Lecture # 7

We have established for a cylinder, of period $L = 2\pi R_0$, where there is a "toroidal" field

$$B_z \hat{\underset{\sim}{z}} = \underset{\sim}{\nabla} \times A_\theta \hat{\theta} = \underset{\sim}{\nabla} \times r \underset{\sim}{A_\theta} \underset{\sim}{\nabla}\theta$$

$$= \underset{\sim}{\nabla}(\underset{\sim}{A_\theta}\, r) \times \underset{\sim}{\nabla}\theta$$

$$=$$

$$B_\theta \hat{\theta} = \nabla \times A_z \hat{\underset{\sim}{z}} = \nabla \times A_z R_0 \underset{\sim}{\nabla}\varphi$$

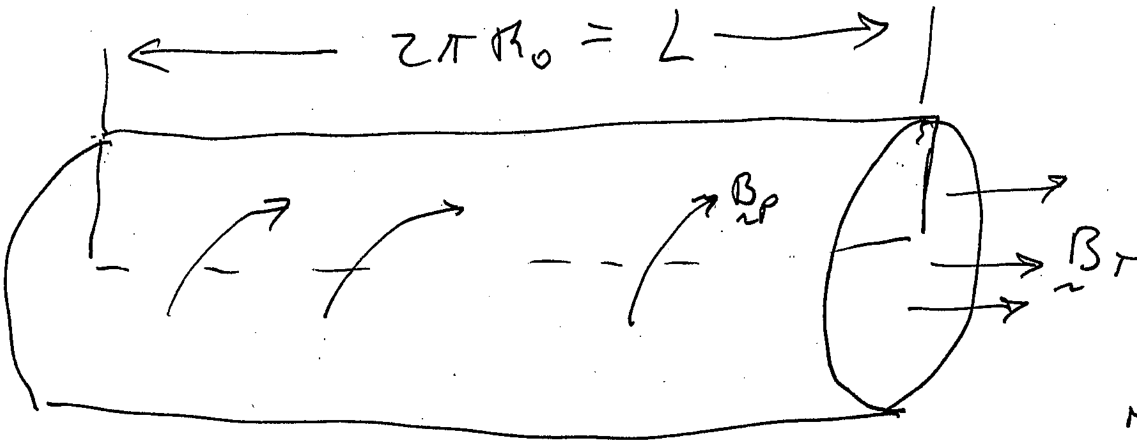
$$R_0 \varphi = z$$

One easily shows

$2\pi r A_\theta \equiv$ Enclosed toroidal flux within radius r $\equiv \Phi_T \equiv 2\pi \psi_T$

$2\pi R_0 A_z \equiv$ Enclosed poloidal flux within radius r

$\equiv \Phi_p \equiv 2\pi \psi_T$



$$\Psi_p = \int B_\theta \, dA = L\int_0^r B_\theta \, dr = 2\pi R_0 \int_0^r B_\theta \, dr$$

$$\Psi_T = \int_0^r \int dr\, r\, d\theta\, B_z = 2\pi \int_0^r dr\, r\, B_z$$

Magnetic field can be written as

$$\underset{\sim}{B} = \underset{\sim}{\nabla} \psi_T \times \underset{\sim}{\nabla}\theta - \underset{\sim}{\nabla}\psi_p \times \underset{\sim}{\nabla}\varphi$$

Also $\quad q(r) = \dfrac{d\psi_p(r)}{d\psi_T(r)}$:

$$\therefore \quad \nabla\psi_p = \frac{\partial \psi_p}{\partial \psi_T}\nabla\psi_T = q\,\nabla\psi_T$$

and we have

$$\underset{\sim}{B} = \nabla\psi_p \times \left(\nabla(\theta q - \phi)\right)$$

This is in fact a general form for $\underset{\sim}{B}$ with nested surfaces &

Generalized Coordinate System

Formed by intersection of three families of ~~cur~~ surfaces

$q^1(x^1, x^2, x^3) \equiv q^1(\underset{\sim}{r})$

$q^2(\underset{\sim}{r})$, $q^3(\underset{\sim}{r})$

Then

$$d\underset{\sim}{r} = \sum_i dq^i \frac{\partial \underset{\sim}{r}}{\partial q^i} \equiv dq^i \frac{\partial \underset{\sim}{r}}{\partial q^i}$$

and

$$dq^i = \frac{\partial q^i}{\partial \underset{\sim}{r}} \cdot d\underset{\sim}{r} = \frac{\partial q^i}{\partial \underset{\sim}{r}} \cdot \frac{\partial \underset{\sim}{r}}{\partial q^j}\, dq^j$$

And we have $\dfrac{\partial q^i}{\partial \underset{\sim}{r}} \cdot \dfrac{\partial \underset{\sim}{r}}{\partial q^j} = \delta^i_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

Now a vector, $\underset{\sim}{A}$, can be expressed in terms of covariant components $\underset{\sim}{A} = A_1 \underset{\sim}{\nabla} q^1 + A_2 \underset{\sim}{\nabla} q^2 + A_3 \nabla q^3$

or contravariant components

$$\underset{\sim}{A} = A^1 \frac{\partial \underset{\sim}{r}}{\partial q^1} + A^2 \frac{\partial \underset{\sim}{r}}{\partial q^2} + A^3 \frac{\partial \underset{\sim}{r}}{\partial q^3}$$

Then ~~we~~ note

$$A^i = \nabla q^i \cdot \mathbf{A}$$

which follows from the relation $\dfrac{\partial q^i}{\partial \mathbf{r}} \cdot \dfrac{\partial \mathbf{r}}{\partial q_j} = \delta^i_j$

Proof

$$\nabla q^i \cdot \left[\frac{\partial \mathbf{r}}{\partial q_1} A^1 + \frac{\partial \mathbf{r}}{\partial q_2} A^2 + \frac{\partial \mathbf{r}}{\partial q_3} A^3 \right]$$

$$= \nabla q^i \cdot \frac{\partial \mathbf{r}}{\partial q^k} A^k = \delta^i_k A^k = A^i$$

Similarly $A_i = \dfrac{\partial \mathbf{r}}{\partial q^i} \cdot \mathbf{A}$

Proof

$$A_i = \frac{\partial \mathbf{r}}{\partial q^i} \cdot \mathbf{A} = \frac{\partial \mathbf{r}}{\partial q^i} A_j \nabla q^j$$

$$\frac{\partial \mathbf{r}}{\partial q^i} \cdot \left[A_1 \nabla q^1 + A_2 \nabla q^2 + A_3 \nabla q^3 \right]$$

$$= \frac{\partial \mathbf{r}}{\partial q^i} \cdot A_j \frac{\partial q^j}{\partial \mathbf{r}} = A_j \delta^j_i = A_i$$

Further, we observe that

$$\frac{\partial \underset{\sim}{r}}{\partial q^1} = \frac{\underset{\sim}{\nabla} q^2 \times \underset{\sim}{\nabla} q^3}{\nabla q^1 \cdot \nabla q^2 \times \nabla q^3}$$

as we then fulfill the conditions:

$$\frac{\partial \underset{\sim}{r}}{\partial q^1} \cdot \underset{\sim}{\nabla} q^1 = 1$$

$$\frac{\partial \underset{\sim}{r}}{\partial q^1} \cdot \underset{\sim}{\nabla} q^{2,3} = 0$$

Similarly:

$$\frac{\partial \underset{\sim}{r}}{\partial q^2} = \frac{\underset{\sim}{\nabla} q^3 \times \underset{\sim}{\nabla} q^1}{\nabla q^1 \cdot (\nabla q^2 \times \nabla q^3)}, \quad \frac{\partial \underset{\sim}{r}}{\partial q^3} = \frac{\underset{\sim}{\nabla} q^1 \times \underset{\sim}{\nabla} q^2}{\underset{\sim}{\nabla} q^1 \cdot (\underset{\sim}{\nabla} q^2 \times \underset{\sim}{\nabla} q^3)}$$

$$\frac{\partial \underset{\sim}{r}}{\partial q^i} = \frac{\epsilon_{ijk} \underset{\sim}{\nabla} q^j \times \underset{\sim}{\nabla} q^k}{2 \underset{\sim}{\nabla} q^1 \cdot (\underset{\sim}{\nabla} q^2 \times \underset{\sim}{\nabla} q^3)}$$

Also recall that

$$\underset{\sim}{\nabla} q^1 \cdot \underset{\sim}{\nabla} q^2 \times \underset{\sim}{\nabla} q^3 = \det\left(\frac{\partial q^i}{\partial x^k}\right) \equiv \mathcal{J}$$

$= \text{Jacobian}$

$$dV = dx^1 dx^2 dx^3 = dq^1 dq^2 dq^3 \mathcal{J}^{-1}$$

(c)

One way to see this must be the case is if we form a Volume element

$$dV = d\underset{\sim}{r}^1 \times d\underset{\sim}{r}^2 \cdot dr^3$$

$$d\underset{\sim}{r}^1 = \frac{dq^1 \underset{\sim}{\nabla} q^2 \times \underset{\sim}{\nabla} q^3}{\underset{\sim}{\nabla} q^1 \cdot \underset{\sim}{\nabla} q^2 \times \underset{\sim}{\nabla} q^3} = \frac{dq^1 \nabla q^2 \times \nabla q^3}{\mathcal{J}}$$

$$d\underset{\sim}{r}^2 = dq^2 \underset{\sim}{\nabla} q^3 \times \underset{\sim}{\nabla} q^1 / \mathcal{J}$$

$$d\underset{\sim}{r}^3 = dq^3 (\underset{\sim}{\nabla} q^1 \times \underset{\sim}{\nabla} q^2)/\mathcal{J}$$

Then

$$dV = d\underset{\sim}{r}^1 \cdot d\underset{\sim}{r}^2 \times d\underset{\sim}{r}^3 =$$

$$= \frac{dq^1 dq^2 dq^3}{\mathcal{J}^3} \left[(\underset{\sim}{\nabla} q^2 \times \underset{\sim}{\nabla} q^3) \cdot \left[(\underset{\sim}{\nabla} q^3 \times \underset{\sim}{\nabla} q^1) \times (\underset{\sim}{\nabla} q^1 \times \underset{\sim}{\nabla} q^2) \right] \right]$$

$$\left[(\underset{\sim}{\nabla} q^2 \times \underset{\sim}{\nabla} q^3 \cdot \underset{\sim}{\nabla} q_1)(\underset{\sim}{\nabla} q^3 \times \underset{\sim}{\nabla} q^1) \cdot \underset{\sim}{\nabla} q^2 \right]$$

$$dV = \frac{dq^1 dq^2 dq^3}{\mathcal{J}} = \frac{dq^1 dq^2 dq^3}{\det\left(\frac{\partial q^i}{\partial x^j}\right)}$$

$$\mathcal{J} = \begin{vmatrix} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^2}{\partial x^1} & \frac{\partial q^3}{\partial x^1} \\ \frac{\partial q^1}{\partial x^2} & \frac{\partial q^2}{\partial x^2} & \frac{\partial q^3}{\partial x^2} \\ \partial q^1/\partial x^3 & \partial q^2/\partial x^3 & \partial q^3/\partial x^3 \end{vmatrix} = \frac{\partial q^1}{\partial \underset{\sim}{r}} \cdot \left(\frac{\partial q^2}{\partial \underset{\sim}{r}} \times \frac{\partial q^3}{\partial \underset{\sim}{r}}\right)$$

d

We also note that

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{A} = \mathcal{J} \frac{\partial}{\partial q^i} \left(\frac{A^k}{\mathcal{J}} \right)$$

because $\quad \underset{\sim}{\nabla} = \nabla q^1 \frac{\partial}{\partial q^1} + \nabla q^2 \frac{\partial}{\partial q^2} + \dots$

$$\underset{\sim}{\nabla} \cdot \left[A^k \underbrace{\underset{\sim}{\nabla} q^2 \times \underset{\sim}{\nabla} q^3}_{\mathcal{J}} + \dots \right]$$

since $\underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} q^2 \times \underset{\sim}{\nabla} q^3) = 0$

$$\nabla q^2 \times \nabla q^3 \cdot \nabla q^1 \frac{\partial}{\partial q^1} \left(\frac{A^1}{\mathcal{J}} \right) + \dots$$

$$= \mathcal{J} \frac{\partial}{\partial q^i} \left(\frac{A^i}{\mathcal{J}} \right)$$

$$\underset{\sim}{\nabla} q^i \cdot \underset{\sim}{\nabla} \times \underset{\sim}{A} = \mathcal{J} \epsilon^{ijk} \frac{\partial}{\partial q^j} (A_k)$$

follows from

$$\nabla q^i \cdot \underset{\sim}{\nabla} \times \left[A_1 \underset{\sim}{\nabla} q^1 + A_2 \underset{\sim}{\nabla} q^2 + A_3 \underset{\sim}{\nabla} q^3 \right]$$

using: $\underset{\sim}{\nabla} \times \underset{\sim}{\nabla} q^i = 0$

Metric Tensor

$$d\underset{\sim}{r} = dq^i \frac{\partial \underset{\sim}{r}}{\partial q^i}$$

$$d\underset{\sim}{r} \cdot d\underset{\sim}{r} = dq^i\, dq^j\, \frac{\partial r}{\partial q^i} \cdot \frac{\partial r}{\partial q^j} \equiv g_{ij}\, dq^i dq^j$$

$$g_{ij} = \frac{\partial \underset{\sim}{r}}{\partial q^i} \cdot \frac{\partial \underset{\sim}{r}}{\partial q^j}$$

$$g \equiv \det g_{ij} = \det\left(\frac{\partial x^k}{\partial q^i} \frac{\partial x^k}{\partial q^j} \right)$$

$$= \det\left(\frac{\partial x^k}{\partial q^i} \right) \det\left(\frac{\partial x^k}{\partial q^j} \right)$$

$$= \frac{1}{J} \cdot \frac{1}{J} \equiv \frac{1}{J^2}$$

$$\frac{dq^1 dq^2 dq^3}{J} = \sqrt{g}\, dq^1 dq^2 dq^3 = dV$$