Lecture # 37

Implications of Quasi-Linear Theory
Let us discuss the mode saturation in the bump-on-tail instability in some more detail.

We saw that the prediction when there is a continuum of modes, the distribution flattens.

\[ f_0(v_a) = f_0(v_b) \]

\[ f_{\text{final}} = f_0(v_b) \Theta(v_b - v) \Theta(v - v_a) \]

\[ \int dv f_0(v) = f_0(v_b) (v_b - v_a) \]

Further, the difference in momentum of final state compared to initial state resides in excited wave momentum.
\[ \Delta P = \rho_i - \rho_f \]
\[ = m \sum_{v_b} dV (v - v_a) \left( f_0(v) - f_0(v_a) \right) \]
\[ = \sum_k k \frac{\partial E_k}{\partial \omega} \frac{2 |E_k|^2}{8\pi} \]
\[ E = \sum_k E_k \exp \left[ -i\omega_k t + ikx \right] + c.c. \]

Also recall that quasi-linear equation for the tail particles was

\[ \frac{df}{dt} - \frac{\partial}{\partial V} \int \frac{dk}{2\pi} \frac{E_k}{m} \left[ \Re S(\omega_k - kV) \right] \frac{df}{dV} = 0 \]

Recall that \[ \Re S(\omega_k - kV) \]

came from \[ \lim_{k \to 0} \frac{\delta_k}{(\omega_k - kV)^2 + \delta_k^2} \]

We will come back to this discussion in a short while.

Now let us consider what happens with a discrete mode.
If we had a single mode

\[
\frac{d^2 x}{dt^2} = \frac{2eF}{m} \sin(kx-\omega t)
\]

\[
\frac{d^2 (kx-\omega t)}{dt^2} = \frac{2eF}{m} \sin(kx-\omega t)
\]

Let \( \psi = kx-\omega t \)

\[\omega_b^2 = \frac{2eF}{m} \equiv \text{square of trapping frequency of a deeply trapped particle} \]

\[
\frac{d^2 \psi}{dt^2} + \omega_b^2 \sin \psi = 0
\]

\[
\frac{d}{dt} \frac{d \psi}{dt} = \frac{1}{2} \frac{d \psi^2}{dt} = -\omega_b^2 \sin \psi = \frac{d}{dt} \omega_b \cos \psi
\]

\[
\psi^2 - \omega_b^2 \cos \psi = E \equiv \text{constant}
\]

\[
\frac{\psi^2}{2} - \omega_b^2 \cos \psi = \frac{\psi^2}{2} - 2(E + \omega_b^2 \cos \psi) > 0
\]

Note that if \(-\omega_b^2 < E < \omega_b^2\)

Particle is trapped in \( \psi \)

\[
\psi_{\text{max}} = \cos^{-1}\left(\frac{-E}{\omega_b^2}\right)
\]
Distribution within trapping region mixes.

Distribution outside trapping regions just distorts without mixing.

We can estimate saturation level of a single mode by comparing the momentum lost from mixing, compared with momentum of wave that is excited.
Momenta in mixed region before mixing and after mixing:

\[ \psi = k \nu - \omega t \]

\[ \int_{\text{upper}}^{\text{lower}} 2 \omega^2 / (1 + \cos \psi) \frac{d\psi}{k} + \omega / k = \text{upper} \]

\[ \int_{\text{lower}}^{\text{upper}} \frac{1}{2} \frac{d\psi}{k} \int dV \left( f_0(V) - f_0'(\omega / k) \right) \]

\[ = \frac{2m}{k} \int d\psi \int dV \left( f_0(\omega / k) + (V - \omega / k) f_0'(\omega / k) - f_0'(\omega / k) \right) \int dV \left( 1 - \nu / k \right) \]

\[ \text{Only term to persist} \]

\[ = \frac{8}{3} \frac{\pi}{k^4} \int_0^{\infty} \int d\psi (1 + \cos \psi)^{3/2} \omega_b^3 \frac{df(\omega / k)}{dV} \left( 1 + \cos \psi = 1 + \cos \frac{\psi}{2} \right) \]

\[ = \frac{12 \pi}{3} \frac{m \omega_b^3}{k^4} \frac{df(\omega / k)}{dV} = \frac{12 \pi}{3} \frac{m^2}{k^4} \frac{\Delta E \delta \omega_b^3}{e^2 \Delta \omega} \]

\[ \text{or} \quad \frac{\Delta E}{\Delta \omega} = \frac{m^2}{k^4} \frac{\Delta E \delta \omega_b^3}{e^2 \Delta \omega} \]

\[ \text{MW} = \frac{k^2 E_h^2 \chi \tau}{8 \pi \Delta \omega k_0} = \frac{m^2}{e^2} \frac{1}{8} \frac{\omega_0^4}{\Delta \omega} \]
There is a major difference from momentum released by "non-overlapped" modes and "overlapped" modes.

From our estimate of momentum released from single modes (non-overlapped):

\[ W_{M_{SM}} \approx m \frac{dv}{dv} \left( \frac{\omega_0}{k} \right)^3 N < m \frac{dv}{dv} \left( \frac{\Delta V}{N} \right)^3 N \]

\[ W_{M_{OL}} \approx m \frac{dv}{dv} \left( \Delta V \right)^3 \]

\[ \frac{W_{M_{SM's}}}{W_{M_{OL}}} < \frac{1}{N^2} \approx \left( \frac{Y_L}{kAV} \right)^2 N = \# \text{ of Modes} \]

filling unstable space
This disparity actually infers that explosive behavior should occur in a system whose sources produce linear unstable distribution functions, and whose additional damping mechanisms, such as due to collisions, exist.

In quasi-linear, the wave energy that is present after a flatterened distribution arises, damps away. The sources then slowly build up the unstable distribution function. However, initially, when only a finite number of modes can be excited, quasi-linear theory not applicable since mode overlap doesn't occur. Modes saturate at single mode limits. However, when overlap is triggered, modes may suddenly grow, which causes damped "free" energy.
We also note that a universal way of writing quasi-linear equation is

\[ \frac{\partial f}{\partial t} - \frac{e}{c} \sum_k \frac{\partial f(\omega - kv)}{\partial \omega} \frac{\partial \omega}{\partial \sigma_k} = 0 \]

\[ \gamma f(\omega - kv) \text{ came from } \frac{\delta L}{(\omega^2 - kv) + \kappa^2} \]

\[ = \frac{1}{\pi} \frac{\omega}{(\omega^2 - k^2)} \]

where \( \int d\omega \omega (\omega^2 - k^2) = 1 \)

with \( kv = \frac{e}{m} \frac{kF_n}{\omega_0^2} = \omega_0^2 \)

\[ \frac{\partial f}{\partial t} - \frac{e}{c} \sum_{k,j} \frac{\partial f}{\partial \omega_j} \frac{\partial \omega_j}{\partial \sigma_k} = 0 \]

This is the uniform form that is applicable to most physical systems, when appropriate wave trapping frequency \( \omega_0^2 \) is defined.

We will see this shortly (next lecture) in drift wave problem.