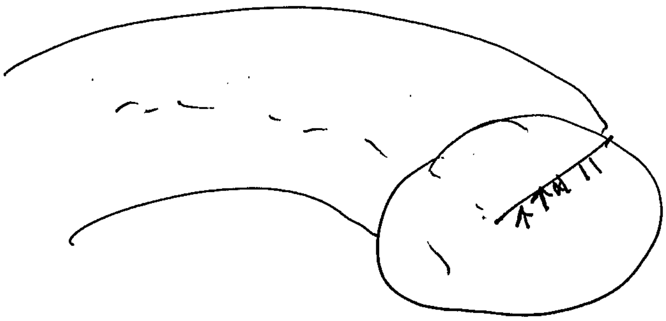
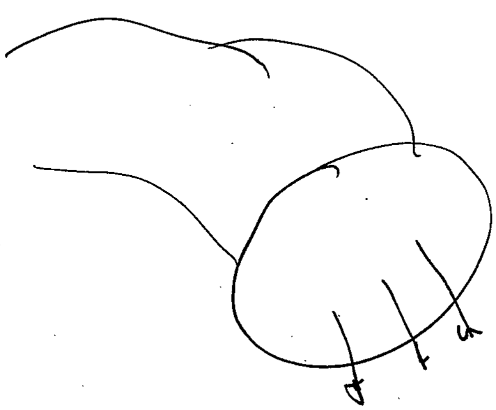
Demonstration that ψ_T and ψ_P are toroidal and poloidal fluxes

$$\underset{\sim}{B} = \underset{\sim}{\nabla}\psi_P \times \left[\underset{\sim}{\nabla}\left(\frac{d\psi_T}{d\psi_P}\theta - \zeta\right)\right]$$



$\Phi_P = \int \underset{\sim}{B}\cdot d\underset{\sim}{A} = 2\pi\,\psi_P$

$dA \propto d\psi_P\, d\zeta$



$\psi_T = \int \underset{\sim}{B}\cdot d\underset{\sim}{A} = 2\pi\,\Phi_T$

$dA \propto d\theta\, d\psi_P$

Take Poloidal Flux $(\mathcal{J} = \nabla\psi_T\cdot(\nabla\theta\times\nabla\zeta))$

$$d\underset{\sim}{A} = d\zeta(\nabla\psi_P\times\nabla\theta)\times d\psi_P(\underset{\sim}{\nabla}\theta\times\nabla\zeta)/\mathcal{J}^2$$

$$\underset{\sim}{B} = \nabla\psi_P\times\left[\frac{d\psi_T}{d\psi_P}\nabla\theta - \nabla\zeta\right]$$

$$d\underset{\sim}{A} = -d\zeta\, d\psi_P\, \mathcal{J}\,\underset{\sim}{\nabla}\theta$$

$$\int \underset{\sim}{B}\cdot d\underset{\sim}{A} = \int -(\underset{\sim}{\nabla}\psi_P\times\underset{\sim}{\nabla}\zeta)\cdot\underset{\sim}{\nabla}\theta\, d\psi_P'\, d\zeta\, \mathcal{J}$$

$$= \int_0^{\psi_P} 2\pi\, d\psi_P' = 2\pi\,\psi_P$$

Similiarly for toroidal flux

$$\int \underset{\sim}{B} \cdot d\underset{\sim}{A} \Rightarrow \int_0^{\psi_p} \frac{d\psi_T}{d\psi_p}\, d\theta\, d\psi_p = 2\pi\, \psi_T$$

In Flux coordinates

$$\frac{B^\zeta}{B^\theta} = q(\psi) \text{ , independent of}$$

θ, ζ

$$B^\zeta = \underset{\sim}{\nabla}\zeta \cdot \underset{\sim}{B} = \underset{\approx}{\nabla}\zeta \cdot \left[\underset{\sim}{\nabla}\psi_p \times \left(\nabla(q(\psi_p)\theta) - \nabla\zeta\right)\right]$$

$$= q(\psi_p)\ \underset{\sim}{\nabla}\zeta \cdot \underset{\sim}{\nabla}\psi_p \times \underset{\sim}{\nabla}\theta$$

$$B^\theta = \underset{\sim}{\nabla}\theta \cdot \underset{\sim}{B} = \underset{\sim}{\nabla}\theta \cdot \left[\underset{\sim}{\nabla}\psi_p \times \left(\nabla\left(q\theta - \zeta\right)\right)\right]$$

$$= -\underset{\sim}{\nabla}\theta \cdot \left(\underset{\sim}{\nabla}\psi_p \times \underset{\sim}{\nabla}\zeta\right)$$

$$\boxed{\therefore\ \frac{B^\zeta}{B^\theta} = q(\psi_p)}$$

An example of nice property of magnetic coordinates

$$\underset{\sim}{B} = \underset{\sim}{\nabla}\psi_T \times \underset{\sim}{\nabla}\theta - \underset{\sim}{\nabla}\psi_p \times \underset{\sim}{\nabla}\varsigma$$

Now take ψ_T, θ, ς as coordinates, and $\psi_p = \psi_p(\psi_T, \theta, \varsigma)$

We will see ψ_p is ~~the~~ the Hamiltonian of a two degree of freedom system with ς the "time", and ψ_T and θ conjugate coordinates

Along a field line we have

$$d\underset{\sim}{r} = d\psi_T \frac{\partial \underset{\sim}{r}}{\partial \psi_T} + d\varsigma \frac{\partial \underset{\sim}{r}}{\partial \varsigma} + d\theta \frac{\partial \underset{\sim}{r}}{\partial \theta} = d\lambda \underset{\sim}{B}$$

Take dot product with $\underset{\sim}{\nabla}\psi_T, \underset{\sim}{\nabla}\varsigma, \underset{\sim}{\nabla}\theta$ respectively

$$d\psi_T = d\lambda \underset{\sim}{B}\cdot\nabla\psi_T, \quad d\varsigma = d\lambda \underset{\sim}{B}\cdot\underset{\sim}{\nabla}\varsigma, \quad d\theta = d\lambda \underset{\sim}{B}\cdot\nabla\theta$$

$$\therefore \quad \frac{d\psi_T}{d\varsigma} = \frac{d\lambda \underset{\sim}{B}\cdot\nabla\psi_T}{d\lambda \underset{\sim}{B}\cdot\underset{\sim}{\nabla}\varsigma} = \frac{-\underset{\sim}{\nabla}\psi_p \times \underset{\sim}{\nabla}\varsigma\cdot\underset{\sim}{\nabla}\psi_T}{(\underset{\sim}{\nabla}\psi_T \times \underset{\sim}{\nabla}\theta)\cdot\underset{\sim}{\nabla}\varsigma} =$$

$$= \frac{\underset{\sim}{\nabla}\psi_T \times \underset{\sim}{\nabla}\varsigma\cdot\left[\frac{\partial\psi_p}{\partial\psi_T}\underset{\sim}{\nabla}\psi_T + \frac{\partial\psi_p}{\partial\theta}\underset{\sim}{\nabla}\theta + \frac{\partial\psi_p}{\partial\varsigma}\underset{\sim}{\nabla}\varsigma\right]}{(\underset{\sim}{\nabla}\psi_T \times \underset{\sim}{\nabla}\theta)\cdot\nabla\varsigma} = -\frac{\partial\psi_p}{\partial\theta}$$

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Similarly, we find

$$\frac{d\theta}{d\rho} = \frac{\partial \psi_p(\psi_T, \theta, \rho)}{\partial \psi_T}$$

This type of system is often used in calculations.

$$\frac{d\psi_T}{d\rho} = -\frac{\partial \psi_p}{\partial \theta} \; ; \quad \cancel{d\psi_T}\, \frac{d\theta}{d\rho} = \frac{\partial \psi_p}{\partial \psi_T}$$

$\theta \Rightarrow$ coordinate

$\psi_T \Rightarrow$ momentum

$\rho \Rightarrow$ time

If ψ_p is a flux surface

$$\psi_p = \psi_p(\psi_T)$$

$$\frac{d\theta}{d\zeta} = \frac{\partial \psi_p}{\partial \psi_T} = \frac{1}{q}, \quad \frac{d\psi_T}{d\zeta} = -\frac{\partial \psi_p}{\partial \theta} = 0$$

we will discuss other cases later

The choice of suitable magnetic coordinates is quite flexible. In a tokamak, when there is magnetic symmetry

One can choose the geometric angle ζ as one coordinate ($\nabla \zeta = \frac{1}{R}$), or others, as long as the field line label $q\theta - \zeta = q\theta' - \zeta'$ is preserved, where i.e

$$\theta' = \theta + K(\psi_p, \theta, \zeta)/q$$

$$\zeta' = \zeta + K(\psi_p, \theta, \zeta)$$

K is periodic in θ and ζ.

Hamada Coordinates

$$dV = \frac{d\psi_p\, d\theta\, d\zeta}{J} = \frac{d\psi_p\, d\theta\, d\zeta}{\nabla\psi_p \cdot \nabla\theta \times \nabla\zeta}$$

We have the freedom to choose θ and ζ so that J is independent of θ and ζ.

Then

$$B^\zeta = \nabla\zeta \cdot \underset{\sim}{B} = \nabla\zeta \cdot \nabla\psi_p \times \nabla\theta$$

$$B^\theta = \nabla\theta \cdot \underset{\sim}{B} = -\nabla\theta \cdot (\nabla\psi_p \times \nabla\zeta) = \nabla\zeta \cdot \nabla\psi_p \times \nabla\theta$$

are constant on a flux surface (a convenient property)

A frequent manipulation we have is "flux surface average" which is particular convenient in Hamada coordinates

Flux Surface Average

Consider a volume integral of a quantity $A(\underline{r})$ within an enclosed flux volume

$$\int_{\psi_p} d^3r\, A(r) = \int^{\psi_p} d\psi_p'\, d\theta\, d\varsigma\, A(\psi_p, \theta, \varsigma)/\mathcal{J}$$

$$\langle A\rangle_{\psi_p} = \frac{\partial}{\partial V}\int_{\psi_p} d^3r A = \int \frac{d\theta\, d\varsigma\, A(\psi_p, \theta, \varsigma)}{\mathcal{J}(\theta \varsigma \psi_p)}\frac{\partial \psi_p}{\partial V}$$

$$= \frac{\int d\theta\, d\varsigma\, A/\mathcal{J}}{\int d\theta\, d\varsigma/\mathcal{J}} = \langle A\rangle_{\psi_p} = \frac{\int d\theta\, d\varsigma\, A(\psi_p, \theta, \varsigma)/\mathcal{J}_H(\psi_p)}{\int d\theta\, d\varsigma/\mathcal{J}_H(\psi_p)}$$

where

$$\frac{\partial V}{\partial \psi_p} = \int d\theta\, d\varsigma/\mathcal{J}$$

~~The spatial volume enclosed in ψ_p is~~

$$\cancel{V = \int \frac{d\psi_p\, d\varsigma\, d\theta}{\mathcal{J}_H} = (2\pi)^2 \int \frac{d\psi_p}{\mathcal{J}_H}}$$

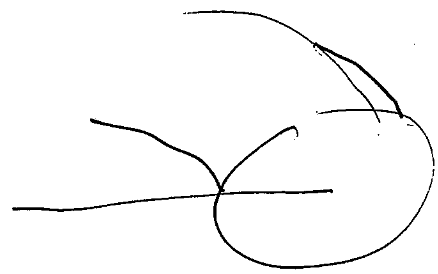
$$\cancel{\frac{dV}{d\psi_p} = \frac{2\pi^2}{\mathcal{J}(\psi_p)}}$$

$$\therefore \langle A\rangle_p = \int \frac{d\theta\, d\varsigma}{(2\pi)^2} A(\psi_p, \theta, \varsigma)$$

Symmetry Coordinates

We can choose for ρ the geometrical coordinate where

$$|\underset{\sim}{\nabla} \rho|^2 = \frac{1}{R^2}$$



In this case

$$g^{\rho\rho} = \underset{\sim}{\nabla}\rho \cdot \underset{\sim}{\nabla}\rho = \frac{1}{R^2}$$

$$g_{\rho\rho} = \frac{\partial \underset{\sim}{r}}{\partial \rho} \cdot \frac{\partial \underset{\sim}{r}}{\partial \rho} = \frac{\partial (R\hat{r})}{\partial \rho} \cdot \frac{\partial (R\hat{r})}{\partial \rho} = R^2$$

The only non-diagonal components at

$$\underset{\sim}{\nabla}\theta \cdot \underset{\sim}{\nabla}\psi_p = g^{21} = g^{12}$$

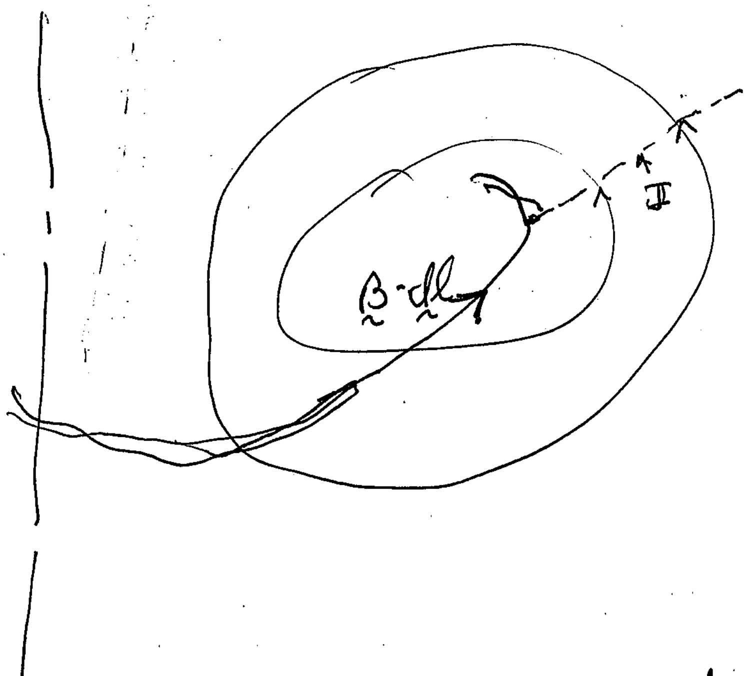
$$\frac{\partial \underset{\sim}{r}}{\partial \theta} \cdot \frac{\partial \underset{\sim}{r}}{\partial \psi_p} = g_{12} = g_{21}$$

We will work out a specofic case later

An important property for symmetric coordinates is that

$\mathcal{J} R^2$ independent of θ & ζ

Follow From Ampere's Law



$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J}$$

$\therefore$

$$\int \underset{\sim}{B}\cdot d\underset{\sim}{r} = \mu_0 \, I(\psi_p)$$

$I(r) \equiv$ ~~total~~ poloidal current outside enclosed flux volume

$$\underset{\sim}{B}\cdot d\underset{\sim}{r} = B_i \, dr^i = B_3 \, d\zeta \quad \text{(in our case)}$$

$$\oint B\cdot dr = 2\pi B_3 = \mu_0 I(\psi_p)$$

$$B_3 = \frac{\mu_0 I(\psi_p)}{2\pi} = B_\zeta R$$

$$B^3 = g^{33} B_3 = \frac{B_3}{R^2} = \mu_0 \frac{I(\psi_p)}{2\pi R^2}$$

But

$$\underset{\sim}{B}^3 = \underset{\sim}{\nabla}\rho \cdot \underset{\sim}{B}$$

$$= \underset{\sim}{\nabla}\rho \cdot \left[\underset{\sim}{\nabla}\psi_T \times (\underset{\sim}{\nabla} q\theta - \nabla\zeta)\right]$$

$$= q\mathcal{J} \equiv \frac{\mu_0 I(\psi_p)}{2\pi R^2}$$

$$\therefore \quad \mathcal{J} = \frac{\mu_0 I(\psi_p)}{2\pi q(\psi_p) R^2}$$

Thus: we will find

$$\langle A \rangle_S = \frac{\int d\theta\, d\zeta\, R^2(\theta)\, A(\psi_p, \theta, \zeta)}{2\pi \int d\theta\, R^2}$$

Relatively simple metric weighting