Lecture #39

Quasilinear Spatial Diffusion

&

Neoclassical Theory

We considered the effect of electron transport from drift waves in a sheared magnetic field

$$\underset{\sim}{B} = B_0 \left(\hat{\underset{\sim}{z}} + \frac{x}{L_s} \hat{\underset{\sim}{y}} \right)$$

$$\phi = \hat{\phi} \exp\left[-i\omega t + i k_y y + i k_z z \right]$$

$$\delta f_e = \frac{c \dfrac{k_y \phi_k}{B} \left.\dfrac{\partial F}{\partial X_g}\right|_{E'}}{\omega - k_\parallel(x) v_\parallel} \quad ; \quad E' = E - \frac{\omega}{k_y} P_y$$

$$= E - \frac{\omega}{k_y} m \omega_{ce} (x - x_0)$$

$$v_\parallel k_\parallel(x) = \left(k_z + k_y \frac{x}{L_s} \right) v_\parallel \equiv k_y \left(\frac{x - x_0}{L_s} \right) v_\parallel$$

$$\left(\frac{x_0}{L_s} k_y = - k_z \right)$$

We then obtained quasilinear equation

$$\frac{\partial F}{\partial t} = 2\pi \sum_{k_y} \frac{\partial}{\partial X_g} \left| \frac{c k_y \phi_k}{B} \right|^2 \delta\left(\omega - \frac{k_y v_\parallel}{L_s} \right) \frac{\partial F}{\partial X_g}$$

$$\frac{\partial}{\partial X_g} \equiv \left. \frac{\partial}{\partial X_g} \right|_{E'}$$

How do we get quasi-linear equation in universal form?

Look at single mode dynamics

Single Mode

$$\frac{dx_g}{dt} = -2c\,\frac{k_y \phi}{B}\,\sin\left(k_y y \underset{H}{+} k_z z - \omega t\right)$$

$$\boxed{\cancel{\frac{k_y (x-x_0)}{L_s}} - \omega t}$$

$$\psi = k_y y + k_z z - \omega t$$

$$\frac{d\psi}{dt} = k_y v_\parallel \frac{B_y(x_g)}{B} + k_z v_\parallel - \omega = \Omega$$

$$= \frac{k_y (x_g - x_0)}{L_s} v_\parallel - \omega$$

$$\frac{d^2\psi}{dt^2} = -2\frac{k_y v_\parallel}{L_s}\, c\,\frac{k_y \phi}{B}\,\sin\Big(\overbrace{k_y y + k_z z - \omega t}^{\psi}\Big)$$

$$\equiv -\omega_b^2 \sin(\psi)$$

$$\therefore\quad \omega_b^2 = 2\,\frac{k_y^2 \phi\, v_\parallel c}{L_s B}$$

For QL Equation

$$\frac{\partial f}{\partial t} = 2\sum_j \pi \frac{\partial}{\partial X_g}\left|\frac{c\phi_k k_y}{B}\right|^2 \cdots \delta(\omega_j - \Omega)\frac{\partial f}{\partial X_g}$$

$$\Omega = k_y x_g v_\parallel / L_g$$

$$= \frac{\pi}{2}\sum_j \frac{}{k_y \frac{v_\parallel}{L_g}} \frac{\partial}{\partial X_g}\left|\frac{2c\phi_k k_y^2 v_\parallel}{L_s B}\right|^2 \delta(\omega_j - \Omega)\frac{1}{k_y \frac{v_\parallel}{L_g}}\frac{\partial f}{\partial X_g}$$

$$\frac{\partial f}{\partial t} = \frac{\pi}{2} \sum_j \underbrace{\frac{\partial}{\frac{k_y v_\parallel}{L_s}\partial x_g}}_{= \frac{\partial}{\partial \Omega_j}} \underbrace{\left| \frac{2c\,\phi_k}{L_s} \frac{k_y^2 v_\parallel}{B} \right|^2}_{= |\omega_{bj}^2|^2} \delta(\omega_j - \Omega_j) \frac{1}{\frac{k_y v_\parallel}{L_s}} \frac{\partial f}{\partial x_g}$$

$$d\Omega = \frac{k_y v_\parallel dx_g}{L_s}$$

as

$$\Omega = k_y (x_g - x_0) v_\parallel / L_s$$

$$\boxed{\frac{\partial f}{\partial t} = \frac{1}{2} \sum_j \frac{\partial}{\partial \Omega} |\omega_{bj}^2|^2 \, G(\Omega - \omega_j) \frac{\partial f}{\partial \Omega}}$$

where

$$\int d\Omega \; G(\Omega - \omega_j) = \pi$$

with $G(\Omega - \omega_j)$ a narrowly peaked function

Back to orbit in a tokamak, with collisions

We saw that in a cylinder the radial diffusion coefficient across a field line is $D_{cl} = \nu_e \rho_{Le}^2$

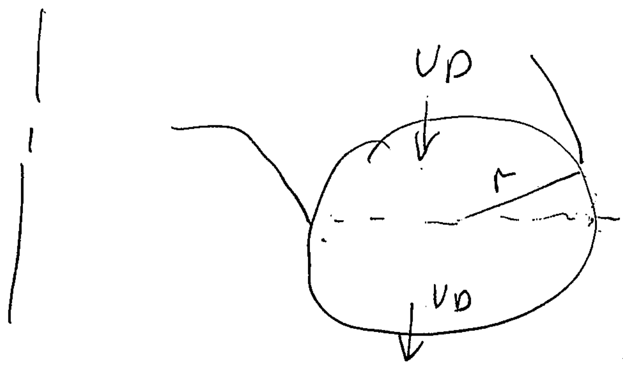
In a tokamak, the fluid (lmf) expression for the radial diffusion is found to be

$$D = 2\nu_e (1 + 2q^2) \rho_{Le}^2 = (1 + 2q^2) D_{cl}$$

Pfirsch - Schluter diffusion

The source for the additional Pfirsch-Schluter diffusion has an interesting interpretation

In the short (but not very short) mean free path ($l_{mfp} = \frac{v_{the}}{2\nu}$) limit ($\rho_L < l_{mfp} < qR$) where qR "connection" length around tokamak)



When a particle is upward part of the tokamak it drifts inwards when in the lower part of the tokamak it is drifting outward.

The diffusion coefficient is

$$D_{ps} = \frac{(\Delta r)^2}{\tau} = \frac{(v_D \tau)^2}{\tau} = v_D^2 \tau$$

$$v_D \simeq \frac{v_{th}^2}{\omega_c R}$$

How long a time τ does the particle remain on top (below) the tokamak when $l_{mfp} < qR$

This is determined by a second diffusion relation

$$(\Delta s)^2 = (qR)^2 = D_{\parallel} \tau = \frac{v_{th}^2}{\nu} \tau$$

$$\therefore \quad \tau = \frac{\nu (qR)^2}{V_{th}^2}$$

Thus

$$D_{ps} \simeq V_D^2 \tau = \left(\frac{V_{th}^2}{\omega_c R}\right)^2 \frac{\nu (qR)^2}{V_{th}^2}$$

$$= \nu q^2 \frac{V_{th}^2}{\omega_c^2} = q^2 \nu \frac{V_{th}^2}{\omega_c^2}$$

$$= q^2 D_{cl}$$

Now let us consider long mean free path diffusion

This diffusion is dominated by trapped particles

Passing particle makes an excursion

$\Delta r \simeq q V^2 / \omega_c V_{\parallel}$ from surface

$\simeq q \frac{V_{th}}{\omega_c}$

$$D = \nu \Delta r^2 \sim q^2 \nu \frac{V_{th}^2}{\omega_c^2} \approx D_{ps}$$

(5)

Trapped particle makes ~~a~~ radial excursion

$$\Delta r \simeq q \frac{V_{th}}{\omega_c} \frac{1}{\epsilon^{1/2}}$$

Effective ~~f~~requency ~~of~~ of losing Δr correlation is ν/ϵ, as a result of small angle collisions. Thus Because in Fokker Planck Equation, the rate to diffused a pitch angle $\theta \simeq \epsilon^{1/2}$ is $\nu_{eff} = \frac{\nu_{90}}{\theta^2}$ $\cancel{= \frac{\nu}{\theta^2}}$ $\cancel{= \frac{\nu}{\theta}}$

Thus diffusion of trapped particles is

$$D_{TR} = \frac{\nu}{\epsilon} \Delta r^2 = \frac{\nu}{\epsilon^2} q^2 r_L^2 = \frac{1}{\epsilon^2} D_{ps}.$$

However only $\epsilon^{1/2}$ of particles are trapped

$$\therefore D_{NC} \simeq \epsilon^{1/2} D_{TR} \simeq \frac{D_{ps}}{\epsilon^{3/2}} = \frac{\nu}{\epsilon^{3/2}} q^2 r_L$$

This long mean free path limit is valid as long as

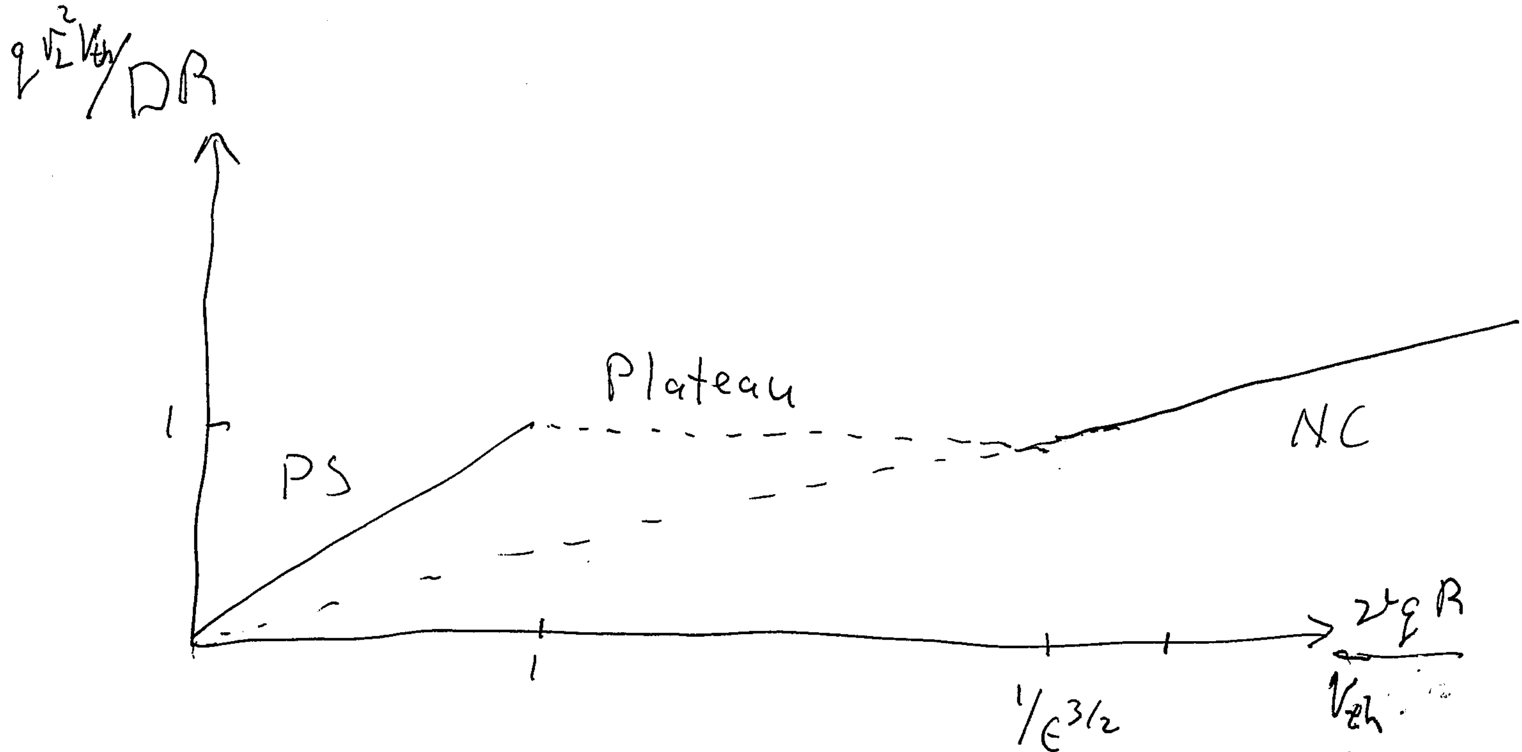
$$\nu_{eff} \equiv \frac{\nu}{\epsilon} > \omega_{Tr} \simeq q \frac{V_{th}}{R} \epsilon^{1/2}$$

$$\text{or} \quad \nu > q \frac{V_{th}}{R} \epsilon^{3/2}$$

Thus we have

$$D_{PS} = \nu q^2 r_L^2 = \frac{\nu}{V_{th}} qR \; \frac{q r_L^2 V_{th}}{R}$$

$$D_{NC} = \frac{\nu q^2 r_L^2}{\epsilon^{3/2}} = \frac{\nu q R}{V_{th}} \; \frac{q r_L^2}{\epsilon^{3/2} R}$$



Plateau regime can be calculated with a collisionless quasilinear theory where B-field modulation is perturbation

Reference

Plateau & Neo-Classical Regime

Galeev & Sagdeev ~ 1967

Ware Pinch

In a magnetic field trapped particle respond to a toroidal magnetic field as if ~~it~~ we absent

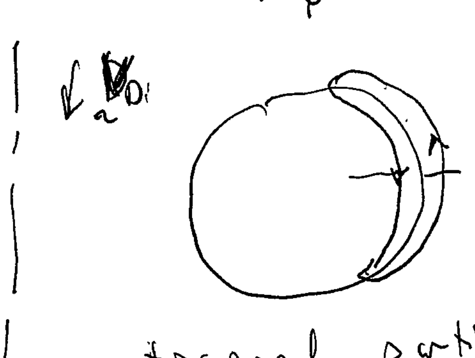
$$V_{Dr} = c\,\frac{E_\phi}{B_\theta}$$

Why is this?

$E_\phi = 0$

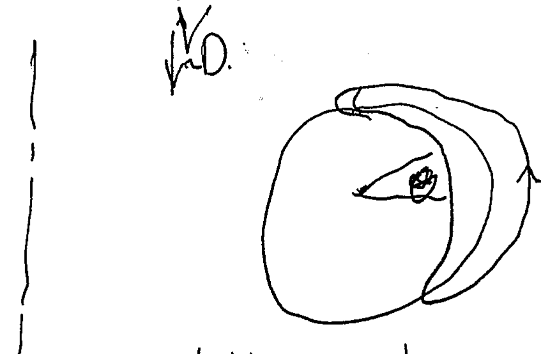
$|V_D| = \frac{V_{th}^2}{\omega_c R}$

E_ϕ = finite



trapped particle orbit without electric field

Time above is the same as time below



orbit with electric field

Orbit displaced along $E_{\parallel}$. Time above longer than below. On a bounce time, particle moves inward

Displacement due to $E_{\parallel}$

$$\frac{dv_{\parallel}}{dt} = -\mu \frac{\partial B}{\partial s} - \frac{e}{m} E_{\parallel} \qquad \left(\mu = \frac{v_{\perp}^2 B_0}{2}\right)$$

$$E_{\parallel} \simeq E_{\phi}$$

as $\quad v_{\parallel} = qR \dfrac{d\theta}{dt}, \quad ds = qR\, d\theta, \quad B_0 = B_0(1-\epsilon\cos\theta)$

$$\frac{d^2\theta}{dt^2} = -\frac{\mu B_0 \epsilon \sin\theta}{q^2 R^2} - \frac{e}{m} \frac{E_{\phi}}{qR}$$

leads to a displacement in θ

$$\delta\theta \simeq \frac{e}{m} \frac{E_{\theta}\, qR}{\epsilon\, V_{th}}$$

$$V_{\phi} \simeq \delta\theta \cdot V_{D}$$

$$= \frac{e}{m} \frac{E_{\theta}\, qR}{\epsilon\, V_{th}} \frac{V_{th}^2}{\omega_c R} \simeq c\frac{E_{\phi}}{B_{\theta}}$$

However of $\epsilon^{1/2}$ of particles are trapped

Ware Pinch

$$\Gamma_i = -n_0 \frac{cE_{\theta}}{B_{\theta}} \epsilon^{1/2}$$

Boot strap current

Recall source of diamagnetic current

Downward current because anti-parallel velocities at a fixed point have different values of f

g for guiding center

$$f^+(r) = f_g\left(x + \frac{|v_y|}{\omega_c}\right) = f_g(x) + \frac{|v_y|}{\omega_c}\frac{\partial f}{\partial x}$$

$$f^-(r) = f_g\left(x - |v_y|/\omega_c\right) = f_g(x) - \frac{|v_y|}{\omega_c}\frac{\partial f}{\partial x}$$

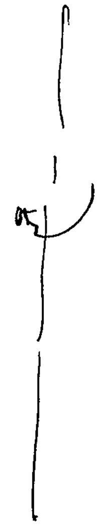
$$v_y f^+(r) - v_y f^-(r) = 2\frac{v_y}{\omega_c}\frac{\partial f}{\partial x}$$

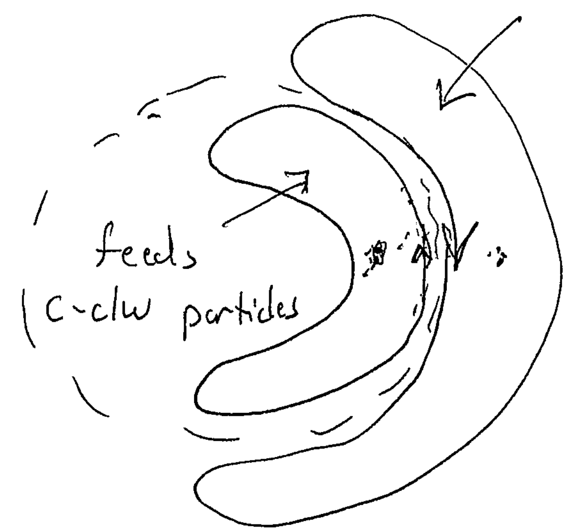
Current is

$$e\int f\, v_y\, d^3v = e\int d^3v\, \frac{v_y^2}{\omega_c}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(n\, v_{th}^2 / B\right)$$

diamagnetic current

Boot strap current is similar for trap particle orbit widths but with a twist





$$\Delta_b \simeq \frac{V_{th}}{\omega_c} q/\epsilon^{1/2} = \frac{V_{\parallel}}{\omega_{c\theta}}$$

Distribution of matching "anti" currents differ by

$$\Delta_b \frac{\partial f}{\partial r} = \frac{V_{th}}{\omega_c} \frac{q}{\epsilon^{1/2}} \frac{\partial f}{\partial r}$$

Expected parallel current then

$$\overline{e V_{\parallel} \Delta_b \frac{\partial f}{\partial r}} \simeq \overline{e V_{th}\, \epsilon^{1/2} \Delta_b \frac{\partial f}{\partial r}}$$

$$j_{\parallel} \simeq e \underset{\uparrow}{\epsilon^{1/2}} \frac{\partial}{\partial r} \frac{V_{th}^2}{\omega_c} \frac{q}{\epsilon^{1/2}} \epsilon^{1/2} \simeq \epsilon^{3/2} \frac{\partial p}{\partial r} / B_\theta$$

fraction of trapped partic

However, passing particles remember "banana" region they came from, and weight difference from f^+ and f^- will persist but $v_\parallel$ now covers all phase space for orbit widths $\Delta_b \simeq \frac{v_{th}}{\omega_c} q/\epsilon^{1/2}$

Then

$$\overline{e\, v_\parallel \Delta_b \frac{\partial f}{\partial r}} \simeq e\, v_{th}\, \Delta_b \frac{\partial f}{\partial r}$$

$$\boxed{j_\parallel \simeq \frac{\epsilon^{1/2}}{B_\theta} \frac{\partial p}{\partial r}}$$

$\boxed{\text{Bootstrap Current}}$

Saviour of Tokamak?