Lecture # 9

Straight Pinches

Let us consider how a magnetic field holds in a plasma. We will describe the plasma by an isotropic pressure, that satisfies the fluid equation

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V} = -\nabla P + \mathbf{j}\times\mathbf{B}\left(\frac{1}{c}\right)$$

The plasma is quasi-neutral so $\rho \mathbf{E} = 0$ $(\rho = 0)$.

When there is no flow, $\mathbf{V} = 0$ and equilibrium condition is

$$\nabla P = \mathbf{j} \times \mathbf{B}$$

Further:

$$\mathbf{j} = \mu_0 \nabla \times \mathbf{B} \; ; \quad \therefore \quad \nabla\cdot\mathbf{j} = 0$$

$$\left(\mu_0 = \frac{4\pi}{c}, \; 1, \; \ldots\right)$$

$$\underset{\sim}{\nabla} P = \underset{\sim}{j} \times \underset{\sim}{B}$$

$$= (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B}$$

$$= -\underset{\sim}{\nabla} \frac{B^2}{2} + \underset{\sim}{\nabla} \cdot \underset{\sim}{B}\,\underset{\sim}{B} \qquad (\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0)$$

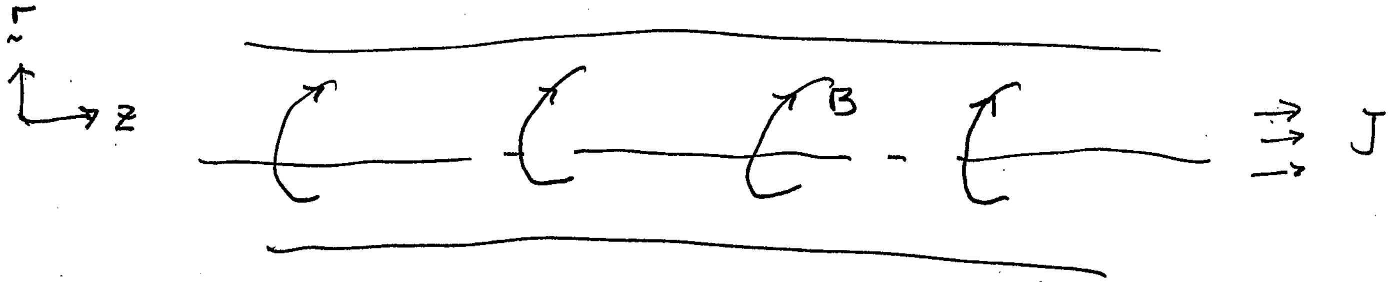
$$\underset{\sim}{\nabla} \left(P + \frac{B^2}{2} \right) = \underset{\sim}{b}\,\underset{\sim}{b} \cdot \underset{\sim}{\nabla} \frac{B^2}{2} + \underset{\sim}{\kappa} B^2 \qquad (\underset{\sim}{\kappa} = \underset{\sim}{b} \cdot \nabla \underset{\sim}{b})$$

$$\underset{\sim}{\nabla}_\perp \left(P + \frac{B^2}{2} \right) = \underset{\sim}{\kappa} B^2$$

$$\underset{\sim}{b} \cdot \underset{\sim}{\nabla} P = 0$$

In ideal MHD, pressure uniform along field lines

In 2-D it is straight-forward to see that magnetic fields will confine plasma



Z - Pinch , $\underset{\sim}{\kappa} = -\frac{\hat{r}}{r}$

$$\frac{\partial}{\partial r}\left(p + \frac{B^2}{2}\right) = -\frac{B^2}{r} \quad ; \quad B = B_\theta$$

$$\therefore \quad \frac{\partial}{\partial r}\left(\frac{B^2}{2}\right) + \frac{B^2}{r} = -\frac{\partial p}{\partial r}$$

$$\frac{\partial}{\partial r}\left(B^2 r^2\right) = -2 r^2 \frac{\partial p}{\partial r}$$

$$\boxed{B^2 = \frac{-2}{r^2} \int_0^r dr' \, \frac{\partial p}{\partial r'}(r') \, r'^2}$$

Example: $p = p_0 \left(1 - r^2/a^2\right); \; r < a$

$$B^2(r) = p_0 \frac{r^2}{a^2} \quad ; \quad J_z = \frac{1}{r}\frac{\partial (B_\theta r)}{\partial r}$$

$$P = P_0\left(1 - \frac{r^2}{a^2}\right); \quad r < a; \quad P = 0 \quad (r > a)$$

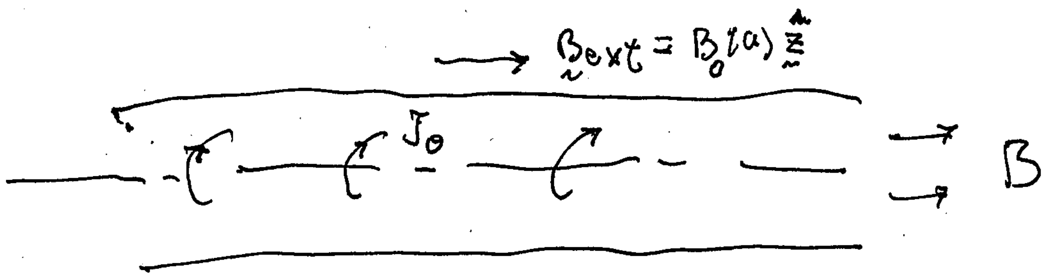
$$J_z = \frac{1}{r}\frac{\partial (B_\theta r)}{\partial r} = 2\sqrt{P_0}\,\frac{1}{a} \equiv \text{const} \quad (r < a)$$

$$\text{const} = \frac{I_0}{\pi a^2}$$

$$B_\theta = \frac{I_0}{2\pi r}, \qquad r > a$$

Another solution:

θ- pinch $\qquad \varkappa = 0$



$$\frac{\partial}{\partial r}\left(p + \frac{B^2}{2}\right) = 0$$

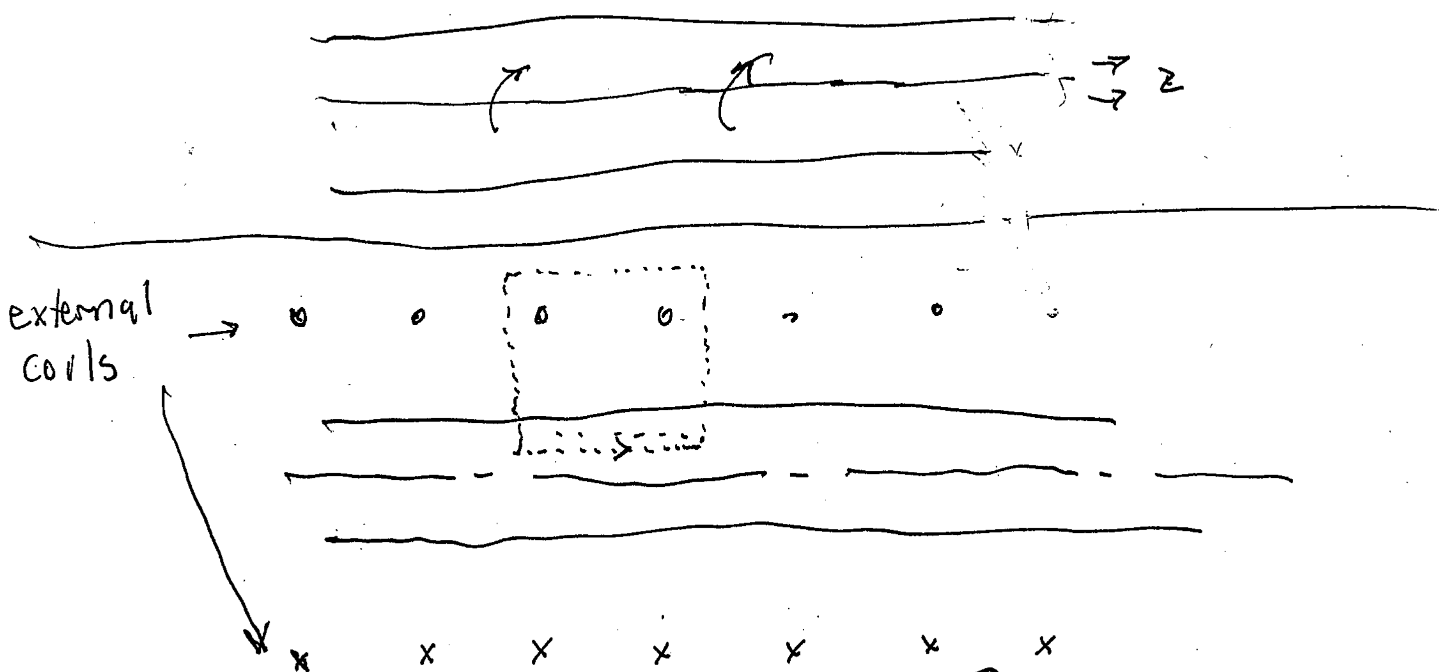
$$\frac{B^2}{2}(r) + p(r) = \frac{B^2(a)}{2} \qquad (\text{where } p = 0)$$

$$\beta \equiv \frac{2p(0)}{B^2(a)},$$

$B = B_0$ outside plasma.

Screw Pinch

Combination of θ & z pinch



$$\underset{\sim}{\kappa} = -\frac{B_\theta^2}{B^2 r}\,\underset{\sim}{\hat{r}}$$

$$\frac{\partial}{\partial r}\left(p + \frac{B_\theta^2}{2} + \frac{B_z^2}{2}\right) = + B^2 \kappa$$

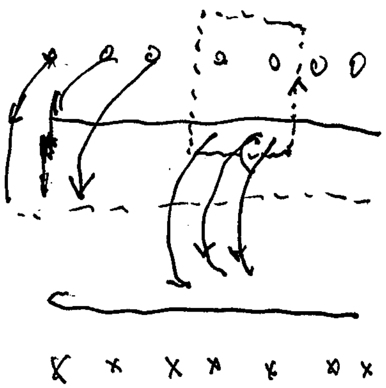
$$= -\frac{B_\theta^2}{r}$$

$$\underset{\sim}{B}_p = \underset{\sim}{\nabla} z \times \underset{\sim}{\nabla}\psi = B_\theta \hat{\theta}$$

$$\therefore \quad B_\theta = \frac{\partial \psi}{\partial r}$$

$$B_\theta \frac{\partial B_\theta}{\partial r} + \frac{B_\theta^2}{r} = -\left[\frac{\partial p}{\partial r} + B_z \frac{\partial B_z}{\partial r}\right]$$

Now $\quad B_z \hat{z} = I_\theta(\psi)\hat{z}$



$\equiv$ enclosed poloidal current in shown circuit

$$p = p(\psi)$$

$\therefore$ Divide by $B_\theta = \dfrac{\partial \psi}{\partial r}$

$$\frac{\partial}{\partial r}\frac{\partial \psi}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial r} = -\frac{\partial p}{B_\theta \partial r} - \frac{I_\theta(\psi)\,\partial I_\theta(\psi)}{B_\theta\, \partial r}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \psi}{\partial r} \equiv \nabla^2 \psi = -\frac{\partial p(\psi)}{\partial \psi} - I_\theta(\psi)\frac{\partial I_\theta(\psi)}{\partial \psi}$$

Grad-Shafranov Equation for a cylinder: Consequence of Force Balance

If $P(\psi)$ and $I_\theta(\psi) \equiv I(\psi)$ are specified function of ψ, a solution can be found

For example

$$\frac{\partial P}{\partial \psi} = \frac{\partial P}{\partial \psi}\bigg|_0 \quad \text{if} \quad \psi < \psi_0$$

$$I(\psi) = I_0\Big(1 + \beta(\psi - \psi_0)\Big), \quad \psi < \psi_0$$

$$= 0 \quad \psi > \psi_0$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \psi}{\partial r} = \frac{\partial P}{\partial \psi_0} + \beta I_0^2 (1 + \beta(\psi - \psi_0))$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \psi}{\partial r} + \beta I_0 \psi = -\frac{\partial P}{\partial \psi_0} + \beta I_0^2 (1 + \beta \psi_0)$$

$$\underset{\Lambda}{|||}$$

let $k^2 = \beta I_0$,

$$\psi = A/k^2 + \psi_1 J_0(kr)$$

Family of solutions can be found that will match $B_{zv} = I_0$ at plasma boundary $\psi(a) = \psi_0$, which would be

(6)

related to the total plasma current I_z

$$I_z = 2\pi a B_\theta(a) = 2\pi a \frac{\partial \psi(a)}{\partial r}$$

We can seek a solution specifying (assume $p(a) = 0$)

(1) B_{zV} , , a, $p(0)$

Note: There can be "force free" solution (i.e $p(0) \equiv 0$)

Current along field line then determined.

Note $B_{zV} = 0$, means no external coils neeeded to have confined z-d plasma.

These equilibrium still unbounded.

We can ask:

Is it possible to have a classically magnetically confined system in 3-d without external coils.

We can use virial theorem to prove this cannot be found

E&M + Plasma Stress Tensor $\underset{\approx}{T}$

$$\underset{\sim}{T} = \left(\frac{B^2}{2} + p\right) \underset{\approx}{I} - \underset{\sim}{B}\,\underset{\sim}{B}$$

$\nabla \cdot \underset{\approx}{T} = 0$ is just

$\underset{\sim}{j} \times \underset{\sim}{B} = \underset{\sim}{\nabla} P$, using: $\underset{\sim}{J} = \underset{\sim}{\nabla} \times \underset{\sim}{B}$

Now consider

$$\int d^3r\, \underset{\sim}{\nabla} \cdot (\underset{\approx}{T} \cdot \underset{\sim}{r}) = \int d\underset{\sim}{S} \cdot \underset{\approx}{T} \cdot \underset{\sim}{r}$$

$$\parallel$$

$$= \int d^3r \left[\underbrace{\underset{\sim}{\nabla} \cdot \underset{\approx}{T}}_{= 0}\, \underset{\sim}{r} + \underset{\approx}{T} : \underset{\approx}{I} \right]$$

$$= \int d^3r\ \mathrm{Trace}\,(T) =$$

$$= \int d^3r \left[3\left(p + \frac{B^2}{2}\right) - \underset{\sim}{B} \cdot \underset{\sim}{B} \right]$$

$$= \int d^3r \left[3p + \frac{B^2}{2} \right] > 0$$

$\therefore$

$$\int d^3r \left[\frac{3p}{2} + \frac{B^2}{2}\right] =$$

$$= \int dS \left[\left(\frac{B^2}{2} + p\right) \underset{\sim}{r} \cdot \underset{\sim}{n} - \underset{\sim}{B} \cdot \underset{\sim}{n}\, \underset{\sim}{r} \cdot \underset{\sim}{B}\right]$$

If system contained

$p \to 0 \quad B \propto \frac{1}{R^3}$

Then surface integral $S \sim R^2$

$$\sim R^2 \frac{1}{R^6} R \sim \frac{1}{R^3} \to 0$$

Thus we have impossibility

$\therefore$ a 3-d classical magnetic system needs external forces for confinement