \in G$.

The induced homomorphism of Lie algebras defines the *coadjoint* action, $ad^* : \mathfrak{g}^* \longmapsto End\ (\mathfrak{g}^*)$ by $ad^*_\xi : \mathfrak{g}^* \longmapsto \mathfrak{g}^*$,

$$\langle ad^*_\xi\mu, \eta\rangle = -\langle \mu, ad_\xi\eta\rangle = -\langle \mu, [\xi,\eta]\rangle, \tag{4.14}$$

for the 1-parameter group $g(t) = \exp t\xi$.

The *orbits* of the natural G-action on $\mathfrak{g}^*$,

$$\Omega(\mu) \equiv G \cdot \mu = \{Ad^*_g\mu |\ \ \forall g \in G\} \tag{4.15}$$

are called the *Coadjoint* orbits.

Kirillov - Kostant construction

Fix $\mu \in \mathfrak{g}^*$, and consider a Coadjoint orbit through it, $G \cdot \mu$. The tangent space to the orbit at μ is spanned by vectors of the form $ad^*_{\xi_i}\mu = X_i$, $i = 1, \ldots, \text{corank } (G)$. Then the unique symplectic form on the leaves is given by

$$\omega_\mu(X_i, X_j) = \langle \mu, [X_1, X_2]_\mathfrak{g} \rangle, \tag{4.16}$$

where the identification $T_\mu \mathfrak{g}^* \simeq \mathfrak{g}$ has been made. The fact that this is a symplectic form follows from the observation that the corank of a group is the number of roots, which is always even. The rank of the group is the number of Casimirs of the Lie-Poisson bracket (cf. Subsection 3.1.2) whose dual away from the origin is precisely the above symplectic form.

The Annihilator theorem [61]

Let $\mathfrak{g}$ be a Lie algebra, $\mu \in \mathfrak{g}^*$, and

$$\mathfrak{g}_\mu = \mathfrak{L}(G_\mu) = \{\xi \in \mathfrak{g} |\ ad^*_\xi \mu = 0\}$$

the annihilator of μ in $\mathfrak{g}$ (or the Lie algebra of the isotropy group of the coadjoint action of G on $\mathfrak{g}^*$, at the point μ).

The linear approximation of the *transverse* Poisson structure to the orbit of the coadjoint action at the point μ is canonically isomorphic to the linear Poisson structure on the dual space of the annihilator $\mathfrak{g}_\mu$, namely $\mathfrak{g}^*_\mu$, i.e.,

$$\mathfrak{g}^*/ad^*_\mathfrak{g}\mu \longrightarrow \mathfrak{g}^*_\mu$$

as isomorphism of spaces. Recall that the coadjoint orbit itself is isomorphic to the orbit space of the group with respect to the isotropy subgroup (i.e., $G \cdot \mu \simeq G/G_\mu$), so the above identification is justified on dimensional grounds.

As a corollary to the above result, since the rank of a Lie algebra is the same as the *corank* of its Poisson structure, the rank of $\mathfrak{g}_\mu$ at $\mu \in \mathfrak{g}^*$ is not less than rank $\mathfrak{g}$; it follows that a semisimple Lie algebra of rank r is contained in an r-dimensional commutative subalgebra. This is one of the many results in representation theory of semisimple Lie groups that are easy to see from the viewpoint of adjoint orbit geometry. (Kirillov [62], [63] pioneered a method of using orbit geometry in representation theory as follows: Define the dual $\hat{G}$ of G to be the equivalence classes of unitary irreducible representations of G. Then his idea was to identify $\hat{G}$ with the set of orbits in $\mathfrak{g}^*$, namely $\mathfrak{g}^*/G$, where G is simply connected and nilpotent.)

Duflo & Vergne's theorem [61]

The following result gives a criterion to distinguish regular points of a Poisson manifold that is locally isomorphic to the coalgebra to a semisimple Lie algebra (the *linearizable* ones according to Weinstein's criterion), from the nonregular points.

If the annihilator of $\mu \in \mathfrak{g}^*$ is commutative, then the rank of the Poisson structure at that point is locally constant (and hence, by lower semicontinuity of rank, maximal). Thus, a nonabelian $\mathfrak{g}_\mu$ signals a change in the leaf genericity from regular to nonregular (which includes exceptional and singular leaves).

This criterion is obvious in view of the above two results, as the rank of $\mathfrak{g}$ is the same as the dimension of the annihilator at any point.

Coadjoint orbits parametrization [64]

The coadjoint orbits are parametrized as follows. Let $\mathfrak{t} \subset \mathfrak{g}$ be the Cartan subalgebra, and B the Killing form (G-invariant and positive definite) on $\mathfrak{g}$. Let $\mathfrak{t}^* \subset \mathfrak{g}^*$ correspond to $\mathfrak{t}$ via B, which identifies $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. Recall that the Weyl group $\mathcal{W}$ is defined as $\mathcal{N}(G)/G$, where $\mathcal{N}(H) = \{g \in G |\ gHg^{-1} = H\}$ is the normalizer for $H \subseteq G$. Every coadjoint orbit intersects $\mathfrak{t}^*$ in a Weyl group orbit. This gives a one to one correspondence between coadjoint orbits in $\mathfrak{g}^*$ and Weyl group orbits in $\mathfrak{t}^*$.

Singular coadjoint orbits have the property that they pass through the intersection of the walls of the Weyl chamber [37].

4.1.5 The moment map

Now, we let M be symplectic, with symplectic form ω, and P Poisson, with the Poisson structure J. Let G be a d-dimensional Lie group that acts on M and P in such a way that the symplectic or Poisson structures are preserved (i.e., we assume that the action maps have those properties, called symplectic and Poisson actions, respectively).

There is an intimate connection between Casimirs of the *reduced* space of a Poisson manifold P/G, and the Casimirs of the Lie coalgebra $\mathfrak{g}^*$. The latter is a Poisson manifold with Lie-Poisson bracket, so its Casimirs are just

the ad^*-invariant functions, as will be shown later. The connection is made using the concept of a *moment map*.

Let G act on M: for $\xi \in \mathfrak{g}$, $\xi \longmapsto \xi_M(x)$, $x \in M$, by flows.

Since G acts by symplectomorphisms (i.e., action is symplectic), we have that

$$\begin{aligned} \mathfrak{L}_{X_\xi}\omega = 0 &= d(i_{\xi_M}\omega) + i_{\xi_M}(d\omega) \\ \text{or} \qquad i_{\xi_M}\omega &= d\rho(\xi), \qquad \text{if } \mathrm{H}^1(M,\mathbb{R}) = 0. \end{aligned}$$

I.e., there is a correspondence $\xi \longmapsto \rho(\xi)$, $\forall \xi \in \mathfrak{g}$, and at each point $x \in M$.

(Similar considerations apply for Poisson morphisms of G on a Poisson manifold P.)

The *moment map* is a collection of dim $(G) = d$ Hamiltonians, whose flows, under the vector fields they define, generate the orbits of the symmetry group on the manifold. If $\boldsymbol{\rho} = (\rho^1, \ldots, \rho^d)$ is the moment map, then it is defined by setting $\boldsymbol{\rho} : M \longrightarrow \mathfrak{g}^* \longrightarrow \mathbb{R}$ such that, with

$$\langle \boldsymbol{\rho}_x, \quad \xi \rangle = \rho_x(\xi) \in C^\infty(\mathfrak{g}^*) \quad \forall \xi \in \mathfrak{g}, \tag{4.17}$$

we have:

$$\omega(\xi_M(x), \quad \cdot \quad) = d\rho_x(\xi) \tag{4.18}$$

$$J \cdot d\rho_x(\xi) = \xi_P(x) \tag{4.19}$$

for symplectic and Poisson manifolds, respectively. Here, ξ_M or ξ_P is the flow generator of the symplectic or Poisson action corresponding to the element

$\xi \in \mathfrak{g}$. (If ξ is written $\xi = \xi^a e_a$, in a basis $\{e_a\}$ for the Lie algebra, then the moment map components ρ^a can be mapped to the components ξ^a, for $a = 1, \ldots, d$.)

The moment map has the important property of equivariance with respect to the action, that is, the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\Theta(g,\cdot)} & M \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ \mathfrak{g}^* & \xrightarrow[Ad_g^*]{} & \mathfrak{g}^* \end{array} \tag{4.20}$$

Infinitesimally, this can be viewed as a closure condition for the moment functions, ρ^a, $a = 1, \ldots, d$, with respect to the Poisson bracket of functions (which exists for the symplectic case by nondegeneracy, and in the Poisson case, by definition):

$$\{\rho_x(\xi), \rho_x(\eta)\} = \rho_x([\xi, \eta]), \tag{4.21}$$

where $\xi, \eta \in \mathfrak{g}$ and $x \in M$ or P.

Inserting the definition of the moment map into the above equation (4.21), and using the Jacobi identity, we find that

$$[\xi_M, \eta_M]_{\mathfrak{X}(M)} = -([\xi, \eta]_{\mathfrak{g}})_M.$$

Thus infinitesimal equivariance is equivalent to the antihomomorphism property.

Moreover, from the definition of the moment map, (4.18) (4.19), we can find a condition that connects together the action, moment map, and the symplectic or Poisson structure. Denoting $\Theta' : \mathfrak{g} \times M \longrightarrow M$ the infinitesimal form of the action, we have

$$(\Theta'_{(0)a}(x))^i = J^{ji}(x)\frac{\partial \rho^a(x)}{\partial x^j},$$

for $a = 1, \ldots, d$. This specifies the i^{th} component of $D\Theta_g|_{g=e_G}$ and by the usual existence and uniqueness, given $\Theta_e = Id_M$, this implies a unique solution. We also see that singularitis of $D\,\boldsymbol{\rho}$ are zeroes of J and vice-versa, presuming the action Θ is well defined and proper.

Noether's theorem [65]

If f is a G-invariant function on P, then $\boldsymbol{\rho}$ is conserved on trajectories of X_f. (We state the Poisson case, as the symplectic one is a special case.)

Indeed, G-invariance of f means that

$$f(\Theta(g(t), x)) = f(x) \tag{4.22}$$

where $g(t) = \exp t\xi$, $\xi_P = (\xi \cdot \Theta'|_{t=0})(x)$, $\xi \in \mathfrak{g}$, $x \in P$. Expanding (4.22)

about $t = 0$,

$$\begin{aligned} f(x + t\xi \cdot \Theta'|_0(x) + \cdots) &= f(x), \\ \text{or} \quad f'(x) \cdot \xi_P(x) &= 0 \\ \text{or} \quad df_x \cdot \xi_P(x) &= 0 \\ &= \{f, \rho(\xi)\} \\ &= -\{\rho(\xi), f\} \\ &= -d\rho(\xi) \cdot X_f. \end{aligned}$$

Thus, f conserved along group orbits means $\boldsymbol{\rho}$ is conserved along flows of f.

Smale's criterion

We call $x \in M$ a regular value for $\boldsymbol{\rho}$ if $d\,\boldsymbol{\rho}_x \simeq \mathfrak{g}^*$. Let $\boldsymbol{\rho}$ be a (not necessarily G-equivariant) moment map. Let G_x be the isotropy of $x \in M$ and $\mathfrak{g}_x$ its Lie algebra. Then x is a regular value for $\boldsymbol{\rho}$ iff $\mathfrak{g}_x$ is trivial (i.e., G_x is discrete).

For a proof, one observes that if $x \in M$ were regular, then for $v \in T_x M$,

$$\begin{aligned} \langle d\,\boldsymbol{\rho}_x \cdot v, \xi \rangle &= 0 \quad \forall \xi \in G \\ \Longrightarrow \qquad d\rho_x(\xi) \cdot v &= 0 \\ \text{or} \qquad \omega(\xi_M, v)|_x &= 0 \end{aligned}$$

Since $\omega(x)$ is nondegenerate, this happens iff $\xi_M(x) = 0$ or in other words, $\mathfrak{g}_x$ is trivial (which is implied by G_x discrete).

Clearly, the proof does not go through for the case of a Poisson manifold P, as the final step could imply *either* that $J(x)$ is degenerate, or that $\xi_P(x) = 0$. But, away from the singular points of J, the theorem would apply to a Poisson manifold (P, J) as well as a symplectic manifold (M, ω).

In closing, we give an example of a moment map, to show how it justifies its name. Let $M = T^*\mathbb{R}^3 \simeq \mathbb{R}^6$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, presymplectic 1-form $\theta \in T_x\mathbb{R}^3$ by $\theta = \sum_{i=1}^3 y_i dx_i$, where $y = (y_1, y_2, y_3) \in T_x\mathbb{R}^3$ are fiber coordinates; then the symplectic form on M is just $\omega = d\theta = \sum_{i=1}^3 dx_i \wedge dy_i$.

Let G be the group of translations on the base, $\mathbb{R}^3$. This induces translation on the fiber in order to keep ω invariant, $\mathfrak{L}_X\omega = 0$, where $X \in \mathfrak{g}$, the Lie algebra of G. The moment map, $\boldsymbol{\rho}$ is easily seen from the definition to correspond to linear momentum, that is, $\boldsymbol{\rho}(x,\ y) = y \in \mathfrak{g}^*$. Likewise, if G is the group of rotations on the base, symplectic invariance permits the lift of it to the whole cotangent bundle, and the moment map is seen now to be the angular momentum, $\boldsymbol{\rho}(x,\ y) = x \times y \in \mathfrak{g}^*$. (The duals of $\mathbb{R}^n$ and itself are here identified implicitly.)

Moment maps and Casimirs [66]

Let (P, J) be a Poisson manifold and G act as a Poisson map, with $\boldsymbol{\rho} : P \longrightarrow \mathfrak{g}^*$ the equivariant moment map. Define the Casimir function on the orbit space P/G, $C : P/G \longrightarrow \mathbb{R}$ s.t. $J(dC, \cdot) = 0$. Let $F \in \mathfrak{g}^*$ be an ad^*-equivariant function. Then, F is a Casimir for the Lie-Poisson bracket, and

$F \circ \boldsymbol{\rho} = C \circ \Pi$, where Π is the orbit map, $\Pi : P \longrightarrow P/G$. The commutative diagram below illustrates this theorem (whose proof can be found in the cited reference).

$$\begin{array}{ccccc} & & P & \xrightarrow{\boldsymbol{\rho}} & \mathfrak{g}^* & \xrightarrow{F} & \mathbb{R} \\ & & \Pi \downarrow & & \downarrow \Pi & & \\ \mathbb{R} & \xleftarrow[C]{} & P/G & \xrightarrow[\hat{\boldsymbol{\rho}}]{} & \mathfrak{g}^*/G & & \end{array}$$

Essentially, there is a direct map between the Casimirs to be found on the space $\mathfrak{g}^*$, and those on the orbit space of the Poisson manifold, and hence, via the orbit map, on the Poisson manifold itself.

4.1.6 Geometric reduction

The G-equivariant moment map defined above is used to facilitate the reduction process, whereby the symmetry of the dynamical system is removed. For the various reduction procedures to work, restrictions on the action need to be imposed. The action shall be taken to be *proper*, and G to be compact. The results of the following Subsection 4.2.1 then allow various types of reduction to be defined.

Reduction schemes that start with M as the basic object, and use the moment map associated to the group action to mod out the symmetries, will be considered. Several other schemes of reduction [67], [68], start from algebras of functions, and while these are more general, the geometrical interpretation of the reduced algebra is usually lacking.

Within the first approach, or *geometrical* reduction, two cases can be considered in turn: regular and singular reduction. These turn out to be well defined for the case of locally free proper actions (and in particular, free proper actions).

The basic idea behind the reduction philosophy is the following: If a Lie group G acts on M symplectically, with moment map $\boldsymbol{\rho} : M \longrightarrow \mathfrak{g}^*$, then M/G has a Poisson structure. Consider the coadjoint orbit $\Omega(\mu) \subset \mathfrak{g}^*$; then $\boldsymbol{\rho}^{-1}(\Omega(\mu))//G$ is a symplectic leaf of M/G. (An even more prototypical scheme for reduction is possible if stated in terms of generalized distributions and their integrability [69].)

We first describe the terms 'regular', 'clean' (or weakly regular) and 'singular' [70]. The manifold could be either symplectic or Poisson, but for simplicity, we take the former case.

Let (M, ω) be a symplectic manifold, and $\boldsymbol{\rho}$ the associated moment map for a symplectic G-action on M. $\mu \in \mathfrak{g}^*$ is a regular value for $\boldsymbol{\rho} : M \longrightarrow \mathfrak{g}^*$ if $\boldsymbol{\rho}^{-1}(\mu)$ is such that for $x \in \boldsymbol{\rho}^{-1}(\mu)$,

$$\begin{aligned} \boldsymbol{\rho}'(T_x M) &= T_\mu \mathfrak{g}^*, \\ T_x M &= T_x\, \boldsymbol{\rho}^{-1}(\mu) + T_x(M/G). \end{aligned}$$

Since under $\boldsymbol{\rho}$, the whole orbit $\boldsymbol{\rho}^{-1}(\mu)$ gets mapped to a point in $\mathfrak{g}^*$, we have that

$$\ker\,(\boldsymbol{\rho}') = T_x(\boldsymbol{\rho}^{-1}(\mu)), \tag{4.23}$$

and

$$\text{im } (\boldsymbol{\rho}') = T_x(M/G). \tag{4.24}$$

So, $\mu \in \mathfrak{g}^*$ a regular value $\Longrightarrow$ im $(\boldsymbol{\rho}') \equiv T_\mu \mathfrak{g}^* =$ range $(\boldsymbol{\rho}')$.

An element $\mu \in \mathfrak{g}^*$ is called *clean* if (4.23) is satisfied, but not (4.24). The image of $\boldsymbol{\rho}'$ consists of all vectors in T_xM that generate motion on M upon changing the value of the moment map on $\mathfrak{g}^*$, away from the value at μ. (This justifies the name 'moment map' in analogy with angular momentum being the generator of rotations.) A clean value need not be regular since it is not required for it to have the property that the image of $\boldsymbol{\rho}'$ maps onto $T_\mu \mathfrak{g}^*$; i.e., it is possible that singular orbits may be accounted for as well.

The kernel of $\boldsymbol{\rho}'$ maps the tangent to the level submanifold $\boldsymbol{\rho}^{-1}(\mu)$ to zero in $T_\mu \mathfrak{g}^*$. However, for a clean value, the normal to the submanifold need be mapped only into a subset of $T_\mu \mathfrak{g}^*$. When the normal space to the level set of the moment map gets mapped *onto* $T_\mu \mathfrak{g}^*$, the map is regular. Clearly, regular implies clean but not conversely. In either case, $\boldsymbol{\rho}^{-1}(\mu)$ is an imbedded submanifold. Note that regularity or cleanliness also implies that the symplectic form restricted to the level set has constant rank, and that the characteristic distribution of ω, namely its kernel, is fibrating (so that $\boldsymbol{\rho}^{-1}(\mu)/\ker\,(\omega)$ is an immersed Hausdorff submanifold of the level set).

A value for the moment map is singular when it is not regular nor weakly regular. In this event, $\boldsymbol{\rho}^{-1}(\mu)$ is not an imbedded submanifold, or the

restriction of the symplectic form is not of constant rank, or the characteristic distribution is nonfibrating.

Regular reduction theorem ([71, 72], [73])

Let $\mu \in \mathfrak{g}^*$ be (weakly) regular, and $\boldsymbol{\rho}^{-1}(\mu)$ a smooth manifold. If G_μ acts freely and properly on $\boldsymbol{\rho}^{-1}(\mu)$, then the *reduced* manifold is defined by $M_\mu := \boldsymbol{\rho}^{-1}(\mu)/G_\mu$. Then $\exists!$ ω_μ on M_μ such that

$$i_\mu^* \omega = \pi_\mu^* \omega_\mu$$

with

$$i_\mu : \boldsymbol{\rho}^{-1}(\mu) \longrightarrow M$$
$$\text{and} \quad \pi_\mu : \boldsymbol{\rho}^{-1}(\mu) \longrightarrow M_\mu.$$

We have a few observations about this theorem, which are not usually pointed out in the literature.

1. First, the fact that the reduced manifold is even dimensional follows from purely counting arguments as well. (In their original paper [71], this fact is deduced by the presence of a nondegenerate symplectic form on M_μ, which of course is a constructive argument.)

   Since the reduced space comes as a double quotient, $M_\mu = M/G/G_\mu :=$

$M//G$, where $\mu \in \mathfrak{g}^*$, we have

$$\begin{aligned}
\dim M_\mu &= \dim M - \dim G - \dim G_\mu \\
&= \dim M - (\dim G - \dim G_\mu) - 2\dim G_\mu \\
&= 2n - (d - r) - 2r \\
&= 2n - 2 \times \#\text{ of positive roots} - 2r \\
&= \text{an even integer.}
\end{aligned}$$

This counting argument does not prove that there exists a nondegenerate closed 2-form: that is the content of the reduction theorem.

2. Secondly, the Marsden-Weinstein weakly regular (or clean) version of reduction as stated above implicitly assumes that G acts on M freely, or at most locally freely (i.e., with discrete isotropy), at each $x \in M$. This is very restrictive, for most interesting actions are effective but not free.

   In the event that $G_x \neq e_G$ or a discrete subgroup $\Gamma \subset G$, we note that the invariant slice that is G_x-invariant, adds to the thickness of the moment level set, $\boldsymbol{\rho}^{-1}(\mu)$, in order to compensate for the thinness of the orbit (which now goes from $G \cdot x$ to $\Omega(x) = (G/G_x) \cdot x$). Therefore, the quotient of the thickened moment level by G_μ ought to leave one with a M_μ of larger even dimension.

   The dimension formula follows from the following counting consideration. From the observation in the previous paragraph, we can write

$$\dim T_x(\boldsymbol{\rho}^{-1}(\mu)) + \dim T_x(\Omega(x)) = \dim M$$

and by equivariance, it follows that $T_x(\boldsymbol{\rho}^{-1}(\mu)) \cap T_x(\Omega(x)) = T_x(G_\mu \cdot x)$, for $x \in \boldsymbol{\rho}^{-1}(\mu)$, and $\mu \in \mathfrak{g}^*$. Now, since two points in the moment level are related by G_x action, we would need to mod out by G_x the isotropy orbit while considering the reduced space. With this in mind, we can write using the above equality,

$$\begin{aligned} \dim\, M_\mu &= \dim\, \boldsymbol{\rho}^{-1}(\mu) - \dim\, G_\mu + \dim\, G_x \\ &= \dim\, M - (\dim\, G - \dim\, G_x) - \dim\, G_\mu + \dim\, G_x \\ &= \dim\, M - \dim\, G - \dim\, G_\mu + 2\dim\, G_x. \end{aligned}$$

Since we have seen in the previous note that the free action reduced space is even dimensional, it follows that adding 2dim G_x would keep it even dimensional.

Orbit reduction [65]

It can be easily seen, from $G/G_\mu \simeq G \cdot \mu$, that

$$M_\mu = \boldsymbol{\rho}^{-1}(\mu)/G_\mu = \boldsymbol{\rho}^{-1}(\Omega(\mu))/G$$

where $\Omega(\mu)$, for $\mu \in \mathfrak{g}^*$ is a coadjoint orbit. This version is useful when orbit geometry comes into play. The orbit map, $\Pi : M \longrightarrow M/G$ takes the image of the moment level set to the reduced space: $M_\mu = \Pi(\boldsymbol{\rho}^{-1}(\Omega_\mu))$.

Singular reduction theorem [60]

Let (M, ω) be a symplectic manifold, and let G act properly on it. Suppose that $\Omega(\mu)$ is locally closed. Then $M_\mu = \boldsymbol{\rho}^{-1}(\Omega(\mu))/G$ is a Poisson manifold with a *nondegenerate* Poisson bracket, and as a topological manifold, M_μ is a locally finite union of symplectic manifolds called 'symplectic pieces'.

There are some interesting connections between singular reduction and invariant theory [74, 75], [60]. There are also many fascinating connections between moment maps and *geometric* invariant theory [76], [77]. It turns out that geometric invariant theory of complex projective varieties is governed by the geometry of the the level sets of the moment map. The symplectic quotients which arise upon (in general singular) reduction serve to characterize the many notions of stability of a vector, on the tangent space to the orbit of an algebraic variety.

For both the regular and the singular cases, the isotropy may be a discrete subgroup Γ of the Lie group G. In this case, we would obtain not a symplectic manifold, but instead a symplectic *orbifold*, with the leaves of the Poisson manifold $\boldsymbol{\rho}^{-1}(\mu)$ being a stratification by symplectic manifolds of different dimensions. (Orbifolds and stratifications are considered in greater detail in Section 4.2 below.)

Further developments in the theory of singular reduction for symplectic spaces may be found in [78], [79], [80].

Poisson reduction theorem [65]

Let G act on P by Poisson maps (that is, Poisson bracket preserving), and assume that P/G is a well defined smooth manifold, $\Pi : P \longrightarrow P/G$ the canonical projection (the orbit map). If functions $f, h \in C^\infty(P/G)$ are extended to $F, H \in C^\infty(P)$ by the orbit map, $F = f \circ \Pi$, $H = h \circ \Pi$, then there is a unique Poisson structure on P/G defined by

$$\{f, h\}_{P/G} \circ \Pi = \{F, H\}_P.$$

For example, as indicated in Subsection 3.1.2, the Lie-Poisson bracket for the dual to a semisimple Lie algebra comes from a Poisson reduction, where the cotangent bundle to the Lie group goes to its orbit space by Poisson reduction: $\mathfrak{g}^* \simeq T^*G/G$.

4.2 Geometry of Strata

We define the notion of properness of an action, and the important construct called a *slice* for the action. These concepts will permit the existence of orbit spaces, which are seen to be in general *orbifolds*. The tangent and normal spaces of the various strata (each stratum corresponds to one equivalence class of isotropies) both in the space M and the orbit space M/G are described; these invariant spaces shall play a role in parametrizing the orbits.

Notation: For this section, the orbit through $x \in M$ shall be denoted by $\Omega(x)$, and not $G \cdot x$.

4.2.1 Proper actions and slices

We list a few definitions and properties of proper actions and then describe the slice construction [60].

A continuous map Θ from a topological space A to another space B is *proper* if $\Theta^{-1}(K)$ is compact in A for every compact subset $K \subset B$.

An action $\Theta : G \times M \longrightarrow M$ is said to be proper if the map

$$\begin{aligned} G \times M &\longrightarrow M \times M \\ (g, x) &\longmapsto (g \cdot x, x) \end{aligned}$$

is proper. For a proper action of a Lie group G on manifold M, the isotropy G_x at any point $x \in M$ is compact, by applying the above definition.

A proper action is locally *linearizable*. That is, for $x \in M$, $\exists$ a G_x-invariant neighborhood U_x and a diffeomorphism $\Psi : U_x \longrightarrow T_xM$ such that $\Psi(x) = 0$, $d\Psi(x) = Id_{T_xM}$, and Ψ is G_x-equivariant:

$$\Psi(\Theta_g(y)) = T_x\Theta_g(\Psi(y)),$$

$y \in U$, $g \in G_x$. This follows from the splitting theorem below.

This linearizability property is due to Bochner [59], and among other things, allows the action to be represented by the Adjoint of G_x.

Slice. Let G act on M via action map Θ, with orbits $\Omega(x)$ through a point $x \in M$. Let $G_x \subset G$ be the isotropy subgroup of x, and $\mathfrak{g}_x$ its Lie algebra. Then, if $H \subset G$ such that

$$\mathfrak{g} = \mathfrak{g}_x + T_eH,$$

and $S \subset M$ a submanifold transversal to the orbit $\Omega(x)$

$$T_x M = T_x S + T_x \Omega(x),$$

then $\exists$ open neighborhood $U_e \times V_x$ of $(e, x) \in H \times S \subset G \times M$ such that $\Theta|_{U \times V}$ is a diffeomorphism onto an open neighborhood of $x \in M$.

If the action is proper, then S is called a slice for the action if it is G_x-invariant; if for any point $y \in S$, the orbit through y, $\Omega(y)$, is transversal to the submanifold S; and if the only elements of G that leave S invariant at any point y in it are elements of G_y.

The Figure 4.1 below illustrates the slice construction.

4.2.2 Orbit type

Consider, for $H \subset G$ a compact subgroup, the subset of M given by

$$M_H = \{x \in M |\ H = G_x\}.$$

By the linearization property, and by the slice construct, it follows that M_H is a submanifold of M.

Define

$$M_{(H)} = G \cdot M_H,$$

the set of points of points of M of *orbit type* (H). That is,

$$M_{(H)} = \{x \in M |\ \exists g \in G\ s.t.\ gG_x g^{-1} = H\}.$$

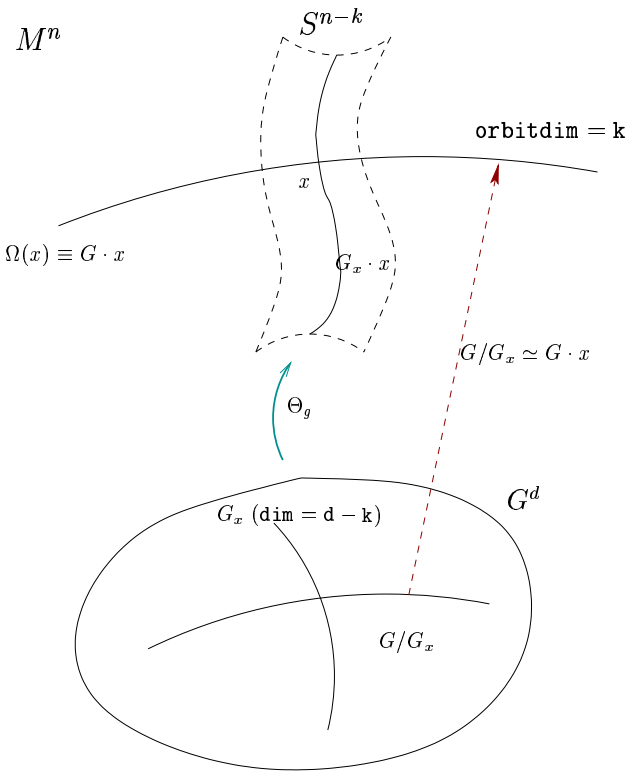

Figure 4.1: Slice construction.

On the set of all closed subgroups of G, define a partially ordered set by the relation $<$: $K < H$ iff K is conjugate to H in G. Then $<$ is an equivalence relation on the set of all closed subgroups of G. Also, since orbits are homeomorphic to the orbit spaces in G by the stabilizing subgroup (i.e., $G \cdot x \simeq G/G_x$), this equivalence relation (saying that H is conjugate to a subgroup of G) is equivalent to there being an equivariant map $G/H \longrightarrow G/K$.

As H varies over the conjugacy classes of closed subgroups of G, the orbit types partition M into an orbit type decomposition. (This decomposition is locally finite if the action is proper. Local finiteness ensures that any neighborhood of an orbit intersects surrounding orbits in only a finite number of pieces. Thus, for example, irrational flows on a torus are ruled out in this scheme of things. If such flows from a group action are allowed, then it can be shown that Casimirs do not determine the various orbits of a proper action [9].)

Clearly, orbit types can vary only *across* the orbits, changing as the foliation goes from regular to singular leaves, while remaining constant along an orbit. Also, smaller orbit types are associated to fatter orbits, which is to say, the generic orbits have the least residual symmetry. Singular orbits have smaller dimension and hence, larger isotropy, and orbit types. Later it will be seen that the orbit type of a generic leaf of a foliation in the case of a Poisson manifold is always *abelian*, while that of the non-generic leaf is nonabelian.

For any G-orbit, Ω, the orbit type it belongs to shall be denoted (Ω) and set equal to (G_x), for any $x \in \Omega$. The partial ordering $<$ on orbit types

can be used to compare the residual symmetry of the orbits themselves; we say $(\Omega) < (\Omega')$ if an element of (G_x) is contained strictly in $(G_{x'})$, where $x \in \Omega$, $x' \in \Omega'$. Clearly, by the above remarks, if $H < K$, then the types of G/H and G/K satisfy the relation $(G/H) \geq (G/K)$. Hence, *the smaller the isotropy type, the larger the orbit.*

This idea plays an important role in what follows, as physical systems tend to settle down into states of greatest symmetry. (For example, a *dead* top, that is at rest (or has lost all its energy through a dissipative mechanism [1]), would have fuller symmetry than a top steadily spinning about an axis. While the former top is represented by the origin of any solution by extremizing a potential, the latter is represented by a flat direction.)

For example, consider the action of $SO(3)$ on $\mathfrak{so}(3)^*$. The coadjoint orbits are all 2-dimensional away from the origin, with the differential of the Casimir, dC (where $C = (\mu^1)^2 + (\mu^2)^2 + (\mu^3)^2$, and $\{\mu^i\}_{i=1,\dots,3}$ form a basis for the vector space $\mathfrak{so}(3)^*$) generating the transverse slice to any point on the orbit. The isotropy subgroup, $SO(2)$ is compact, abelian, and its conjugacy class defines a orbit type for the union of all 2-spheres of non-zero radii. The only closed (nontrivial) subgroup of $SO(3)$ that is not conjugate to $SO(2)$ is $SO(3)$ itself, and this is the isotropy of the origin where $\mu^i = 0, \forall i = 1, \dots, 3$. Thus M, which in this case is $\mathfrak{so}(3)^*$, is a disjoint union of the two orbit types, the origin of type $(SO(3))$ and everything else of type $(SO(2))$.

[1] Such a dissipative mechanism may be modeled using the idea of *Metriplectic* systems [81].

Now, we can define the notion of an *orbit space*. Formally, these are reduced spaces M/G, where the group action is modded out. That is, each orbit gets mapped to a point in the orbit space. Let G act properly on M. In general, this ensures that M/G is a (Hausdorff) topological space.

Orbits: Say $x \sim y$ if they both belong to the same orbit, that is $\Omega(x) = \Omega(y)$. This equivalence relation partitions M into G-orbits, M/G. The canonical projection $\Pi : M \longrightarrow M/G$ is called the *orbit map*, and it sends each point of M onto its G-orbit. The topology of M/G is the usual quotient topology inherited from that on M: the set $U \subseteq M/G$ is open iff $\Pi^{-1}(U) \subset M$ is open.

Slices: By definition, the slice S at $x \in M$ is transverse to the orbit, $\Omega(x)$. For a neighborhood $V_x \subseteq S$, $\Pi(V)$ is open in M/G. As only elements of G that belong to $G \backslash G_x$ can move the slice along the orbits, the orbit spaces $(G \cdot S)/G$ and S/G_x are homeomorphic, $x \in S$. (This shall play a role while defining the invariant normal slice below, where we consider tubular neighborhoods and the Palais' retraction map.)

For the special case of a *free* action (i.e., $G_x = e \; \forall x \in M$), the orbit space is a smooth manifold, and in fact forms the base space of a principal G-bundle, $\Pi : M \longrightarrow M/G$.

If the isotropy subgroup is discrete, then the (proper) action is said to be *locally free*. In this event, M/G can be given the structure of a mildly singular manifold called an orbifold. [An orbifold chart associated to an open

set $\widetilde{U}$ in a Hausdorff topological space X is a triple (U, G, ϕ) s.t. $U \subset \mathbb{R}$ is an open set, $G \subseteq GL(U)$, $\phi : U \longrightarrow \widetilde{U}$ is a continuous map that is G- invariant and onto, which induces the chart homeomorphism

$$\widetilde{\phi} : U/G \longrightarrow \widetilde{U}.$$

Orbifold transition maps and atlases are similarly defined, and give the orbifold structure to X.]

These orbifolds also appear when the reduced space of a symplectic or Poisson manifold is considered for the case of a locally free action (as in the Subsection 4.1.6 above on clean, singular and Poisson reductions).

4.2.3 Principal and singular strata

For simplicity of notation, although it can be done in full generality, we assume in Chapter 5 that the manifold is a *linear* vector space $\mathbb{V}$. In the literature for symmetry breaking, the fields are assumed to live in a vector space (the fiber of an associated bundle to the principal G-bundle), with the underlying base space usually not explicitly considered. Thanks to Whitney's embedding theorems (stated below), this could always be done with no loss of generality.

Further, if G is a compact Lie group, then in virtue of the existence of a G-invariant Haar measure, it can be shown [58] that the linear G-action could always be replaced by an orthogonal action [2]. So, let $\mathbb{V}$ be a G-manifold,

[2]We emphasize here that there is little loss in generality by doing this. Orthogonal

and $\mathbb{V}/G$ the space of orbits. The orbits in $\mathbb{V}$ are points, under the orbit map $\Pi : \mathbb{V} \longrightarrow \mathbb{V}/G$, in the image in $\mathbb{V}/G$. Due to the classification of orbits by *type*, we consider a stratification of $\mathbb{V}$ and $\mathbb{V}/G$ by orbit types. Similarly, for a G-manifold M, we have the stratification into orbit types, although the geometry of the orbits and their parametrization could become more complicated.

Stratifications are a bit more general than generalized foliations, as they do not require the strata to be generated by differentiable distributions. Recall, from Appendix 2 to Chapter 2, that a foliation of a differentiable manifold M is a collection of all maximal integral submanifolds of an integrable system of vector fields on M, which are guaranteed to exist by a generalized version of Frobenius' theorem (to include nonconstant rank distributions). The submanifolds are arranged as disjoint *leaves*, with appropriate 'box' coordinates on any locally constant rank neighborhood. Stratifications assemble the *strata* without reference to their being generated by an integrable distribution. They are more topological in nature.

First we give a vector space definition and then one for the manifold [59], [8].

Stratification

---

decompositions of tangent and normal spaces are convenient in the symmetry breaking context, but not germane to the general picture we are trying to elucidate. Most of what follows applies to noncompact group actions as well, cf. Section 3.3.

1. $\mathcal{E}$, a collection of subsets S_i of $\mathbb{V}$, is a stratification of $\mathbb{V}$ if for any pair S_1, S_2, either $S_1 = S_2$ or $S_1 \cap S_2 = \emptyset$, with each S_i a smooth manifold; and if $S_1 \cap \overline{S_2} \neq \emptyset$ implies $S_1 \subset \partial S_2$.

2. A C^r stratification of a differentiable manifold M is a locally finite partition of M into locally closed, connected C^r submanifolds S_α, $\alpha \in \mathrm{I}$, of M, called 'strata', such that

$$\overline{S_\alpha} = S_\alpha \cup \bigcup_{\beta \in \mathrm{I}_\alpha} S_\beta,$$

with $\mathrm{I}_\alpha = \mathrm{I} \setminus \{\alpha\}$, and dim $S_\beta <$ dim S_α, $\forall \beta \in \mathrm{I}_\alpha$.

All points in a given group orbit have the same symmetry (or isotropy), but all points with the same isotropy subgroup do not belong to the same orbit. The stratification collects all orbits of a given type, so that they have the tangent structure transversal to the orbit. (Strata are also called *orbit structures* [58].)

The stratification of a G-manifold M (or a linear G-vector space $\mathbb{V}$) into strata of various orbit types can be defined next. There are three types of strata:

1. The *principal stratum* Σ_P of maximum orbit type G/H, (so that H is conjugate to a subgroup of each isotropy group for the principal orbits in the stratum), is an open dense subset of M. The image via the orbit map $\Pi : M \longrightarrow M/G$ of the principal stratum, denoted by $\hat{\Sigma}_P$, is connected

in M/G. If Ω_P is a principal orbit, and Ω any other orbit, then $\exists$ G-equivariant map $\Omega_P \longrightarrow \Omega$. If $\Omega_P \simeq G/H$ and $\Omega \simeq G/K$, then $K \supset H$. Then $\Omega_P \longrightarrow \Omega$ is a (principal) bundle projection $G/H \longrightarrow G/K$ with fiber K/H.

2. If dim $(K/H) > 0$ (or dim $\Omega_P >$ dim Ω), then Ω is called a *singular* orbit, denoted by Ω_S or $\widetilde{\Omega}$, and the stratum to which it belongs is called a singular stratum Σ_S. The corresponding image in the orbit space is not necessarily connected, and it is denoted by $\hat{\Sigma}_S$.

3. If dim $(K/H) = 0$ (so that dim $\Omega_P =$ dim Ω) but $\Omega_P \longrightarrow \Omega$ is a nontrivial covering map (so that K/H is finite and nontrivial), then Ω is called an exceptional orbit, the stratum to which it belongs is denoted Σ_E, and its image in the orbit space $\hat{\Sigma}_E$.

It can be shown on general grounds that the disjoint union of singular (and exceptional) strata form the *boundary* of the principal stratum. We shall not consider exceptional strata as these are not rank changing. (They would, however, be of importance in physical situations such as bifurcations, or patterns of discrete symmetry breaking.) Further, it can be shown that the minimum and maximum orbit types are unique.

As an example, consider the rigid body Poisson manifold. It is composed of two strata: the principal stratum Σ_P being a disjoint union of all 2-spheres indexed by their radii, and of isotropy type $(SO(2))$; and a singular stratum Σ_S, namely the origin, of type $(SO(3))$. The corresponding orbit

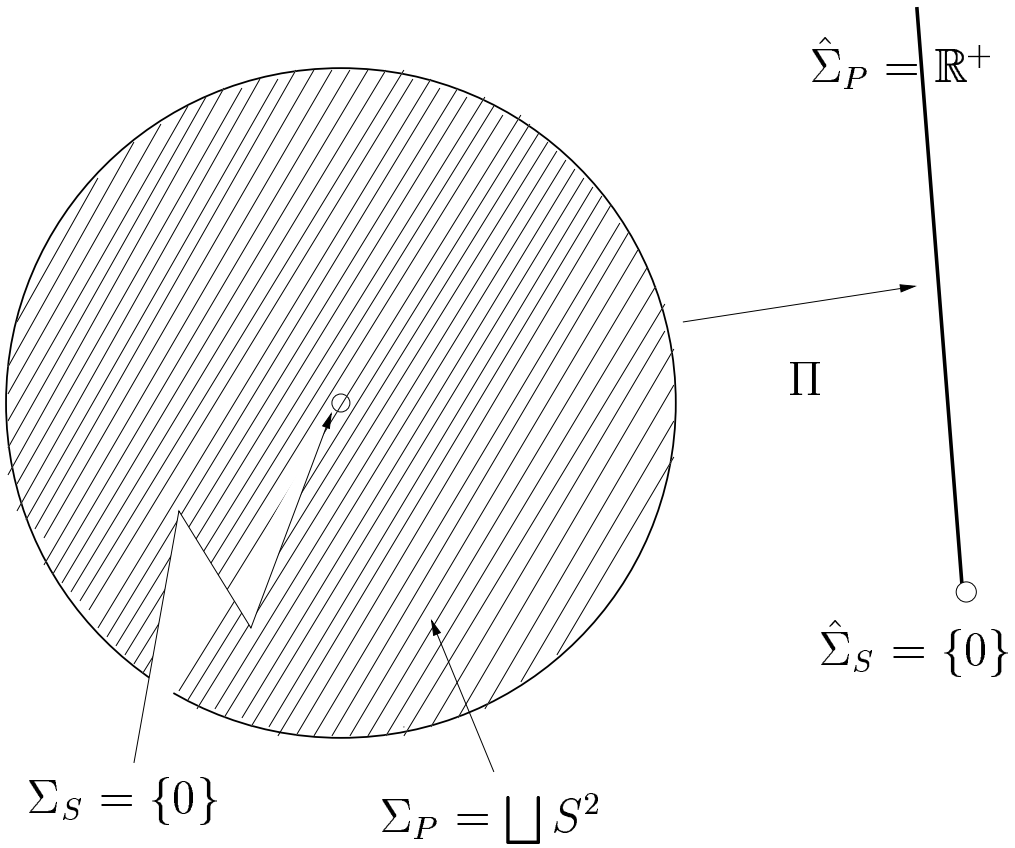

Figure 4.2: Principal and singular strata and their orbit map images

space strata (which are the projections of the orbits along the half real line, which is also the base of a singular fibration with fibers the orbits) are $\hat{\Sigma}_P$ which is $(0, \infty)$ and Σ_S, which is $\{0\}$. (See Figure 4.2.)

4.2.4 Tangents and normals

Let a compact Lie group G act on a linear vector space $\mathbb{V} \simeq \mathbb{R}^n$ orthogonally (this can always be arranged [58]). Let the orthogonal represen-

tation be denoted by $(G, \mathbb{R}^n)$. Recall from Subsections 4.1.2 and 4.1.3 that $G_\phi = \{g \in G |\ g \cdot \phi = \phi\}$ is the isotropy subgroup or symmetry group of a point $\phi \in \mathbb{V}$, and $g_\phi = \{\xi \in \mathfrak{g} |\ \xi \cdot \phi = \phi\}$ is the corresponding Lie algebra. The reference for what follows is [82].

The tangent to an orbit $\Omega(\phi)$ in the orbit space $\mathbb{V}/G$ is generated by $\dim G - \dim G_\phi$ vectors that are the image, via the action map, of as many Lie algebra elements. From the slice construction (Figure 4.1; Subsection 4.2.1) it is clear that the normal directions to the orbit are generated by a slice that is G_ϕ-invariant. Not all the points in the normal direction move under the isotropy, however. The notion of an invariant normal slice plays an important role in orbit geometry. We outline the definitions and properties in steps:

1. The tangent to the orbit, $T_\phi(\Omega) = \{\xi \cdot \phi |\ \xi \in \mathfrak{g}\}$. Clearly, $\dim T_\phi = \dim G - \dim G_\phi$. Also, the tangent is G-invariant, that is $T_{g\cdot\phi} = g \cdot T_\phi$.

2. The normal to the orbit is specified by the orthogonal decomposition, $N_\phi + T_\phi = \mathbb{R}^n$, $\langle N_\phi, T_\phi \rangle = 0$, with $\langle\ ,\ \rangle$ the G-invariant inner product on $\mathbb{R}^n$. That is, $N_\phi = \{w \in \mathbb{R}^n \mid \langle w, \xi \cdot \phi \rangle = 0\ \forall \xi \in \mathfrak{g}\}$, and $\langle \phi, w \rangle = \langle g \cdot \phi, g \cdot w \rangle$. The latter condition, given the antisymmetry of the Lie algebra to $O(n)$, when differentiated with respect to a 1-parameter group $g(t) = \exp t\xi$, at $t = 0$, yields the condition $\langle \phi, \xi \cdot \phi \rangle = 0$, which implies that the normal space contains the point on the orbit regarded now as a position vector emanating from the origin of $\mathbb{R}^n$, $\phi \in N_\phi$. (The zero vector, for the same reason, also lives in N_ϕ for any $\phi \in \mathbb{R}^n$.) Like T_ϕ,

N_ϕ is G-invariant, $N_{g\cdot\phi} = g \cdot N_\phi$.

3. The set of all points in N_ϕ that are fixed under the G_ϕ action, $N_\phi^0 = \{u \in N_\phi |\ h \cdot u = u,\ \forall h \in G_\phi\}$, orthogonally decomposes N_ϕ further into two subspaces,

$$N_\phi = N_\phi^0 \oplus N_\phi^1, \tag{4.25}$$

$$\mathbb{R}^n = T_\phi \oplus N_\phi^0 \oplus N_\phi^1. \tag{4.26}$$

4. The tangent space to a *stratum* Σ is well-defined (thanks to invariant tube (or normal slice translate) neighborhood theorem, in turn guaranteed by the Palais' retraction map) as being the orthogonal sum of the orbit and the invariant normal space:

$$T_\phi(\Sigma) = T_\phi(\Omega) \oplus N_\phi^0. \tag{4.27}$$

   Thus N_ϕ^0 lies in the tangent space to the stratum, and N_ϕ^1 is normal to the stratum. It can be shown that the necessary and sufficient condition that the stratum is a principal one is that $N_{\phi_P}^0$ spans the slice coordinates transverse to an orbit. See Figure 4.1 in Subsection 4.2.2, and Figure 4.3 below.

5. From counting dimensions, (4.27) implies dim Σ = dim G − dim G_ϕ + dim N_ϕ^0. Now, use the orbit map, $\Pi : \mathbb{R} \longrightarrow \mathbb{R}/G$, under which all of the tangent space to an orbit gets mapped to the zero vector in $\mathbb{R}/G$. The corresponding strata also get mapped, $\Pi : \Sigma_{(\Omega)} \longrightarrow \hat{\Sigma}_{(\Omega)}$, where (Ω)

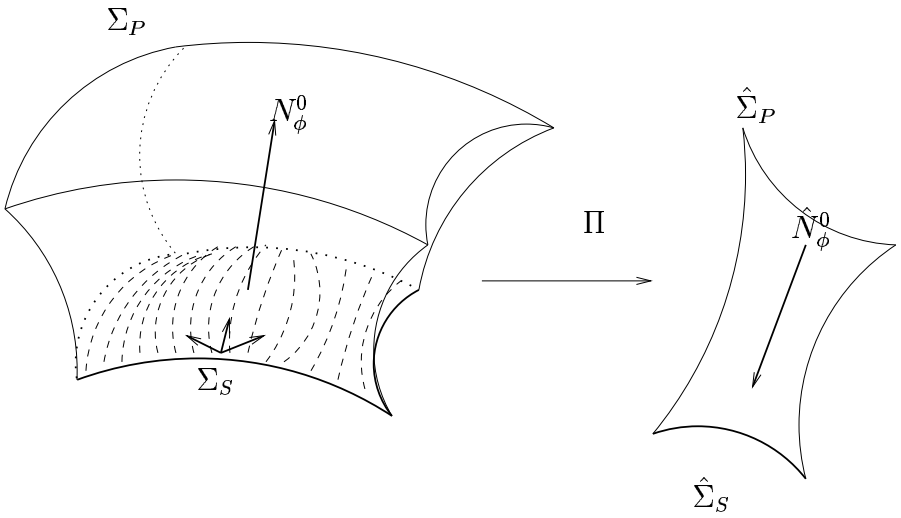

Figure 4.3: Tangents and normals to strata in the space of fields and the orbit space.

indicates the orbit type of a representative orbit of the stratum. Thus, we see that dim $\hat{\Sigma}_{(\Omega)} = \dim N_\phi^0$ [3].

6. By combining the dimensionality relations above with the definitions of the various types of strata, we have a criteron for principal orbits: The necessary and sufficient condition that Ω is principal is that $N_\phi^1 = \emptyset$ $\forall \phi \in \Omega$, in which event, the principal stratum to which the orbit belongs is full dimension, dim $\Sigma_P = n$.

[3]From this follows a famous theorem in particle physics called Michel's theorem: the sufficient condition that dim $\hat{\Sigma}_{(\Omega)} = 1$ is that the restriction of $(G_\phi, \mathbb{R}^n)$ to $(G, \mathbb{R}^n)$ contains only one singlet of G_ϕ.

4.2.5 Whitney's embedding theorems

We state the two embedding theorems of Whitney here, for arbitrary differentiable manifolds and G-manifolds respectively [58] [8]. For G-spaces, that is for continuous actions on topological (metric) spaces, an embedding theorem was proved by Mostow. Recall that a map $f : M \longrightarrow N$ induces a tangent map $df_x : T_xM \longrightarrow T_{f(x)}N$; if df_x is injective, then f is called an immersion. If in addition, f is a homeomorphism onto its image, then f is called an embedding (we give the image $f(M)$ the subspace topology relative to N). Also, a map from one G-space to another, $f : M \longrightarrow N$ is said to be G-equivariant if $f(\Theta_g(x)) = \Psi_g(f(x))$, where Θ and Ψ are the G-actions on M and N respectively.

Standard Whitney embedding theorem

Any smooth differentiable manifold M^n can be embedded into Euclidean space of dimension $2n + 1$.

Equivariant Whitney embedding theorem

Given any G-manifold M^n there exists a G-equivariant diffeomorphism of it into a linear G-space $\mathbb{V}$, embedding M into $\mathbb{V}$.

In other words, we could as well replace a complicated, nonlinear action on a differentiable manifold by a linear action of the group on a (larger) linear vector space.

4.3 Invariants of Orbits

In this section, we briefly indicate how invariant theory is used to parametrize orbits of a group action. The action shall be assumed to be a regular representation (that is, the matrix representing it has finite entries), and shall be denoted by ϱ. If $(\varrho, \mathbb{V})$ is a regular representation, we say a subset $\mathbb{W} \subset \mathbb{V}$ is G-invariant if $\varrho(g)\,\mathbb{W} \subset \mathbb{W}$. A representation is *reducible* if there exists a G-invariant subspace $\mathbb{W} \subset \mathbb{V}$ such that $\{0\} \neq \mathbb{W}$ and $\mathbb{V} \neq \mathbb{W}$. A representation is irreducible if it is not reducible (i.e., there exists no nontrivial proper G-invariant subspace of $\mathbb{V}$).

4.3.1 Classical invariant theory

The problem of *classical* invariant theory may be stated as follows: Given a representation of a classical Lie group, $(\mathbb{V}, \varrho)$, describe the elements of the fixed point tensor subspaces with respect to the representation:

$$(\otimes^k \mathbb{V})^G = \{u \in \otimes^k \mathbb{V} \mid \varrho^{\otimes k}(g)(u) = u\}.$$

Let $\mathcal{P}(\mathbb{V}^{*k} \times \mathbb{V}^m)$ denote the algebra of polynomials of k covector (1-form) arguments and m vector (derivative) arguments. Then a polynomial $p \in \mathcal{P}$ is said to be G-invariant if

$$g \cdot p(\phi_1^*, \ldots, \phi_k^*, \phi_1, \ldots, \phi_m) = p(\phi_1^* \cdot \varrho(g), \ldots, \phi_k^* \cdot \varrho(g), \varrho(g^{-1})\phi_1, \ldots, \varrho(g^{-1})\phi_m),$$

$\forall g \in G$. Then a related question is the following one: Describe the elements

of $P(\mathbb{V}^{*k} \times \mathbb{V}^m)^G$ for a pair (G, ϱ) [4].

A constructive procedure to answer this question was given by the founders of the subject of representation theory, Weyl [83], Schur, et al. In this subsection, we lay forth some preliminary notation and state the so-called FFT (First Fundamental Theorem) of invariant theory. We follow mainly the notes of Procesi [55] (see also [56], [84]), and leave the details to these references.

A *relative invariant* of a reductive group G (G is said to be reductive if every finite dimensional linear representation of G is completely reducible into a direct sum of irreducible representations) of weight χ is a map $f : \mathbb{V} \longrightarrow \mathbb{K}$ (with $\mathbb{K}$ any field) such that $\forall\ g \in G$ and $\phi \in \mathbb{V}$, $f(g \cdot \phi) = \chi(g) f(\phi)$, with $\chi : G \longrightarrow \mathbb{K}^*$ is a homomorphism called a *character map.* If $\chi(g) = 1$, then f is called an *absolute invariant.*

The first fundamental theorem (FFT) of invariant theory gives the condition for multilinear invariants of $GL(n, \mathbb{K})$ of k vectors in $\mathbb{V} \simeq \mathbb{K}^n$.

FFT

Let $f : \mathbb{V}^{\otimes k} \longrightarrow \mathbb{K}$ be a relative invariant and $x_1, \ldots, x_n \in \mathbb{V}$ vectors with $x_i = \xi_i^j e_j$. Let $[x_1, \ldots, x_n] := \det\,(\xi_i^j)$ of weight 1. Then multilinear invariants of $GL(n, \mathbb{K})$ of k vectors exist only if k is a multiple of n, $k = tn$, where t is the weight. Each such invariant is a linear combination of invariants of the

[4] The *existence* of such elements is the content of Hilbert's basis theorem (see below).

form $[x_{i_1}, \ldots, x_{i_n}]$, $[x_{i_{n+1}}, \ldots, x_{i_{2n}}]$, $\ldots$, $[x_{i_{k-n+1}}, \ldots, x_{i_k}]$.

Define $T^p_q(\mathbb{V}) = \mathbb{V}^{\otimes p} \otimes (\mathbb{V}^*)^{\otimes q}$, a tensor of order (p, q) with action on $\mathbb{V}^*$ occuring by the contragredient representation, and define the G-invariant inner product between $\mathbb{V}$ and $\mathbb{V}^*$ as follows:

$$\langle x, g \cdot x^* \rangle = \langle g^{-1} \cdot x, x^* \rangle$$

The classic theorem of Weyl gives the answer to the first question that began the subsection.

Weyl's theorem

Every multilinear invariant of $GL(n, \mathbb{K})$ on T^p_q is a linear combination of invariants of three types:

1. contravariant brackets $[x^1, \ldots, x^n]$ of weight 1
2. covariant brackets $[y_1, \ldots, y_n]$ of weight -1
3. scalar products $\langle x, y \rangle$ of weight 0.

4.3.2 Orbit and strata parametrization by invariants

In a footnote to the previous subsection, it was mentioned that Hilbert's basis theorem guarantees the existence of a finite basis of invariants. Here, we shall review aspects of this theory, and obtain a natural parametrization for strata for a group action on a vector space.

As before, let $\mathbb{V}$ be a vector space (and for definiteness, let $\mathbb{K} \equiv \mathbb{R}$ be the field), on which a compact Lie group G acts linearly (hence by orthogonal representation) on $\mathbb{V}$. A polynomial $p : \mathbb{V} \longrightarrow \mathbb{R}$ is called G-invariant if $p(g \cdot \phi) = p(\phi)\ \forall g \in G,\ \phi \in \mathbb{V}$.

Hilbert's basis theorem[85, 8]

P^G, the ring of G-invariant polynomials under the operations of polynomial addition and multiplication, is a finitely generated algebra under $\mathbb{R}$.

The algebra P^G is a graded algebra, graded by the degree of the invariant polynomial. The basis guaranteed by the above theorem is called an *integrity* basis. There is a minimum number q, for a given representation $(G, \mathbb{V})$; this is called a minimal integrity basis, or Hilbert basis. We denote the generators of P^G by $(\theta_1, \ldots, \theta_q) \in \mathbb{R}^q$. Any polynomial $p \in P^G$ can be written, $\phi \in \mathbb{V}$, $p(\phi) = \hat{p}(\theta_1(\phi), \ldots, \theta_q(\phi))$. When a Hilbert basis is formed by algebraic independent polynomials, it is called *free* and the corresponding representation $(G, \mathbb{R}^n)$ is called *cofree*. A non-cofree basis would have a certain number of nontrivial algebraic identities between the basis elements. (All cofree representations of complex semisimple Lie groups have been classified by G. W. Schwarz [86].)

Next we define the *Hilbert map* (sometimes called the orbit map; but we reserve that term for $\Pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n / G$) as a map that goes from the orbits to its invariant parametrization, $\mathfrak{H} : \mathbb{V}^n \longrightarrow \mathbb{R}^q$, $\phi \longmapsto (\theta_1(\phi), \ldots, \theta_q(\phi))$. The

image $\mathfrak{H}(\mathbb{V})$ is a *semi-algebraic* variety $\mathcal{S} \in \mathbb{R}^q$. Via the orbit map Π, $\mathbb{R}^n/G$ becomes immersed as a semi-algebraic variety [5] $\hat{\mathcal{S}} \in \mathbb{R}^q$ as well.

The main theorem that allows orbits to be parametrized by invariants is the following:

Theorem

1. $\mathfrak{H}$ is a proper map (namely, inverse image of a compact set is compact also).

2. If $\phi \notin \Omega(\phi')$, then $\mathfrak{H}(\phi) \neq \mathfrak{H}(\phi')$; points from distinct orbits are separated by the Hilbert map.

3. $\mathcal{S} \simeq \hat{\mathcal{S}}$, that is, there is a 'commutative triangle',

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\mathfrak{H}} & \mathcal{S} \in \mathbb{R}^q \\ \Pi \downarrow & & \| \\ \mathbb{R}^n/G & \xrightarrow[\hat{\mathfrak{H}}]{} & \hat{\mathcal{S}} \in \mathbb{R}^q \end{array}$$

   where the map $\hat{\mathfrak{H}}$ is a homeomorphism.

The above result for polynomials has been generalized to the *smooth* category by G. W. Schwarz. The proof essentialy uses Weierstraß's approximation theorem. A smooth function may also be written as a function of a finite number of basis elements that generate the ring of smooth G-invariant functions.

[5] An algebraic variety is a zero locus of a finite set of polynomials. A semi-algebraic variety is defined in part by inequalities.

Examples

As an example, let us consider $SU(n)$ in the adjoint representation, defined by $Tr\phi = 0$. A basis for the invariants are just the Casimirs, $\theta_1 = Tr\phi^2, \ldots, \theta_{n-1} = Tr\phi^n$. Similarly for $\mathfrak{so}(\mathfrak{n})$, the Lie algebra of skew-symmetric matrices, $X^T = -X$, for which the invariants are $S_k = TrX^k$, $k = 1, \ldots, [n/2] = l$. The characteristic polynomial is the expression

$$\begin{aligned} \Delta_X(\lambda) &= |X - \lambda \mathbf{1}| \\ &= \lambda^n - p_1\lambda^{n-1} - \cdots - p_n, \end{aligned}$$

The various p_i in the expansion are the i^{th}-symmetric function of the eigenvalues of X. Let the polynomial ring over the real numbers generated by these p_i be denoted by $\mathbb{R}[p_1, \ldots, p_n]$. Clearly, the symmetric functions form an integrity basis for an arbitrary symmetric polynomial in n variables. There is an *isomorphism* between $\mathbb{R}[p_1, \ldots, p_n]$ and $\mathbb{R}[S_1, \ldots, S_l]$, made explicit through a set of recursive relations known as the *Newton formulas*,

$$\begin{aligned} S_i - p_1 S_{i-1} + p_2 S_{i-2} - \cdots + (-1)^{i-1} p_{i-1} S_1 + (-1)^i p_i &= 0, \ i = 1, \ldots, n \\ S_i - p_1 S_{i-1} + p_2 S_{i-2} - \cdots + (-1)^n p_n S_{i-n} &= 0, \\ & \quad i = n+1, n+2, \cdots . \end{aligned}$$

It is in fact possible [37] to use these relations in order to parametrize singular orbits of the $SO(n)$ coadjoint action on the Lie coalgebra $\mathfrak{so}(\mathfrak{n})^*$,

and in general, for any semisimple Lie group. The various symmetric functions $p_1, \ldots, p_{n-k}$ could be written as the sum over squares of Pfaffians (essentially determinants of minors obtained by deleting the complementary indices). Setting them equal to zero amounts to looking at lower rank subalgebras of $X \in \mathfrak{so}(\mathfrak{n})$, or on the group level, at homogeneous spaces of the form (for $n = 2m$ case as an example) $SO(2m)/(SO(2k) \times U(1)^{m-k}$. From general considerations of orbit types (Subsections 4.2.2, 3.1.2) and symmetry breaking, this amounts to a partial breaking of symmetry. The subcasimirs that arise in the process are simply the coordinates of the embedding specified by the above system of equations. (The representation in this case is called *non cofree* as the equations are tied up with determinantal identities.)

4.4 Appendix: Equivariant and L-P Cohomology

The subject of Equivariant Cohomology was developed in part to give a satisfactory description of actions that are not free. For free actions of a compact group G on a manifold M, as we have seen in this chapter, M/G can be given the structure of the base of a principal G-bundle, with the projection map $\Pi : M \longrightarrow M/G$ serving as the orbit map. For non-free actions, the topology of the orbit structure required separation into orbit types, principal and non-principal orbits, etc. Another approach [87] is to replace the base space with some other space that is *homotopic* to it, and on which the group action is indeed free. This construction is outlined briefly below.

It is then possible to 'measure' cohomology of the normal bundle to

the orbit in question, and from its size, deduce the nature of the orbit. (For the case of a Poisson manifold, the related notion of L-P cohomology serves a similar purpose: when jumps in orbit type are made, the rank change of the Poisson structure J, is signalled by the relevant L-P *leaf* cohomology. Thus, another way to state the assumptions made about the Poisson structures we consider is to say that the L-P cohomology and equivariant cohomology both capture the same phenomenon.)

Consider the fixed point set of the (left) group action of G on M, call it M^G. Two spaces X and Y are isomorphic if there exist maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g$ is homotopic to Id_Y and $g \circ f$ is homotopic to Id_X. So we replace each $m \in M^G$ by a space EG of the same homotopic type as m, but on which G acts freely. Then $EG/G = BG$ is a proper quotient (and $\Pi : EG \longrightarrow BG$ is a principal G-bundle), called the *classifying space* for G, and $EG \longrightarrow BG$ is called a *model* for the free action of G on the contractible space EG. (Note: This notation is not descriptive enough, for it hides the role of the action $\Theta : G \times M \longrightarrow M$; different actions will give rise to different models.)

Next, we pass to a diagonal action of G on $EG \times M$, which we mod out by G to give the *homotopy theoretic quotient*:

$$M_G = (EG \times M)/G := EG \times_G M.$$

Define the Equivariant Cohomology of M to be the (ordinary) cohomology of

the quotient space M_G,

$$H^*_G(M) := H^*(M_G), \tag{4.28}$$

with $H^*_G := H^*_G(point) \equiv H^*(BG)$.

Recall the definition of an associated bundle [59]. Let X and Y be two manifolds and H act on them via the same (left) action. Assume that action on X is free, so that $X \longrightarrow X/H$ is a principal H-bundle. Define the action on $X \times Y$ by $(h,(x,y)) \longmapsto (x \cdot h^{-1}, h \cdot y)$ for $h \in H$ and $(x,y) \in X \times Y$. Note that this definition makes sense, since left action by $h \in H$ is equivalent to a right action by $h^{-1} \in H$ (see Subsection 4.1.1). This product action is proper and free. Define the quotient to be the smooth manifold by

$$X \times_H Y := \{\{(x \cdot h^{-1}, h \cdot y) |\ h \in H\} |\ (x,y) \in X \times Y\},$$

so that $X \times Y \longrightarrow X \times_H Y$ is a principal H-bundle. Consider the following commutative diagram:

$$\begin{array}{ccc} X \times Y & \xrightarrow{\Pi} & X \times_H Y \\ {\scriptstyle \Pi_1}\downarrow & & \downarrow{\scriptstyle \widetilde{\Pi}_1} \\ X & \xrightarrow[\Pi]{} & X/H \end{array} \tag{4.29}$$

If such a projection on the first factor can induce one on the quotient space so that the above diagram commutes, then we say that $\widetilde{\Pi}_1 : X \times_H Y \longrightarrow X/H$ is an *associated bundle* to the principal bundle $\Pi : X \longrightarrow X/H$.

Using this definition, it is clear that the homotopic theoretic quotient is likewise accociated to the principal G-bundle $\Pi : EG \longrightarrow BG$, via the following commutative diagram:

$$\begin{array}{ccc} EG \times M & \xrightarrow{\Pi} & EG \times_G M \simeq M_G \\ {\scriptstyle \Pi_1}\downarrow & & \downarrow{\scriptstyle \widetilde{\Pi}_1} \\ EG & \xrightarrow[\Pi]{} & EG/G \simeq BG \end{array} \tag{4.30}$$

This means, in particular, that $H^*(M_G)$ is a module over $H^*(BG)$. We define $[H^*(BG)]$ to be the characteristic class of the bundle (not a principal fiber bundle in general) $M \longrightarrow M/G$.

The other half of the above commutative diagram completes the so-called *mixing diagram*:

$$\begin{array}{ccc} M & \xrightarrow{\Pi} & M/G \\ {\scriptstyle \Pi_2}\uparrow & & \uparrow{\scriptstyle \widetilde{\Pi}_2} \\ EG \times M & \xrightarrow[\Pi]{} & M_G \end{array} \tag{4.31}$$

The application to orbit geometry [88] is as follows: if the isotropy group of any point $m \in M$ is denoted by G_m, then from the above commutative diagrams, the fiber over an orbit $G \cdot m$ is a classifying space of G_m,

$$\widetilde{\Pi}_2^{-1}(G \cdot M) = EG/G_m \simeq BG_m.$$

Then, by definition (4.28), we have that

$$H^*_G(G/G_m) = H^*_{(G_m)} = H^*(BG_m),$$

so that the *normal bundle* to the orbit (or the generator of the invariant normal slice tube neighborhood) has its characteristic classes lying in $H^*(BG_m)$, which is obviously a function of both $m \in M$ and the isotropy at that point, G_m. (Recall that the slice is invariant with respect to precisely this isotropy.) Thus, we have a direct connection between the orbit types of a G-manifold, and the equivariant cohomology of the transversal to the orbit [89].

For the case of a Poisson manifold, and for the case that the Poisson manifold is a Lie-Poisson group (i.e., the manifold is the dual $\mathfrak{g}^*$ to a Lie algebra $\mathfrak{g}$, with the Lie-Poisson bracket (2.9)), it can be shown that the L-P cohomology is simply the canonical (Lie algebra) cohomology tensored with the algebra of Casimirs [90],

$$H^*_{L-P}(\mathfrak{g}^*, J) = H^*(\mathfrak{g}) \otimes \{\text{Casimirs of } (\mathfrak{g}^*, J)\}, \tag{4.32}$$

which clearly depends on the structure of the Poisson tensor J. This information would coincide with the one provided by the equivariant cohomology as far as rank change is concerned, if the general Poisson foliation — defined by its characteristic distribution — is indeed (as it is assumed throughout this thesis) one that coincides with the orbits structure of a compact Lie group action on the Poisson manifold P.

L-P cohomology

The local structure of a Poisson manifold depends strongly on its Poisson structure. One of the ways to measure this dependence is through a cohomology that depends also on J. It was first introduced by Lichnerowicz, and subsequently studied by many authors. In this subsection, we note the definition of L-P cohomology for a Poisson manifold (P^m, J) as it appears in the literature. (See, eg., [15, 91], [92, 93, 94], [95].) Deformations of Poisson structure, and rank changes across leaves, are measured by L-P cohomology.

Notation: Let $\Lambda^\star(P)$ denote the tensor subspaces of antisymmetric (contravariant) forms of an unspecified degree, and $\Lambda^{-\star}(P)$ denote the tensor subspaces of antisymmetric *covariant* tensors of an unspecified degree[6].

(For instance, the Poisson tensor $J \in \Lambda^{-2}(P)$, since by definition (2.5), we have $\{f, g\} = J(df, dg) \in \mathbb{R} \Longrightarrow J = J^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$. Here, i, j run over $1, \ldots, m$.)

$\Lambda^{-p}(P)$ is a *graded* vector space, graded by the degree of the tensors, and is furthermore a Lie algebra by the exterior product $\wedge$ (just as for forms). The Lie derivative is a derivation on the algebra for the case $p = 1$.

We now define the *Schouten Bracket*. Given $A \in \Lambda^{-k}(P)$, it is defined as an index raising operator,

$$D : \Lambda^{-k}(P) \longrightarrow \Lambda^{-(k+1)}(P)$$

[6]These notations are explained also at the end of Section 1.3.

by

$$\begin{aligned} DA &= -[J, A] \\ &\sim (J \cdot \partial)A + (A \cdot \partial)J. \end{aligned}$$

More precisely, for $\alpha_i \in \Lambda^1(P)$, we have the expansion

$$\begin{aligned} (DA)(\alpha_1, \ldots, \alpha_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} (J\alpha)_i A(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_{k+1} \\ &+ \sum_{i<j=1}^{k+1} (-1)^{i+j} A(\{\{\alpha_i, \alpha_j\}\}, \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_{k+1}), \end{aligned}$$

with

$$\{\{\alpha_i, \alpha_j\}\} = \mathfrak{L}_{(J\alpha)_i} \alpha_j - \mathfrak{L}_{(J\alpha)_j} \alpha_i - d(J(\alpha_i, \alpha_j)),$$

or, for the case of exact forms, $\alpha_i = df_i,\ f_i \in \Lambda^0(P)$,

$$\{\{\alpha_i, \alpha_j\}\} = d\{f, g\} = dJ(df, dg).$$

From the last two expressions, we see that that Schouten bracket is in essence a generalization of the Lie derivative.

It can be shown that the *Jacobi identity* of the Poisson algebra reduces to the simple expression of the fact that the Schouten bracket of J with itself is identically zero, i.e., $[J, J] \equiv 0$. This implies, in turn, that the exterior derivative, D defined above satisfies $D^2 = 0$. Hence, we can form a *cohomology complex* of skew-symmetric tensors,

$$\ldots \xrightarrow{D^{k-1}} \Lambda^{-k} \xrightarrow{D^k} \Lambda^{-(k+1)} \xrightarrow{D^{k+1}} \ldots$$

where $D^{k-1} \circ D^k = 0$, $\mathrm{Im}(D^k) \subseteq \mathrm{Ker}(D^{k+1})$, etc., and the k-th L-P cohomology is defined as

$$H^k_{L-P}(P, J) = \mathrm{Ker}(D^k)/\mathrm{Im}(D^{k-1}).$$

Since J is involved in the definition of the Schouten operator D, it follows that the L-P cohomology would probe the change in the transverse geometry of the Poisson foliation as we cross orbit types. The computation of the L-P cohomology is complicated by its dependence on the specifics of J, however, and aside from the simplest cases, or the case of a Lie-Poisson group ($\mathfrak{g}^*$ in (4.32) above), the higher order L-P cohomologies have not been computed. (It can be shown that third order cohomologies distinguish between the abelian algebra of differentials of Casimirs of a regular leaf, from the non-abelian algebra of differentials of subcasimirs of a singular leaf.) Finally, we note that $H^0_{L-P}(P, J) = \mathrm{Cass}\ (P)$, the algebra of Casimirs; and that for the case that J is nondegenerate, that is, $J^{-1} = \omega$, a symplectic structure, the L-P and de Rham cohomologies coincide.

Chapter 5

Spontaneous Symmetry Breaking: Kinematics-Dynamics connection

In this chapter, we bring together the various threads of ideas and development from the previous chapters, to outline a scheme that relates the Goldstone bosons of symmetry breaking of a suitably chosen potential, and the subcasimirs of a singular leaf. In this way, using *control parameters* present in the potential, we can move from leaf to leaf and study how the orbit geometry for a given representation changes.

Let P be a Poisson manifold and G a group acting on it. By Whitney's embedding theorem (cf. Subsection 4.2.5), this is equivalent to a linear action on a vector space $\mathbb{V}$, which could moreover (if G is compact) be taken to be an orthogonal action [58]. So, without loss of generality, we consider an orthogonal representation on $\mathbb{V}$. By results of orbit geometry and invariant theory, the strata of $\mathbb{V}$ and $\mathbb{V}/G$ are well defined, related via the orbit map, and parametrized by a ring of invariants. These strata are semi algebraic varieties, and their tangent and normal space at each point has a well defined meaning in the sense of algebraic varieties [96] and their tangent and normal varieties.

For the case that $P = \mathfrak{g}^*$, where $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{g}^*$ its dual, this identification is the natural one suggested by the maximal isotropy embedding of a Lie algebra $\mathfrak{h}$ in the larger Lie algebra $\mathfrak{g}$, for the case that symmetry is broken from G down to H [36].

The notion of *patterns* in the breaking of symmetry, namely variation in the subgroups of the original symmetry group that survive according to the variation of control parameters (like coupling constants, or classical masses), played an important historical role in spontaneous and gauge symmetry breaking in particle physics. Here, it was assumed that the (hadronic, for instance) potential had a certain polynomial form (say, quartic), and the problem was to find the minima of the potential with respect to certain parameters. This problem was soon realized to be mostly *geometrical* in character [97, 98, 82, 99, 100, 101], with orbit geometry and invariant theory playing a role in defining the possible orbital strata that minima could occupy. We turn this development around, and use arbitrarily chosen G-invariant potentials and their extrema in order to relate kinematical data of the Poisson bracket, that is, orbit geometry (as defined by the characteristic distribution of $\mathbb{V}$, the vector space modeling the Poisson manifold, P), with dynamical information that comes from symmetry breaking by the potential, to the appropriate size residual symmetry groups.

The first section motivates the idea behind the orbit geometry of symmetry breaking by considering ordinary (bosonic) field theoretic symmetry breaking. Next, we consider how orbit geometry and invariant theory can be

used to relate extrema of the G-invariant potential with the various strata, both principal and non-principal. This is followed by the main idea of the thesis, which is to relate the subcasimirs that appear in the process of moving across orbits with the extra Goldstone bosons that come from the spontaneous breaking and restoration of symmetry, by means of varying the controls in the G-invariant potential.

5.1 Bosonic symmetry breaking

This is the most basic mechanism for the idea of symmetry breaking. The Hamiltonian, or the potential is assumed to be a G-invariant one, where G is a symmetry group (could be discrete or continuous), and a function of a certain number of scalar fields. The fields themselves depend on the spacetime coordinates (so that they live in an associated bundle to a principal G-bundle), and the symmetry group acts on the space of fields. A potential is assumed to be invariant under the group action of the entire group. Spontaneous breaking of symmetry is said to occur when the symmetry of the solutions [1], got by extremizing the potential, is less than the full symmetry group. Typically, the lower symmetry happens to be an isotropy subgroup for the group action (see Subsection 4.1.1, 4.1.2 for the definitions) at the equilibrium point in question. Below, we review the mechanism of bosonic symmetry breaking, and the associated Goldstone modes that arise as a result [102, 103].

[1] We use the terms 'vacuum' and 'equilibrium' interchangably.

Let $\mathbb{V}$ be the space of n scalar fields, $\phi_i : M \longrightarrow \mathbb{R}$, where M is some underlying differentiable manifold. Let G be a compact Lie group that acts properly on $\mathbb{V}$, and let $V(\phi)$ be a scalar potential function on $\mathbb{V}$. Then the condition for a *vacuum* is that

$$\left.\frac{\partial V(\phi)}{\partial \phi_i}\right|_{\phi=\phi_{(e)}} = 0, \qquad (i = 1, \ldots, n),$$

so the Taylor expansion of the potential about equilibria takes the form:

$$V(\phi) = V(\phi_{(e)}) + \frac{1}{2} M_{ij} \chi_i \chi_j + o(\chi^3).$$

In the above expression, the 'mass matrix' $M_{ij} \geq 0$ is defined as

$$M_{ij} = \left.\frac{\partial^2 V(\phi)}{\partial \phi_i \partial \phi_j}\right|_{\phi_{(e)}}, \tag{5.1}$$

and the shifted fields by $\chi_i(x) = \phi_i(x) - \phi_{(e)i}$. The mass matrix, being the second derivative of the potential, governs the stability of the equilibrium. (See [103] for more on stability in field theory systems; we do not concern ourselves with this aspect of the theory, but rather simply use the null spectrum of M_{ij} to study orbit geometry on $\mathbb{V}$, which models the linear Poisson G-manifold.)

If the potential is G-invariant, then it satisfies the invariance condition at the vacuum as well, so

$$V(\phi_{(e)}) = V(\Theta_g \phi_{(e)}) = V(\phi_{(e)}) + \frac{1}{2} M_{ij} \delta\phi_{(e)i} \delta\phi_{(e)j} + \cdots, \tag{5.2}$$

where $\Theta_g : \mathbb{V} \longrightarrow \mathbb{V}$ is the action $\forall g \in G$.

Let $(T^a)^i_j$ denote the representation matrices, where $a = 1, \ldots, d = \dim G$, and $i, j = 1, \ldots, n = \dim \mathbb{V}$, so that $\Theta_g = \exp tT^a\xi_a$, $\{\xi_a\} \in \mathfrak{g}$, $g(t) \in G$ a 1-parameter subgroup, and set the variation $\delta\phi_{(e)}$ to be:

$$\delta\phi_{(e)i} = \left[\left. \frac{\partial \Theta_g}{\partial \xi_a} \right|_{t=0} \phi_{(e)} \right]_i \delta\xi_a.$$

Now, suppose that only a subgroup $H \subset G$ leaves invariant the equilibrium point. We have two possibilities:

1. If $g \in H$, then $\delta\phi_{(e)i} \equiv 0$, and the G-invariance condition for V (5.2) is identically satisfied, that is $V(\phi_{(e)}) = V(\phi_{(e)})$, since $M_{ij}\delta\phi_{(e)i}\delta\phi_{(e)j} + \cdots \equiv 0$.

2. If $g \in G/H$, then $\delta\phi_{(e)i} \neq 0$, and so G-invariance of V now would require that M_{ij} have null eigenvectors:

$$M_{ij}[\Theta'|_0\phi_{(e)}]_j = 0.$$

   There are $\dim G/H = \dim G - \dim H$ null eigenvectors of the form $\{\Theta'|_0\phi_{(e)}\}$, of eigenvalue zero, that represent massless fields, called Goldstone (or Nambu-Goldstone) bosons, and the homogeneous space G/H parametrizes the space of Goldstone bosons via the action map $\Theta : G \times \mathbb{V} \longrightarrow \mathbb{V}$.

Before we move on to the next section, a few comments are in order. The number of massive fields (those with nonzero eigenvalues to the mass matrix) is dim H. If H happens to be the isotropy subgroup $G_{\phi_{(e)}}$ for the

equilibrium point, $\phi_{(e)}$, then these modes are transversal to the orbit of the group action (cf. Subsection 4.2.1 on slices), and represent the 'difficulty' in getting away from the orbit, or the leaf as they are massive, and doing so would cost energy. Suppose now the potential depended on some number of control parameters (these could be coupling constants, classical masses, charges, background field strengths, etc.), that is, we write $V = V(\phi, \gamma)$, where γ are the controls. Extremizing the potential (supposed G-invariant) as above, we would arrive at equilibria that depended on the control parameters, $\phi_{(e)} = \phi_{(e)}(\gamma)$.

As the controls are varied, we are led from solution to solution. Formally, this happens in an infinite dimensional space of solutions of the classical potential [2], but we shall restrict our attention to what goes on in the orbit space decomposition (cf. Section 4.2) of $\mathbb{V}$. For some values of the control parameters, the isotropy group of the solution may change, so we are led from leaves of one dimension to leaves of other dimensions, and across orbit types. Then invariant theory (cf. Sections 4.3 and 3.4) shall relate the gradients of the potential and the null eigenvalues of the mass matrix. Referring all this back to the Poisson manifold we began with, the picture we seek would be complete.

Finally, we note that the Goldstone bosons we consider below are usu-

[2] A generalization to the simple potential extremization we have used would involve a full Hamiltonian or action, that may preclude a splitting into potential and kinetic terms. Then we cannot implicitly assume the kinetic terms are already frozen, or set to zero. Interesting situations like *Killing symmetry* directions could thus arise [104].

ally gone when quantization of the classical theory in question is carried out. In particular, this could mean that any attempts to relate a *quadratic* Poisson structure with such a picture might probably not work, as quadratic structures govern the quantization of a classical theory [17].

5.2 Extrema and strata

In this section, we use the invariants provided by the gradients of the G-invariant potential $V(\phi, \gamma)$, with $\phi \in \mathbb{V}^n$ and $\gamma \in Q^s$, where the latter notation means the control parameters are derived from some s-dimensional manifold of controls. (In what follows, the structure of Q will play little or no role, although its dimension is relevant.) The orbit in $\mathbb{V}$ to which the equilibrium belongs is characterized by the orbital stratum, and the equations of the stratum are carved out by these gradients. In making this relation, the geometry of the tangent space and normal space to an orbit (cf. Subsection 4.2.4) shall be extensively invoked. (We won't repeat the references to works cited there, and refer the reader to the sections and subsections in question for the most part.)

Recall that the tangent space to the principal stratum comprises both the tangent space to the orbit, $T_\phi(\Omega) = \{\xi \cdot \phi |\ \xi \in \mathfrak{g}\}$, and the G_ϕ-invariant normal slice N_ϕ^0 transversal to it. By the results stated in Subsection 4.3.2, the main observation about the generators of N_ϕ^0 is the following: the invariant normal slice is the span of $\partial\theta_a(\phi)$, $a = 1, \ldots, q$. The proof may be found in [100]. Note that q is the minimal number of basis elements $\theta_i(\phi)$, guaranteed

to exist by Hilbert's theorem.

Furthermore, any orbital invariant on $\mathbb{V}$, such as a G-invariant potential for symmetry breaking, can be expressed in terms of the Hilbert basis as a function in the orbit space $\mathbb{V}/G$:

$$V(\phi) = \hat{V}(\theta_1(\phi), \ldots, \theta_q(\phi)) \tag{5.3}$$

$\forall \phi \in \mathbb{R}^n$. It follows at once by differentiating the G-invariance condition for the potential that its gradients span the invariant normal slice: $V(g{\cdot}\phi) = V(\phi) \Longrightarrow \langle \xi \cdot \phi, \partial V(\phi) \rangle = 0$ for every one parameter subgroup $g(t) = \exp t\xi \in G$; by referring to Subsection 4.2.4, this is seen to be the condition for an invariant normal vector. We note that *any* G-invariant function on the orbits has the spanning property; so the orbit geometry is a lot more general than the specifics of a system requires. However, a potential allows for controls, and these can be manipulated to wander across orbit types.

The final step in the process of using gradients to characterize strata is counting dimensions. Let ϕ_P denote a point that belongs to a principal stratum, and ϕ_S to a singular or nonprincipal stratum [3]. We note that from the relation (4.26), it follows that the sum of the dimensions of the orbit tangent T_ϕ and of the two normal components, N^0_ϕ and N^1_ϕ is fixed, at $n = \dim \mathbb{V}$. Since the singular stratum bounds the principal stratum, $\Sigma_S \subset \partial\Sigma_P$, thinner orbits have fatter normal spaces, and these are characterized by the

[3] We rule out exceptionals, as noted in Subsection 4.2.3.

appearance of extra gradients. These gradients will, however, not be G_{ϕ_P}-invariant, that is, they get moved around in the 'extra' dimensions transverse to the thinner orbit. The larger invariance group, $G_{\phi_S} \supset G_{\phi_P}$ means at once two things: a larger number of invariants (that are left invariant with respect to the larger subgroup); and a larger number of dependent relations (that is, the representation when restricted to the singular orbit is no longer cofree).

Next, we refer to the orbit space, $\mathbb{V}/G$, where distinct points are separated by invariants using the Hilbert map (see Subsection 4.3.2); this map takes both the vector space strata and the orbit space strata to the corresponding semialgebraic varieties $\mathcal{S}$ and $\hat{\mathcal{S}}$ both of which live in the Euclidean space $\mathbb{R}^q$.

Define the real, symmetric, positive semi-definite (no eigenvalues less than zero) matrix formed of the gradients of the invariant polynomials,

$$\begin{aligned} R_{ab}(\phi) &= \sum_{i=1}^{n} \partial_i \theta_a(\phi) \partial_i \theta_b(\phi) \\ &= \hat{R}_{ab}(\theta(\phi)), \end{aligned}$$

with $a, b = 1, \ldots, q$ and ∂_i stands for derivative w.r.t. ϕ_i.

The number of independent vectors in $\{\partial\theta_a\}$ is equal to rank $\hat{R}_{ab}(\phi) = \dim N_\phi^0$. The strata and their images in the orbit space are carved out by the locus of the determinants of the principal minors of the above matrix. The gradients of these determinants in the orbit space can be shown to span the null space of the matrix $\hat{R}_{ab}$, defining the 'normal' directions for the image of the stratum to which ϕ belongs.

Introducing a potential that needs to be extremized is the same as finding the null eigenvectors of the R matrix above, since $\partial_i V(\phi) = 0$ implies, by using (5.3) the same as $\sum_{a=1}^{q} \hat{R}_{ab}(\theta)\partial_b \hat{V}(\theta) = 0$. This is a locus of algebraic equations and inequalities in orbit space $\mathbb{V}/G$ (a semialgebraic variety) with the determinants of the minors of corank the dimension of the strata $\hat{\Sigma}_\phi$ serving as the Lagrange multipliers (as in the stability example in Subsection 3.1.1). Extremizing this system of equations yields an optimal set of values for both basis invariants and the control parameters; the mass matrix (5.1) may be then formed out of these [4].

5.3 An interpretation of Goldstone modes in Poisson geometry

For a Poisson manifold P, the Casimirs arise as constraint surfaces that are defined by a generalized distribution associated to the Poisson structure J. Although the procedure for finding subcasimirs reduces to computing the null eigenvectors of the Poisson structure matrix restricted to the orbit in question, our aim here is to connect the subcasimirs with null eigenvectors of the mass matrix (5.1). With controls added, the mass matrix has the form,

$$M_{ij} = \partial_i \partial_j V(\phi, \bar{\gamma})|_{\phi=\phi_{(e)}},$$

$i, j = 1, \ldots, n$. Here, the controls themselves have been solved with the principal minor determinants added to the potential as constraints and extremized

[4] The number of zero eigenvalues of the mass matrix gives the corank of the corresponding orbit Lie-Poisson structure.

collectively (to yield $\bar{\gamma}$) with respect to both the invariant polynomials and the Lagrange multipliers (as many added as there are normal directions to the orbit space stratum, $\hat{\Sigma}$). We implicitly assume that any smooth variation $\delta\gamma$ in the controls changes the vacuum solution *perturbatively*; that is, if $\delta\gamma \longrightarrow 0$, then $\delta\phi_{(e)} \longrightarrow 0$ as well.

Now, it can be shown using the above, that the mass matrix decomposes into two parts: M^1 and M^2, such that $M^1 \cdot N^0_{\phi_{(e)}} = 0$, $M^2 \cdot N^1_{\phi_{(e)}} = 0$, in addition to the usual equations for the scalar Goldstone modes tangent to orbit: $M^1 \cdot T_\phi = 0 = M^2 \cdot T_\phi$.

The former Goldstone bosons correspond to the directions which arise from adding the constraint conditions, and must therefore be regarded as new features that arise due to the orbital structure [5]. In the context that we are working within, these are precisely the subcasimirs that would arise upon a process of partial *symmetry restoration*, where the isotropy group of the equilibrium segues into one of larger dimension (and does so in a continuous manner since it is perturbative). Let $\phi_{(e)}$ denote the generic vacuum, with isotropy $G_{\phi_{(e)}}$, and $\phi_{(e')}$ the new vacuum upon variation of the control parameters, with isotropy $G_{\phi_{(e')}}$, with $G_{\phi_{(e')}} \supset G_{\phi_{(e)}}$. Conversely, when the controls are reversed, these extra Goldstone modes from the normal directions become the Goldstone bosons for a symmetry breakdown from the larger isotropy group $G_{\phi_{(e')}}$ to the smaller one, $G_{\phi_{(e)}}$. The number of extra Goldstone modes that

[5] In physics, one of the components, namely the N^0_ϕ has an interpretation of being associated to pseudo-Goldstone modes, which do not survive radiative corrections [105, 106].

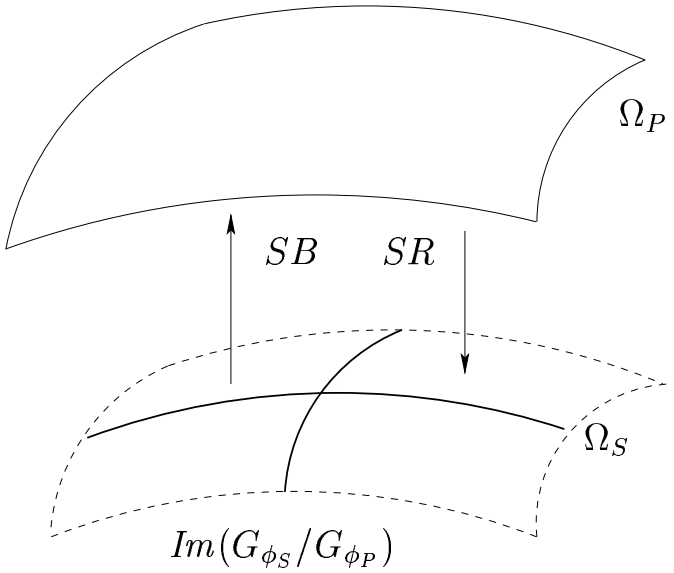

Figure 5.1: Symmetry breaking and restoration between two orbit types.

are needed for this is, by Goldstone's theorem, equal to the difference in the orbit dimensions of the fatter and thinner orbits, which is the dimension of the homogeneous space $G_{\phi_{(e')}}/G_{\phi_{(e)}}$. These new modes may be regarded as the embedding coordinates for the singular leaf into a regular leaf. (See Figure 5.1 below.)

As an illustration, to connect with the $SO(3)$ case, imagine the isotropy of the double well solution to increase from $U(1)$ at the trough (flat direction), all the way to $SO(3)$ at the center, at the top of the hillock (Figure 5.2 below.) Clearly the new equilibrium, for any intermediate hillock shape, is an unstable one, a fact reflected in the degeneracy in the modified mass matrix. (Thus, without the controls, the origin would never become a preferred point of rest. It would, if the shape were modified so much by the controls that it became

$$V(\phi) = \tfrac{1}{2}m^2\phi\cdot\phi + \lambda(\phi\cdot\phi)^2.$$

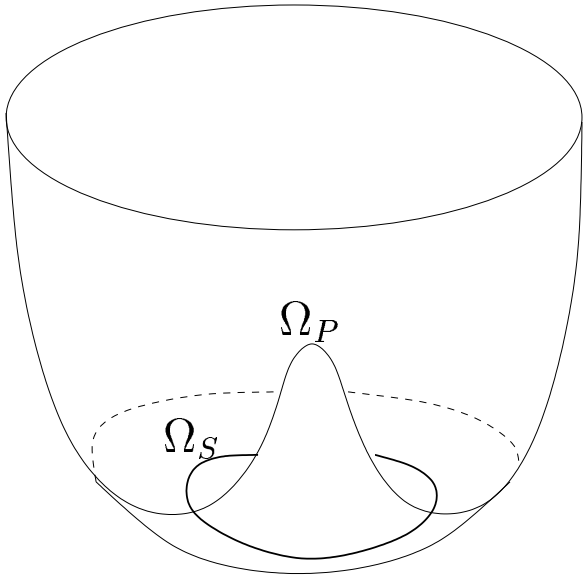

Figure 5.2: Symmetry restoration pattern, rank change, and stability in a simple example of spontaneous symmetry breaking, with control space (m, λ).

favorable to rest there.) In general, if the equlibrium is a stable (or neutrally stable) one, then the pseudo-Goldstone bosons are absent; if the latter manifest themselves, then stability is lost. An observation regarding stability was made concerning the example of a charged rigid body toward the end of Subsection 3.1.1. It would be very interesting to see if such equilibrium analyses for the examples of Chapter 3, using the methods of invariant theory and symmetry breaking as above, would yield a similar simple stability criterion.

Chapter 6

Conclusions and future work

In conclusion, we note a few developments that have not been pursued at all, and could form the basis of future work. First, the geometry of G-structures is most generally examined using the formalism of fiber bundles. For example, in an appendix (Section 4.5), it was indicated how L-P and equivariant cohomologies could distinguish between regular and singular leaves of a foliation. The theory is formulated in terms of *classifying spaces* and characteristic classes. Symmetry breaking and moment reduction are likewise most geometrically seen as the reduction of a principal bundle, and the associated reduced dynamics examined in terms of reduced connections on the bundle [107, 108, 109]. Thus, it would prove useful to have a unified bundle description of the ideas that made up this thesis.

The second point that could use some development is the infinite dimensional version of the problem of rank change, which has not been addressed in this thesis. (The example in Section 3.3 was a reduction to a finite set of moments of an infinite dimensional system, so it does not qualify as a genuinely infinite dimensional illustration of rank change.) A full treatment of this problem would involve the use of the theory of *singular eigenfunctions*.

Since in the functional setting, rank change could happen at isolated points, the subcasimirs would need to be defined in the appropriate functional setting. One fruitful approach that may be taken to characterize rank change in the cosymplectic operator $\mathfrak{J}$, is through the use of *anti-selfadjoint index theory* [110]. The usual Atiyah-Singer theory is done for elliptic Fredholm operators over compact manifolds of even dimension. But there do exist variants of this theory for anti-selfadjoint operators; these aver that the kernel mod 2 is a topological invariant for the operators such as $\mathfrak{J}_0$, which is the tangent space linearization of the full, nonlinear $\mathfrak{J}$. In infinite dimensions, rank does not fall by even integers, as Casimirs are now functionals (see for instance the paragraph following (3.18)); rather the appearance of new subcasimirs might change the index by only 1, so a mod 2 invariant may be useful after all.

Lastly, we review what this thesis was about. Broadly speaking, the problem of rank change in dynamical systems is important for two reasons: systems tend to prefer equilibrium states that are more constrained (by the presence of a larger symmetry); and a full examination of stability of equilibria requires an assessment of subcasimirs that arise upon rank change. We began with a review of Poisson geometry in general, followed by examples that exhibit the rank change properties we wished to study. Ideas from orbit geometry and reduction were invoked next, with the underlying assumption that it is systems with some sort of symmetry that prove physically and geometrically interesting [111]. Invariant theory methods were employed to characterize both the strata of group actions and the invariants (Casimirs and subcasimirs) that

help parametrize them. Symmetry breaking ideas were introduced in order to connect the appearance of new Goldstone modes upon changing the orbit type (using control parameters present in an Hamiltonian), with the extra data provided by subcasimirs to pin down singular orbits.

Now, let us review the main result of the thesis. What we have shown is that the orbit geometries of two pictures coincide, and from this correspondence, derived a dynamical interpretation for a kinematical feature we set out to characterize. On the one hand we have a Poisson manifold whose characteristic distribution lines up with its group action orbits; and on the other hand, we have the orbital dependence of extrema of a symmetry breaking potential (with controls) on a linear vector space. These two very different problems have been connected via orbit geometry, using several results like Weinstein's linearization, Whitney equivariant embedding theorem, and invariant theory. It would be nice to demonstrate explicit formulas for subcasimirs coming from this sort of identification, and this we have not done. But we have provided a geometrical interpretation for the generating differentials of the embedding that makes a singular leaf sit inside a regular leaf, using the process of symmetry breaking and symmetry restoration. The virtue of having a mass matrix is its definiteness, which can distinguish between stable and unstable solutions, while within the Hamiltonian context, these solutions fall into the same class. Therefore, it is hoped that future investigations might yield a stability criterion using this picture, even if the explicit formulae for the subcasimirs could be more easily found by examining the Poisson structure directly.

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