Lecture # 4

Toroidal

Non-Confinement

Magnetic Moment Conservation $(c=1)$

consider $\underset{\sim}{u}_\perp \cdot \left[\dfrac{d(\underset{\sim}{u}_\perp + \underset{\sim}{E}\times\underset{\sim}{B}/B^2)}{dt} = \underset{\sim}{u}\times\underset{\sim}{B} \right]$

$$\underset{\sim}{u}_\perp = \underset{\sim}{u}\cdot[\underset{\sim}{I} - \underset{\sim}{b}\,\underset{\sim}{b}]$$

$$\underset{\sim}{u}\cdot[\underset{\approx}{I} - \underset{\sim}{b}\,\underset{\sim}{b}]\cdot\frac{d}{dt}\left(\underset{\sim}{u} + \frac{\underset{\sim}{E}\times\underset{\sim}{b}}{B}\right) = \underset{\sim}{u}\times\underset{\sim}{B}\cdot[\underset{\sim}{u} - u_\parallel \underset{\sim}{b}] = 0$$

$$\frac{d}{dt}\left(\frac{u_\perp^2}{2}\right) + u_\parallel\, \underset{\sim}{u}_\perp\cdot\frac{d\underset{\sim}{b}}{dt} + \underset{\sim}{u}_\perp\, \underset{\sim}{u}_\perp\cdot\underset{\sim}{\nabla}\left(\frac{\underset{\sim}{E}\times\underset{\sim}{b}}{B}\right) + \text{linear } u_\perp \text{ terms} = 0$$

correction that is needed

$$\frac{d}{dt}\left(\overline{\frac{u_\perp^2}{2}}\right) + u_\parallel\, \overline{\underset{\sim}{u}_\perp(\underset{\sim}{u}_\perp\cdot\nabla)}\, b + \frac{u_\perp^2}{2}(\underset{\approx}{I} - \underset{\sim}{b}\,\underset{\sim}{b}):\nabla\left(\frac{\underset{\sim}{E}\times\underset{\sim}{b}}{B}\right) = 0$$

$$\frac{D}{Dt}\frac{u_\perp^2}{2} + u_\parallel\left(\frac{u_\perp^2}{2}\right)(\underset{\approx}{I} - \underset{\sim}{b}\,\underset{\sim}{b}):\nabla b + \frac{u_\perp^2}{2}\left(\begin{array}{l} \dfrac{\underset{\sim}{\nabla}\times\underset{\sim}{E}\cdot\underset{\sim}{b}}{B} - (\underset{\sim}{\nabla}\times\underset{\sim}{b})\cdot\dfrac{\underset{\sim}{E}}{B} \\ - \dfrac{\underset{\sim}{\nabla}B}{B}\cdot\underset{\sim}{V}_E \\ - \underset{\sim}{b}\cdot\underset{\sim}{E}\times\underset{\sim}{\kappa}/B \end{array}\right) = 0$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{b} = -\underset{\sim}{b}\cdot\underset{\sim}{\nabla}|B|/B, \quad \frac{\partial\vec{B}}{\partial t} = -\underset{\sim}{\nabla}\times\underset{\sim}{E}$$

$$\frac{2}{u_\perp^2}\frac{D}{Dt}\frac{u_\perp^2}{2} + \underbrace{\left[-u_\parallel \frac{\underset{\sim}{b}\cdot\underset{\sim}{\nabla}|B|}{B} - \frac{\partial|B|}{B\,\partial t} - \underset{\sim}{V}_E\cdot\frac{\underset{\sim}{\nabla}B}{B}\right]}_{= -\frac{D}{B\,Dt}B} - \frac{\underset{\sim}{E}}{B}\cdot\underbrace{\left(\underset{\sim}{\nabla}\times\underset{\sim}{b} + \underset{\sim}{\kappa}\times\underset{\sim}{b}\right)}_{=0} = 0$$

$$(\underset{\sim}{\nabla}\times\underset{\sim}{b})_\perp = \underset{\sim}{b}\times\underset{\sim}{\kappa}$$

$$\therefore\ B\frac{D}{Dt}\left(\frac{u_\perp^2}{2B}\right) = 0 \qquad \frac{u_\perp^2}{2B} = \text{constant}$$

on slowly varying time scale

Magnetic moment is

$$\mu = m u_\perp^2 / 2B$$

$$\mu = I A / c \quad ; \quad A = \pi \rho_L^2 = \pi u_\perp^2 / \omega_c^2$$

$$= \frac{e\omega_c}{2\pi} \cdot \pi \frac{u_\perp^2}{\omega_c^2} \frac{1}{c}$$

$$= \frac{e}{2\omega_c} \frac{u_\perp^2}{c} = \frac{m u_\perp^2}{2B}$$

Magnetic Moment is constant in time

Magnetic Moment is also the "action" of the particle

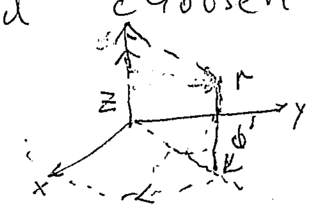
$$J = \oint \underset{\sim}{p} \cdot d\underset{\sim}{r}$$

$$H = \left(p - \frac{e\underset{\sim}{A}}{c}\right)^2 + e\Phi$$

$$\underset{\sim}{E} = -\frac{\partial \underset{\sim}{A}}{c\,\partial t} - \underset{\sim}{\nabla}\phi$$

For uniform B-field choosen in $\hat{z}$ direction

$$\underset{\sim}{A} = +\frac{B_0 r}{2}\hat{\phi}$$



origin at guiding center

$$\dot{\phi} = -\omega_c$$

$$J = \oint P_\phi \frac{d\phi}{2\pi} = P_\phi$$

$$= m r^2 \dot{\phi} + \frac{e}{c}A_\phi = m\rho_L^2\left(-\omega_c + \frac{\omega_c}{2}\right) = \frac{-v_\perp^2 m}{2\,\omega_c}$$

(3)

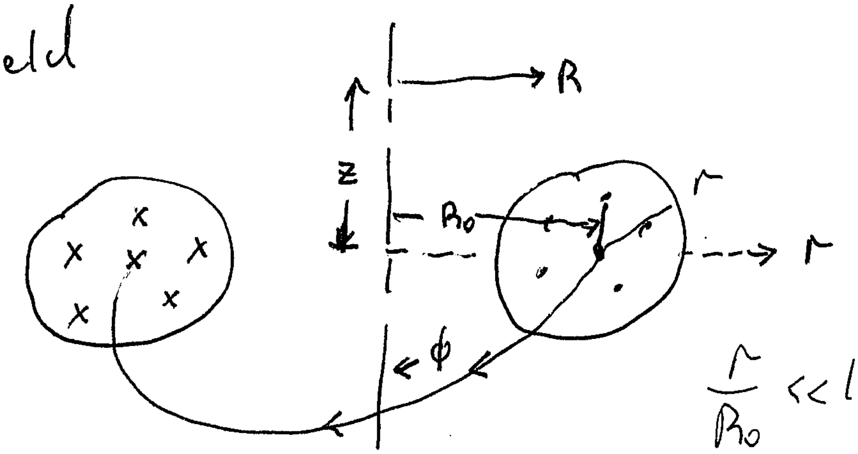
verify $\left| \oint \underset{\sim}{p} \cdot d\underset{\sim}{r} \right| = \frac{v_\perp^2}{2\omega_c}$

In other coordinates.

Cartesian where $\underset{\sim}{A} = B_0 x \hat{y}$

Cylindrical, $\underset{\sim}{A} = -\frac{B_0 r}{2} \hat{\phi}$, but with arbitrary origin(not located at guiding center position).

Let us attempt to confine a plasma in a toroidal field



$$\underset{\sim}{B} = \frac{B_0 R_0}{R} \hat{\phi} \simeq B_0 \left(1 - \frac{R - R_0}{R_0}\right) \hat{\phi}$$

Note $\underset{\sim}{b} \times \underset{\sim}{\kappa} = \underset{\sim}{b} \times \frac{\underset{\sim}{\nabla} B}{B}$ because if

$\underset{\sim}{b} \times (\underset{\sim}{\nabla} \times \underset{\sim}{B}) = 0$ we have

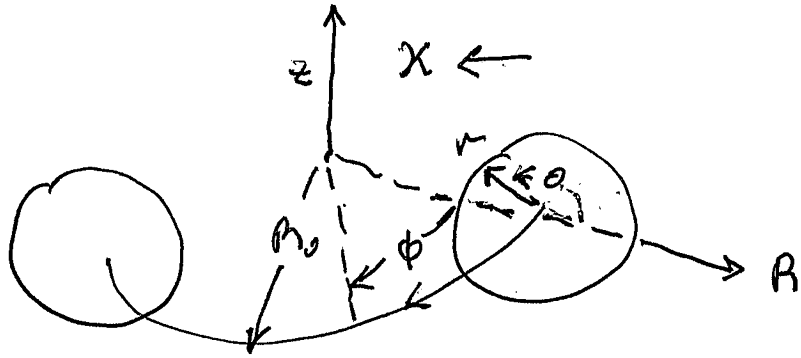
$$\underset{\sim}{B} = |B| \underset{\sim}{b}$$

$$b \times (\underset{\sim}{\nabla} B \times \underset{\sim}{b}) + B \underset{\sim}{b} \times (\underset{\sim}{\nabla} \times \underset{\sim}{b}) = 0$$

recall $\underset{\sim}{b} \times (\underset{\sim}{\nabla} \times \underset{\sim}{b}) = - \underset{\sim}{\kappa}$

i.e. $\frac{\underset{\sim}{\nabla}_\perp B}{B} = \underset{\sim}{\kappa}$ if $\underset{\sim}{b} \times \underset{\sim}{J} = 0$

i.e $\underset{\sim}{J}$ only along field line $\left\{ \begin{array}{l} \text{or zero} \\ \text{i.e. vacuum} \end{array} \right\}$



$$\underset{\sim}{\kappa} = \frac{\underset{\sim}{\nabla} B}{B} \simeq \frac{\underset{\sim}{\nabla} B_0\left(1 - \frac{(R-R_0)}{R_0}\right)}{B} = -\frac{\hat{R}}{R_0}$$

$$\underset{\sim}{V}_D = \left(\frac{V_\perp^2}{2} + V_\parallel^2\right) \frac{\underset{\sim}{b} \times \underset{\sim}{\kappa}}{\omega_c}$$

$$= -\left(\frac{V_\perp^2}{2} + V_\parallel^2\right) \frac{\hat{\phi} \times \hat{R}}{\omega_c R_0} = -\hat{\underset{\sim}{z}}\left(\frac{V_\perp^2}{2} + V_\parallel^2\right)\frac{1}{\omega_c R_0}$$

Hence downward vertical drift of positive charge (unbounded).

but this is not principal response

Downward drift produces charge accumulation at bottom edge giving rise to balancing upward polarization drift to cancel charge build-up

$$\underset{\sim}{V}_p = \frac{c\,\partial(\underset{\sim}{E} \times \underset{\sim}{b}/B)}{\omega_c\,\partial t} = +\hat{z}\,\frac{\frac{V_\perp^2}{2} + V_\parallel^2}{\omega_c R_0}$$

$$\therefore\ \frac{c\underset{\sim}{E} \times b}{B} = \hat{\underset{\sim}{\phi}} \times \hat{\underset{\sim}{z}}\ \frac{\overline{\left(\frac{V_\perp^2}{2} + V_\parallel^2\right)}\,t}{R} \simeq \frac{2\hat{\underset{\sim}{R}}\,V_{thi}^2\,t}{R}$$

Radial expansion of plasma