Demonstration that $\Phi_T$ and $\Phi_p$ are toroidal and poloidal fluxes

$$B = \nabla \Phi_p \times \left[ \nabla \left( \frac{d\Phi_T}{d\Phi_p} \right) - \nabla \Phi \right]$$

$$\Phi_T = \int_B \cdot dA = 2\pi \Phi_T$$
$$dA \propto d\Phi \cdot d\phi$$

Take Poloidal Flux ($J = \Phi_T \cdot (\nabla \times d\phi)$)

$$dA = d\phi (\nabla \Phi_p \times \nabla \phi) \times d\phi (\nabla \Phi \times d\phi) / J$$

$$B = \nabla \Phi_p \times \left[ \frac{d\Phi_T}{d\Phi_p} \right. \nabla \phi - \nabla \Phi$$
$$dA = dS \cdot d\phi \cdot d\theta$$

$$\int B \cdot dA = \int \left( \nabla \Phi_p \times \nabla \phi \right) \cdot \nabla \Phi_T \cdot d\Phi_p \cdot d\phi \cdot d\theta$$

$$= \int 2\pi \Phi_p \cdot d\Phi_p = 2\pi \Phi_p$$
Similarly for toroidal flux

\[ S \mathbf{B} \cdot d\mathbf{A} \Rightarrow \int \frac{d\mathbf{y}_T}{d\phi} d\phi d\rho = \mathbf{4}_T \]
In Fourier coordinates

\[ \mathbf{B}^g = \mathbf{q}(\psi), \text{ independent of } \mathbf{B}^0, \psi, \varphi \]

\[ \mathbf{B}^g = \nabla \varphi \cdot \mathbf{B} = \nabla \varphi \cdot \left[ \nabla \phi \times \left( \phi \left( \phi^t \mathbf{q} \right) \theta - \nabla \varphi \right) \right] \]

\[ = \mathbf{q}(\psi) \nabla \varphi \cdot \nabla \phi \times \nabla \theta \]

\[ \mathbf{B}^0 = \nabla \theta \cdot \mathbf{B} = \nabla \theta \cdot \left[ \nabla \phi \times \left( \phi^t \mathbf{q} \theta - \nabla \varphi \right) \right] \]

\[ = -\nabla \theta \cdot (\nabla \phi \times \nabla \varphi) \]

\[ \therefore \frac{\mathbf{B}^g}{\mathbf{B}^0} = \mathbf{q}(\psi) \]

An example of use

property of magnetic coordinates

\[ B = \nabla \psi \times \nabla \theta - \nabla \rho \times \nabla \phi \]

Now take \( \psi, \theta, \phi \) as coordinates, and \( \psi = \psi (\psi, \theta, \phi) \)

We will see \( \psi \) is the Hamiltonian of a two degree of freedom system with \( \phi \) the "time", and \( \psi, \theta \) and \( \phi \) conjugate coordinates.

Along a field line we have

\[ ds = d\psi \frac{\partial \psi}{\partial \psi} + d\theta \frac{\partial \psi}{\partial \theta} + d\phi \frac{\partial \psi}{\partial \phi} = d\lambda \cdot B \]

Take dot product with \( \nabla \psi, \nabla \theta, \nabla \phi \) respectively

\[ d\psi = d\lambda \cdot B \cdot \nabla \psi, \quad d\theta = d\lambda \cdot B \cdot \nabla \theta, \quad d\phi = d\lambda \cdot B \cdot \nabla \phi \]

\[ \frac{d\psi}{d\phi} = \frac{d\lambda \cdot B \cdot \nabla \psi}{d\lambda \cdot B \cdot \nabla \phi} = \frac{\nabla \psi \cdot \nabla \theta \cdot \nabla \phi}{(\nabla \psi \times \nabla \theta) \cdot \nabla \phi} \]

\[ = \frac{\nabla \psi \times \nabla \theta \cdot \nabla \phi}{(\nabla \psi \times \nabla \theta) \cdot \nabla \phi} \left[ \frac{\nabla \psi \cdot \nabla \phi}{\nabla \psi \cdot \nabla \theta} \theta + \frac{\nabla \psi \cdot \nabla \theta}{\nabla \psi \cdot \nabla \phi} \phi \right] \]

\[ = -\frac{\nabla \psi \cdot \nabla \theta \cdot \nabla \phi}{(\nabla \psi \times \nabla \theta) \cdot \nabla \phi} \]
Similarly, we find
\[ \frac{d\theta}{d\psi} = \frac{\partial \psi}{\partial \psi} \]

This type of system is often used in calculations.
\[ \frac{d\psi_T}{d\theta} = -\frac{\psi}{\partial \psi} ; \quad \frac{\partial \psi}{d\theta} = \frac{\partial \psi}{d\psi} \psi_T \]

\( \theta \) \rightarrow coordinate
\( \psi_T \) \rightarrow momentum
\( \theta \) \rightarrow time
If \( \psi_p \) is a flux surface

\[
\psi_p = \psi_p (\psi)
\]

\[
\frac{d\theta}{ds} = \frac{d\psi_p}{d\psi} = -\frac{1}{q}, \quad \frac{d\psi}{ds} = \frac{d\psi_p}{d\theta} = 0
\]

we will discuss other cases later.

The choice of suitable magnetic coordinates is quite flexible. In a tokamak, when there is magnetic symmetry, one can choose the geometric angle \( \phi \) as one coordinate (\( \phi = \frac{1}{R} \)), or others, as long as the field line label \( \varrho \Theta - \phi \equiv \varrho' \Theta' - \phi' \) is preserved, where i.e.

\[
\Theta' = \Theta + K(\psi_p, \Theta, \phi)/q
\]

\[
\phi' = \phi + K(\psi_p, \Theta, \phi)
\]

\( K \) is periodic in \( \Theta \) and \( \phi \).
Hamada Coordinates

\[ dV = \frac{d\theta \, d\phi \, ds}{\sqrt{\gamma \phi \cdot \gamma \phi}} \]

We have the freedom to choose \( \theta \) and \( \phi \) so that \( J \) is independent of \( \theta \) and \( \phi \).

Then

\[ B^\phi = \nabla \times B, \quad B = \nabla \times \gamma \phi \times \gamma \theta \]
\[ B^\theta = \nabla \phi \times \gamma \phi \times \gamma \theta \]

are constant on a flux surface (a convenient property).

A frequent manipulation we have is "flux surface average" which is particular convenient in Hamada coordinates.
Consider a volume integral of a quantity \( A(r) \) within an enclosed flux volume.

\[
\int_{\psi_0} d^3r \frac{A(r)}{\psi} = \frac{1}{2\pi} \int_0^{\psi_0} d\psi_0 d\theta d\phi A(\psi_0, \theta, \phi) \frac{\psi_0}{\psi}
\]

\[
\langle A \rangle_{\psi_0} = \frac{\int_{\psi_0} d^3r A(r)}{\int_{\psi_0} d^3r} = \frac{1}{2\pi} \int_0^{\psi_0} d\psi_0 d\theta d\phi A(\psi_0, \theta, \phi) \frac{\psi_0}{\psi}
\]

Where the spatial volume enclosed in \( \psi_0 \) is

\[
V = \int_{\psi_0} d\psi_0 d\theta d\phi = \frac{(2\pi)^2}{d\psi} \int d\psi_0
\]

\[
\frac{dV}{d\psi} = \frac{2\pi^2}{d\psi}
\]

\[
\langle A \rangle_{\psi_0} = \frac{\int_{\psi_0} d\theta d\phi A(\psi_0, \theta, \phi)}{(2\pi)^2}
\]
Symmetry Coordinates

We can choose for \( \Phi \) the geometrical coordinate where
\[
\int \frac{1}{\Delta \Phi} = \frac{1}{\beta^2}
\]

In this case
\[
\Phi \frac{\partial \Phi}{\partial \Phi} = \frac{\partial \Phi}{\partial \Phi} = \frac{1}{\beta^2}
\]
\[
\Phi = \frac{1}{\beta^2}
\]
\[
\frac{\partial \Phi}{\partial \Phi} = \frac{2}{\beta^2} \cdot \frac{\partial \Phi}{\partial \Phi} = \frac{2}{\beta^2} \cdot \frac{2}{\beta^2} = \beta^2
\]

The only non-diagonal components are
\[
\partial \Phi, \partial \Phi = \gamma \partial \Phi, \gamma \partial \Phi = \gamma \partial \Phi, \gamma \partial \Phi = \gamma \partial \Phi
\]

We will work out a specific case later.
An important property of symmetric coordinates is that
\[ J R^2 \text{ independent of } \theta + \phi \]

Follow from Ampere's Law

\[ \nabla \times \mathbf{B} = \mathbf{u}_0 \mathbf{I} \]

\[ \oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 I(\phi) \]

\( I(\phi) \) = poloidal current outside enclosed flux volume

\[ \mathbf{B} \cdot d\mathbf{r} = B_r \, dr \, d\phi = B_\phi \, d\phi \, d\psi \] (in our case)

\[ \oint \mathbf{B} \cdot d\mathbf{r} = 2\pi R B_\phi = \mu_0 I(\phi) \]

\[ B_\phi = \frac{\mu_0 I(\phi)}{2\pi} = B_\phi R \]

\[ B^3 = \frac{g^3}{2} B_\phi = \frac{B_\phi^2}{4\pi R^2} = \frac{\mu_0 I(\phi)}{2\pi R^2} \]
But
\[ B^3 = \nabla S \cdot B \]
\[ = \frac{\partial S}{\partial x} \int \left( \nabla \theta - \frac{\partial E}{\partial S} \right) \]
\[ = q \tilde{J} = \frac{\mu_0 I(\phi)}{2\pi R^2} \]
\[ \hat{s} \cdot \tilde{J} = \frac{\mu_0 I(\phi)}{2 \pi + q(\phi) R^2} \]

Thus: we will find

\[ \langle A \rangle_s = \frac{\int d\phi dS R^2 \lambda(\phi, \theta, \psi) A(\phi, \theta, \psi)}{2\pi \int d\phi R^2} \]

Relatively simple metric weighting