“Oh, That.”

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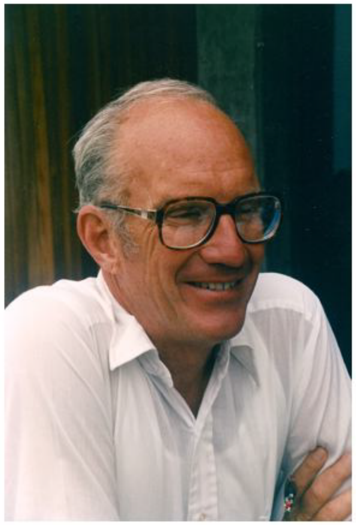
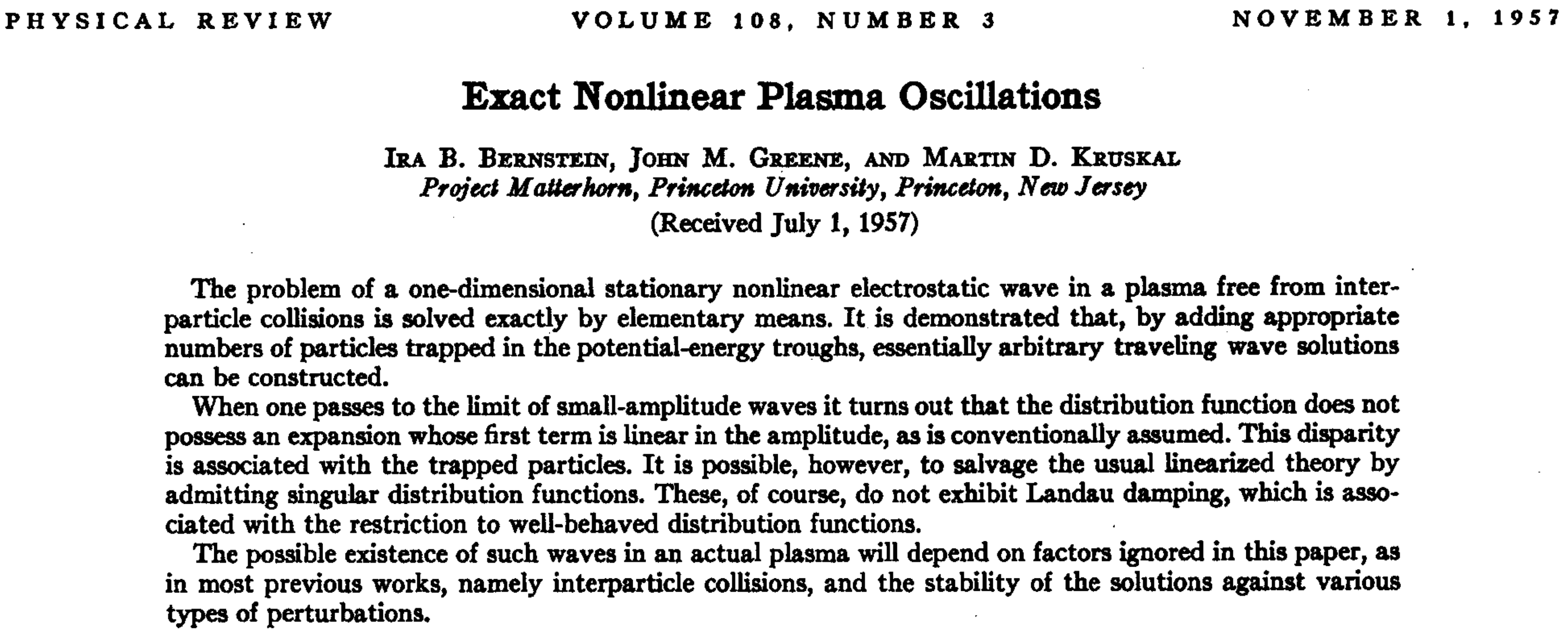
Overview

1. BGK Modes
2. Inverse Scattering
3. Torus Breakup
4. Hamiltonian MHD



John Greene, 1928 - 2007

“Higher-Order Corrections to the Nucleon-Nucleon Potential in Charge-Symmetric Pseudoscalar Theory”
University of Rochester 1956



PHYSICAL REVIEW VOLUME 108, NUMBER 3 NOVEMBER 1, 1957

Exact Nonlinear Plasma Oscillations

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(Received July 1, 1957)

The problem of a one-dimensional stationary nonlinear electrostatic wave in a plasma free from interparticle collisions is solved exactly by elementary means. It is demonstrated that, by adding appropriate numbers of particles trapped in the potential-energy troughs, essentially arbitrary traveling wave solutions can be constructed.

When one passes to the limit of small-amplitude waves it turns out that the distribution function does not possess an expansion whose first term is linear in the amplitude, as is conventionally assumed. This disparity is associated with the trapped particles. It is possible, however, to salvage the usual linearized theory by admitting singular distribution functions. These, of course, do not exhibit Landau damping, which is associated with the restriction to well-behaved distribution functions.

The possible existence of such waves in an actual plasma will depend on factors ignored in this paper, as in most previous works, namely interparticle collisions, and the stability of the solutions against various types of perturbations.

Harbinger: Localized and periodic solutions and Inverse problem

Vlasov Poisson equilibrium in some frame $\Longrightarrow$

$\boxed{\text{distribution function}=f(\mathcal{E})}$ where $\mathcal{E}=mv^2/2+e\phi$ $\Longrightarrow$

$\boxed{\phi_{xx}=-\dfrac{\partial V(\phi)}{\partial \phi}}$ $\longleftarrow$ pseudo-potential $\longrightarrow$ pulse, periodic, double layer, ... BUT $\rightarrow$

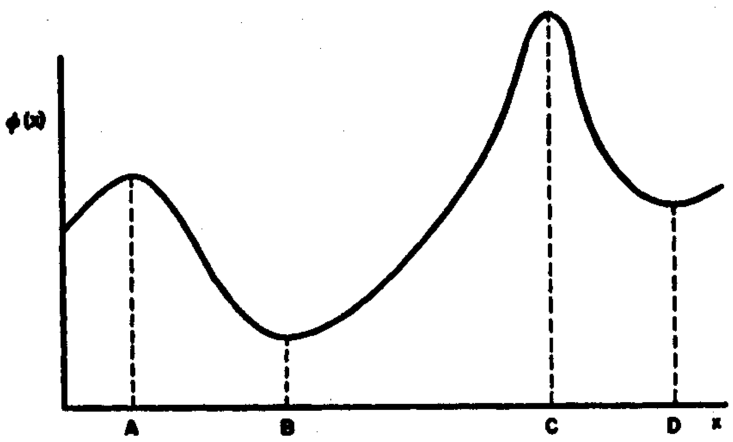
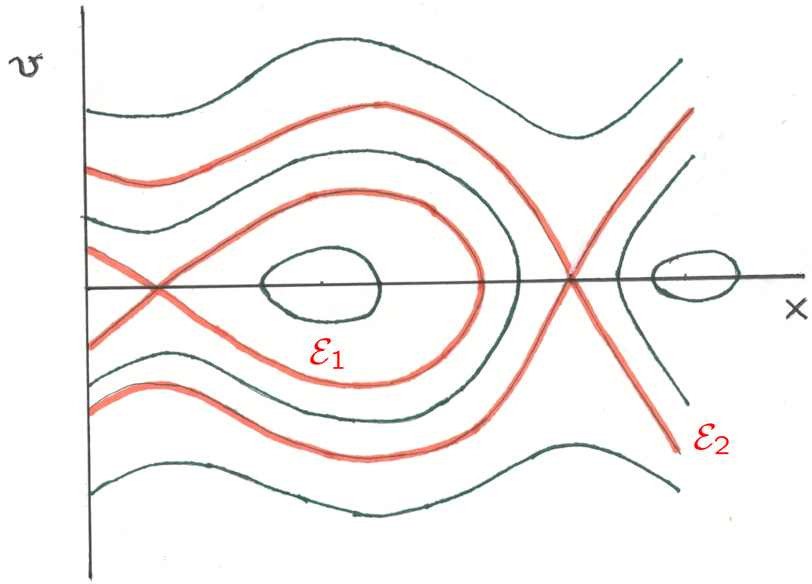


FIG. 2. Potential wave form.



self-consistency loop

$$\boxed{f(\mathcal{E}) \Rightarrow \phi(x) \Rightarrow f(x,v)}$$

$\phi \Rightarrow$ 'topology'

energy partition

$$\boxed{f = \sum_i f_i(\mathcal{E})\chi_i(\mathcal{E})}$$

$\chi_i = 1$ on region i, else 0

orbits with $v = 0$

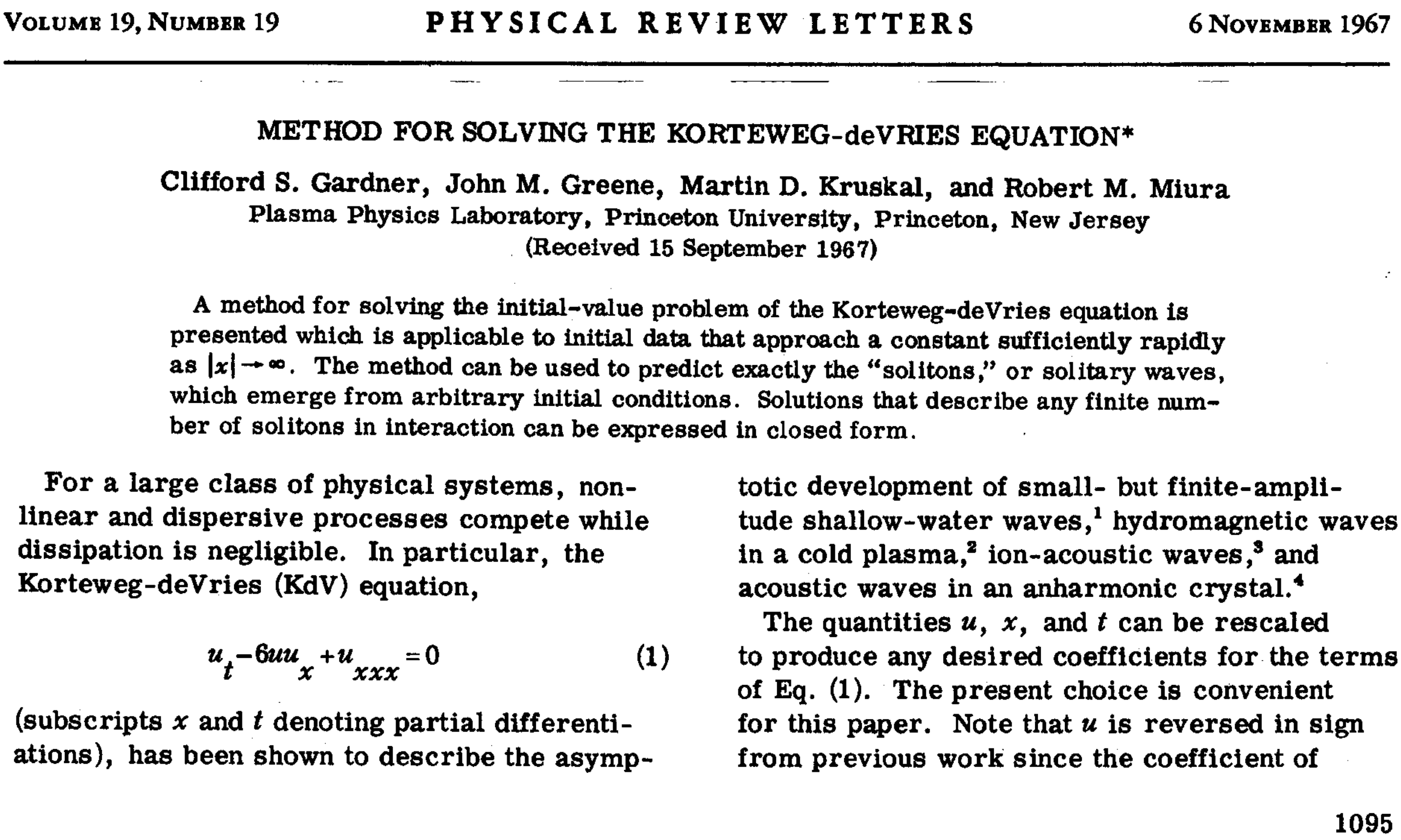
$$\boxed{f^+(\mathcal{E}) = f^-(\mathcal{E})}$$

generalized function sol

$$\boxed{f = f^+(\mathcal{E})H(v) + f^-(\mathcal{E})H(-v)}$$

inverse problem

$$\boxed{\text{choose } \phi \Rightarrow \text{trapped ptles}}$$



Volume 19, Number 19 PHYSICAL REVIEW LETTERS 6 November 1967

METHOD FOR SOLVING THE KORTEWEG-deVRIES EQUATION*

Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura
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(Received 15 September 1967)

A method for solving the initial-value problem of the Korteweg-deVries equation is presented which is applicable to initial data that approach a constant sufficiently rapidly as $|x| \to \infty$. The method can be used to predict exactly the "solitons," or solitary waves, which emerge from arbitrary initial conditions. Solutions that describe any finite number of solitons in interaction can be expressed in closed form.

For a large class of physical systems, nonlinear and dispersive processes compete while dissipation is negligible. In particular, the Korteweg-deVries (KdV) equation,

$$u_t - 6uu_x + u_{xxx} = 0 \qquad (1)$$

(subscripts x and t denoting partial differentiations), has been shown to describe the asymptotic development of small- but finite-amplitude shallow-water waves,[1] hydromagnetic waves in a cold plasma,[2] ion-acoustic waves,[3] and acoustic waves in an anharmonic crystal.[4]

The quantities u, x, and t can be rescaled to produce any desired coefficients for the terms of Eq. (1). The present choice is convenient for this paper. Note that u is reversed in sign from previous work since the coefficient of

Two Problems:

$$\boxed{u_t - 6uu_x + u_{xxx} = 0} \quad \longleftrightarrow \quad \boxed{\psi_{xx} + [\lambda(t) - u(x,t)]\psi = 0}$$

KdV Schrödinger

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XXVII, 97–133 (1974)

Korteweg-deVries Equation and Generalizations. VI. Methods for Exact Solution*

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1. Introduction

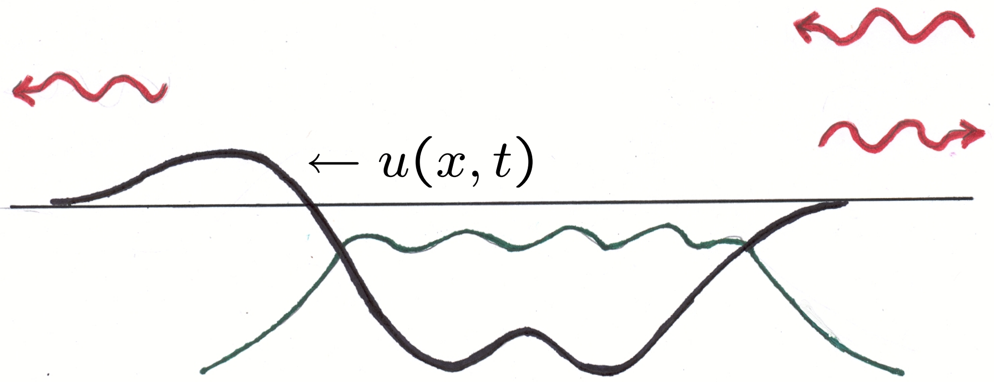
In a previous paper [3], we announced a method for exact solution of the Korteweg-deVries equation (KdV for short)

$$u_t + uu_x + u_{xxx} = 0 . \tag{1.1}$$

This is a nonlinear partial differential equation that describes the long-time evolution of small-amplitude long-wavelength dispersive (nondissipative) waves. Here we give more details of that method and present further results on properties of the solutions.

The equation (in versions trivially equivalent under simple transformation,

Inverse Scattering Problem



$\psi \sim Te^{-ikx}$

$\leftarrow u(x,t)$

$\psi \sim e^{-ikx} + Re^{ikx}$

$\psi_n \sim c_n(t)e^{-\kappa_n x}$

Remarkable Facts

- $\kappa_n(t) = \sqrt{-\lambda_n(t)} = \kappa_n(0)$
- $c_n(t) = c_n(0)e^{4\kappa_n^3 t}$
- $R(k,t) = R(k,0)e^{8ik^3 t}$

$|T|^2 + |R|^2 = 1$

- $T(k,t) = T(k,0)$

Solution

$u(x,0) \Rightarrow$ scattering data $\{\kappa_n(t), c_n(t), T(k,t), R(k,t)\} \Rightarrow B(x,t)$

Gel'fand-Levitan equation

$$\boxed{K(x,y,t) + B(x+y,t) + \int_x^\infty K(x,z,t)B(y+z,t)dz = 0}$$

solution of KdV

$$\boxed{u(x,t) = -2\frac{dK(x,x,t)}{dx}}$$

Whence??

Quotes

- "You are trying to solve the inverse scattering problem."
- "It unfolded like a lily."

A method for determining a stochastic transition

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(Received 6 November 1978)

A number of problems in physics can be reduced to the study of a measure-preserving mapping of a plane onto itself. One example is a Hamiltonian system with two degrees of freedom, i.e., two coupled nonlinear oscillators. These are among the simplest deterministic systems that can have chaotic solutions. According to a theorem of Kolmogorov, Arnol'd, and Moser, these systems may also have more ordered orbits lying on curves that divide the plane. The existence of each of these orbit types depends sensitively on both the parameters of the problem,, and on the initial conditions. The problem addressed in this paper is that of finding when given KAM orbits exist. The guiding hypothesis is that the disappearance of a KAM surface is associated with a sudden change from stability to instability of nearby periodic orbits. The relation between KAM surfaces and periodic orbits has been explored extensively here by the numerical computation of a particular mapping. An important part of this procedure is the introduction of two quantities, the residue and the mean residue, that permit the stability of many orbits to be estimated from the extrapolation of results obtained for a few orbits. The results are distilled into a series of assertions. These are consistent with all that is previously known, strongly supported by numerical results, and lead to a method for deciding the existence of any given KAM surface computationally.

0022-2488/79/061183-19$01.00

Torus Breakup: The Basic 2 DoF Hamiltonian Dynamics Result

$\exists$ Action-Angle Variables: $H(q_1, q_2, p_1, p_2) \rightarrow H(J_1, J_2)$

ϕ_1, ϕ_2 ignorable $\Rightarrow$ foliation by tori

e.g. field lines of tokamak equilibrium

$\nexists$ Action-Angle Variables: (broken tori almost always the case)

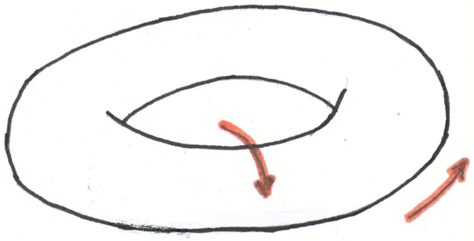
- KAM theorem → applies near integrable
- JMG residue theory → applies near and far from integrable

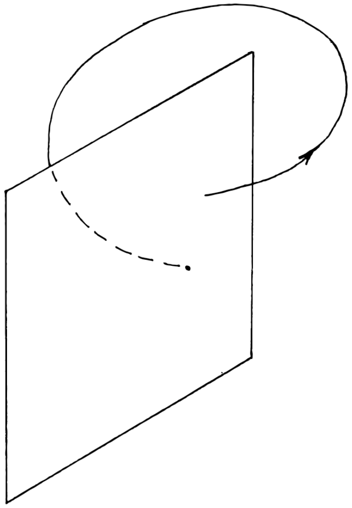
Physically more important than KAM (last barrier to transport).
Conceptually more interesting (describes local behavior).

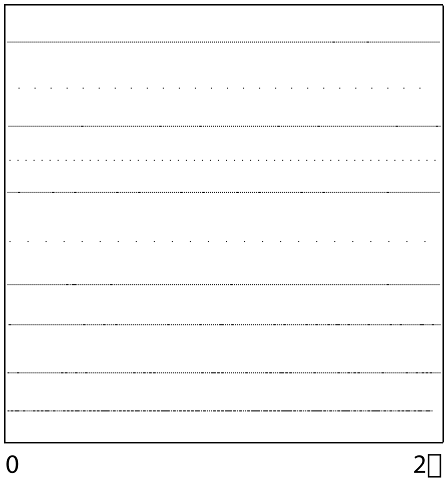
1952 (Kruskal) ⟶ 1968 (JMG) ⟶ 1979 (JMG) ⟶

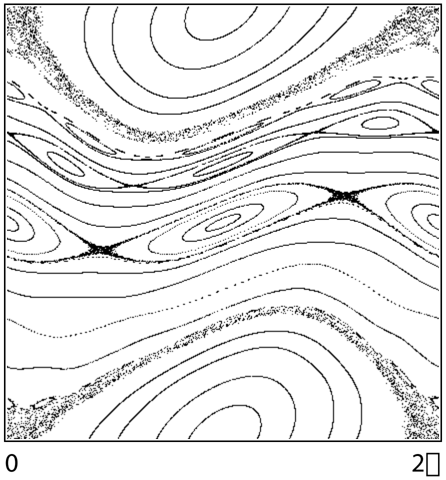
MacKay (renormalization), del-Castillo-Negrete, PJM (nontwist: resiliency of reverse shear torus) ...

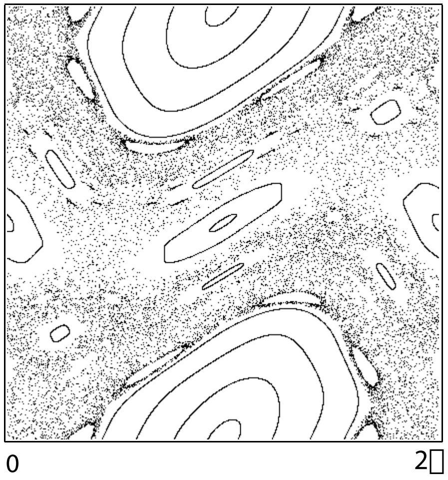
Surface of Section











$k \longrightarrow$

Symplectic Maps

Standard (Twist) Map:

$$\begin{aligned} x' &= x + y' \\ y' &= y - \frac{k}{2\pi}\sin(2\pi x) \end{aligned}$$

Standard Nontwist Map:

$$\begin{aligned} x' &= x + a(1 - y'^2) \\ y' &= y - b\sin(2\pi x) \end{aligned}$$

Parameters:

a measures shear, b, k ripple

Shearless Curve:

$$\text{for } b = 0\,, \qquad \frac{\partial x'}{\partial y} - -2ay' = 0 \Longrightarrow y = 0$$

John's Idea

The sudden change from stability to instability of scads of nearby periodic orbits signals the breakup of invariant tori.

John's Example

For which k-value of standard map is the (last) torus with *rotation number* $\omega^* = 1/\gamma$, the inverse golden mean, critical?

Rotation Number:

$\omega := lim_{n\to\infty}\frac{x_n}{n}$ lifted to $I\!\!R$ $q-$profile $\sim \omega^{-1}$

Extensions of John's method:

$\omega^* :=$ quadratic irrational, e.g. $1/\gamma, 1/\gamma^2$ periodic cf expansion

Shearless curve of standard nontwist map:

For $b = 0$ obvious, for $b \neq 0$

Greene's Method

1. Approxmate invariant torus (far from KAM limit) by sequence of periodic orbits with rotation numbers

$$\omega_i = \frac{n_i}{m_i}, \qquad\qquad n_i, m_i \in \mathcal{Z}$$

such that

$$\lim_{i\to\infty} \omega_i = \omega^*$$

Example: $\gamma =$ golden mean $= (\sqrt{5}+1)/2$ $1/\gamma = [0, 1, 1, 1 \ldots]$ with convergents

$$a_i = \frac{F_i}{F_{i+1}}, \qquad\qquad F_i, F_{i+1} \in Z$$

where $F_i =$ Fibonacci numbers, which are truncations of continued fraction expansion

Higher and higher order ⟶

looks more and more like invariant torus

Greene's Method (Continued)

2. Calculate Residues

$$R := \frac{1}{4}\left[2 - \text{trace} DT^n\right]$$

for sequence of periodic orbits and consider

$$\lim R_i = \begin{cases} 0 & \text{torus exists} \\ \infty & \text{torus does not exist} \\ R_c \sim .25 & \text{torus critical} \end{cases}$$

For the standard map Greene calculated

$$k_c = .971635\ldots$$

for criticality of the $1/\gamma$ torus, the last torus.

How did he do it?

Used involution decomposition to obtain periodic orbits $\sim 10^6$.

Involution Decomposition

Birkhoff, de Voglaere, Greene

Discrete Symmetries (e.g. time reversal) $\Longrightarrow$

$$T = I_1 \circ I_2 \,,$$

where

$$I_1 \circ I_1 = I_2 \circ I_2 = \text{identity map}$$

Reduces 2-dimensional root search to a 1-dimensional search along *symmetry sets*.

- Enables one to obtain periodic orbits of order 10^8 with 13 place accuracy!

- Problem: for nontwist systems, periodic orbits do not exist for all convergents!

Nontwist (Shearless) Results

- We first found (a, b) for $1/\gamma$ by 'intelligent search' such that residue limit to a period-6 cycle at criticality

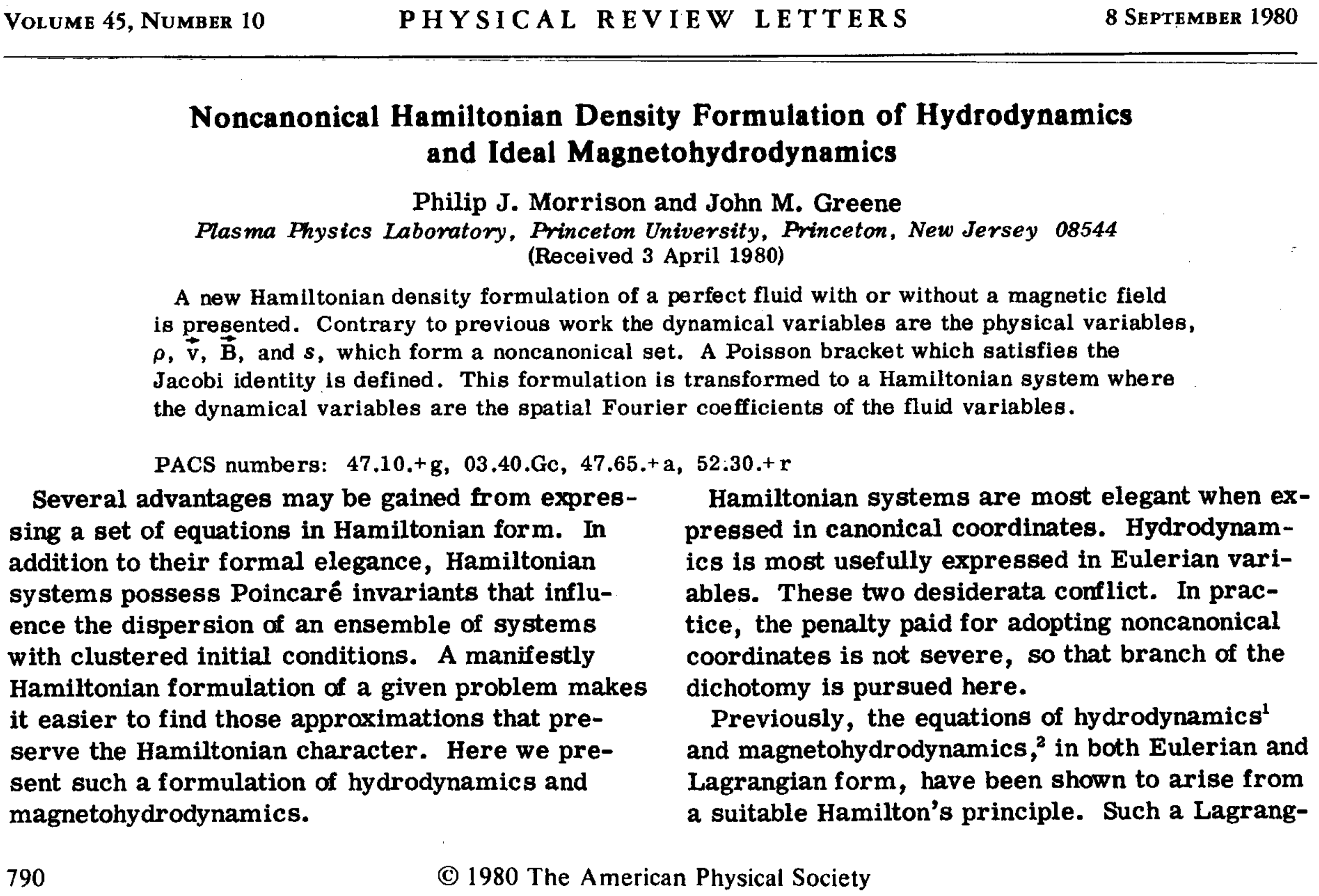
$$\{R_1^*, R_2^*, \ldots, R_6^*\}$$

That is, there exist six convergent subsequences.

Doing this for periodic orbits of order 10^6

$$a \approx 0.686049 \qquad b \approx 0.742497002412$$

- Subsequently, many results related to other shearless tori. Recently for $[2, 2, 2, 2, ...]$ a new kind of breakup.



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Noncanonical Hamiltonian Density Formulation of Hydrodynamics and Ideal Magnetohydrodynamics

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A new Hamiltonian density formulation of a perfect fluid with or without a magnetic field is presented. Contrary to previous work the dynamical variables are the physical variables, ρ, $\vec{v}$, $\vec{B}$, and s, which form a noncanonical set. A Poisson bracket which satisfies the Jacobi identity is defined. This formulation is transformed to a Hamiltonian system where the dynamical variables are the spatial Fourier coefficients of the fluid variables.

PACS numbers: 47.10.+g, 03.40.Gc, 47.65.+a, 52.30.+r

Several advantages may be gained from expressing a set of equations in Hamiltonian form. In addition to their formal elegance, Hamiltonian systems possess Poincaré invariants that influence the dispersion of an ensemble of systems with clustered initial conditions. A manifestly Hamiltonian formulation of a given problem makes it easier to find those approximations that preserve the Hamiltonian character. Here we present such a formulation of hydrodynamics and magnetohydrodynamics.

Hamiltonian systems are most elegant when expressed in canonical coordinates. Hydrodynamics is most usefully expressed in Eulerian variables. These two desiderata conflict. In practice, the penalty paid for adopting noncanonical coordinates is not severe, so that branch of the dichotomy is pursued here.

Previously, the equations of hydrodynamics[1] and magnetohydrodynamics,[2] in both Eulerian and Lagrangian form, have been shown to arise from a suitable Hamilton's principle. Such a Lagrang-

Basic Idea: Plasma models in Eulerian variables, noncanoncal variables, are Hamiltonian field theories in tems of noncanonical Poisson brackets linear in field variables.


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