Lecture # $3\cancel{4}\frac{1}{2}$

Ambipolar Diffusion

Kinetic Equation in Magnetic Field

$$\frac{\partial f}{\partial t} + \underset{\sim}{v}\cdot\nabla f + \frac{e}{m}\underset{\sim}{E}\cdot\frac{\partial f}{\partial \underset{\sim}{v}} - \omega_c\, \underset{\sim}{v}\times\underset{\sim}{b}\cdot\frac{\partial f}{\partial v} = -\hat{\nu}(f - f_M)$$

zeroth order , $\cancel{\underset{\sim}{E} = \hat{\underset{\sim}{z}} E_z + \hat{\underset{\sim}{y}} E_y}$

$$0 = \omega_c \frac{\partial f}{\partial \phi} - \hat{\nu}(f - f_M)$$

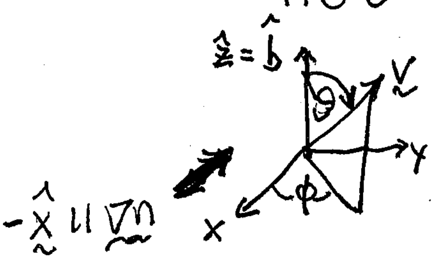
solution $\quad f = f_M = n \exp\left(\frac{mv^2}{2T}\right) \Big/ (2\pi T/m)^{3/2}$

look for stationary solution

take $n(\underset{\sim}{r})$, $T(\underset{\sim}{r})$ functions of magnetic flux.

This is a solution even in long mean free path case

Here we first take 2-D slab geometry



$$\underset{\sim}{v} = v\cos\theta\, \hat{\underset{\sim}{z}} + v\sin\theta\,(\hat{\underset{\sim}{x}}\cos\phi + \hat{\underset{\sim}{y}}\sin\phi)$$

$$v\left[\frac{\partial n}{n\,\partial x}\sin\theta\cos\phi - \frac{eE}{m}\frac{\cos\theta}{T} - \frac{eE}{m}\frac{\sin\theta\sin\phi}{T}\right.$$
$$\left.\frac{\partial T}{T\partial x}\left(\frac{3}{2} - \frac{1}{2}\frac{mv^2}{T}\right)\sin\theta\cos\phi\right] F_M$$

$$= \omega_c \frac{\partial \hat{f}}{\partial \phi} - \hat{\nu} f$$

$$f = \sum_{m=-1}^{1} f_m e^{im\phi}, \qquad f_m = f_m(\theta, v)$$

Now compute response from $\frac{\partial n}{\partial x}\hat{\underset{\sim}{x}}$ only,

$$\underset{\sim}{\Gamma} = \int d^3v\, f^{(1)} \underset{\sim}{v}\, d^3v$$

$$= \int d^3v\, f^{(1)} \left[\begin{array}{l} v\sin\theta\left(\frac{e^{i\phi}+e^{-i\phi}}{2}\hat{x} + \frac{e^{i\phi}-e^{-i\phi}}{2i}\hat{y} \right) \\ +v\cos\theta\, \hat{\underset{\sim}{z}} \end{array} \right]$$

$$= \int d^3v \left[\begin{array}{l} \left[\left(\frac{f_+ + f_-}{2} \right) \hat{\underset{\sim}{x}} + \left(\frac{f_- - f_+}{2i} \right) \hat{\underset{\sim}{y}} \right] \sin\theta\, v \\ + f_0 \cos\theta\, v \end{array} \right]$$

$$= -\frac{1}{4}\int d^3v\, v^2 \sin^2\theta \left[\frac{\partial n}{n\,\partial x}\left(\left[\frac{1}{-\nu - i\omega_c} + \frac{1}{-\nu + i\omega_c} \right]\hat{x} + \left[\frac{-i}{-\nu - i\omega_c} - \frac{-i}{-\nu + i\omega_c} \right]\hat{x} \right) \right] F_M$$

$$= -\frac{1}{4}\int d^3v\, v^2 \sin^2\theta \left[\begin{array}{l} \frac{\partial n}{n\,\partial x} \frac{-2\nu}{\nu^2+\omega_c^2}\hat{\underset{\sim}{x}} \\ \frac{1}{n}\frac{\partial n}{\partial x} \frac{2\omega_c}{\nu^2+\omega_c^2}\hat{\underset{\sim}{y}} \end{array} \right] F_M$$

Completing the velocity integrals gives:

$$\Gamma = -\frac{\partial n}{\partial x}\left(\nu \frac{V_{th}^2}{\omega_c^2+\nu^2}\,\hat{x} - \frac{\omega_c V_{thj}^2}{\omega_c^2+\nu^2}\,\hat{y} \right)$$

We see that the magnetic field introduces an intrinsic anisotropy to the fluxes as there are flows in a different direction than the "thermal" force $\propto \frac{\partial n}{\partial x}$. However this anisotropy is an "old" friend as the flow in the y-direction is the diagmagnetic drift velocity

$$\Gamma_{yj} = \frac{\partial}{\partial x}\frac{(nT_j)}{m_j\omega_{cj}}\frac{1}{(1+\nu_j^2/\omega_{cj}^2)}$$

$$\simeq \frac{\partial}{\partial x}\left(\frac{nT_j}{\omega_c m_j}\right)\left(1+\underbrace{\mathcal{O}(\nu_j^2/\omega_c^2)}_{\text{if small}}\right)$$

(We would have obtained $\frac{\partial T}{\partial x}$ like this had we retained the "force" $\propto \frac{\partial T}{\partial x}$)

The flux in the x-direction is

$$\Gamma_x = -\frac{\partial n}{\partial x}\nu\frac{V_{th}^2}{\omega_c^2+\nu^2} \longrightarrow \begin{cases} -\nu\frac{\partial n}{\partial x}\, l_{mfp}^2, & l_{mfp} << \rho_L \\ -\nu\frac{\partial n}{\partial x}\, \rho_L^2, & l_{mfp} >> \rho_L \end{cases} \quad (3)$$

Let us note that there is a short way to calculate the cross field flux if $\nu \ll \omega_c$,

which is useful in more realistic situations, when we use honest rather than krooked models

Note:

$$\Gamma_x = -\frac{\partial n}{\partial x}\frac{\nu\, V_{th}}{\omega_c^2+\nu^2} \simeq -\frac{\partial n}{\partial x}\nu\frac{V_{th}^2}{\omega_c^2} = -\frac{\partial n}{\partial x}\nu\rho_L^2$$

suggests an expansion in ν/ω_c.

How can we obtain this from a more general kinetic equation:

$$\omega_c\frac{\partial f}{\partial \phi} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla} f = -\nu(f-f_M)$$

To lowest order $f^{(0)} = f_M$

Then (take only density gradient for simplicity)

$$-\omega_c\frac{\partial f^{(1)}}{\partial \phi} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla} F_M = -\nu(f^{(1)}-f_M)$$

But now we use a subsidiary ordering where $\frac{\nu}{\omega_c} \ll 1$.

$$-\omega_c\frac{\partial f^{(1,0)}}{\partial \phi} + \underset{\sim}{v}\cdot\frac{\underset{\sim}{\nabla} n}{n} F_M = 0$$

$$-\omega_c\frac{\partial f^{(1,1)}}{\partial \phi} = -\nu\left(f^{(1,0)}\right)$$

with $\underset{\sim}{v}\cdot\underset{\sim}{\nabla} n = v\sin\theta\cos\phi\,\frac{\partial n}{\partial x}$

$$-\omega_c\frac{\partial f^{(1,0)}}{\partial\phi} + v\sin\theta\cos\phi\frac{\partial n}{n\,\partial x}F_M = 0$$

$$\therefore\ f^{(1,0)} = +\frac{v\sin\theta\,\cancel{\sin}\phi}{\omega_c}\frac{\partial n}{\partial x}\frac{F_M}{n} = -\frac{\underset{\sim}{b}\times\underset{\sim}{v}}{\omega_c}\cdot\underset{\sim}{\nabla} n\ F_{max}$$

But this is all we need to know in calculating the flux

Going back to our kinetic equation we have

$$\frac{e}{mc}\underset{\sim}{v}\times B\cdot\frac{\partial f^{(1)}}{\partial\underset{\sim}{v}} = \hat{\nu}\hat{f}^{(1)} = \frac{\underset{\sim}{v}\cdot\underset{\sim}{\nabla} n_0}{n_0}F_{max}$$

Taking the $\underset{\sim}{v}\times\underset{\sim}{b}$ moment of this equation gives

$$\frac{Be}{mc}\int d^3v\ \underset{\sim}{v}\times b\cdot\frac{\partial}{\partial\underset{\sim}{v}}\left((\underset{\sim}{v}\times\underset{\sim}{b})f^{(1)}\right) = -\int d^3\underset{\sim}{v}\left[\underset{\sim}{v} - v_\parallel\underset{\sim}{b}\right]f^{(1)}\frac{eB}{mc}$$

$$= \int d^3v\ (\underset{\sim}{v}\times\underset{\sim}{b})\hat{\nu}\hat{f}^{(1)}$$

$$\therefore\ \underset{\sim}{\Gamma}_j = \frac{c}{Be_j}\int m_j\underset{\sim}{v}\times\underset{\sim}{b}\ \hat{\nu}f_j^{(1,0)}$$

Now: we need only substitute $f_j^{(1,0)} = -\frac{\underset{\sim}{b}\times\underset{\sim}{v}}{\omega_c}\cdot\underset{\sim}{\nabla} n\ F_{maxj}$ into the collision operator to evaluate the flux.

This is trivial for our model operator, $\hat{\nu} f = \nu f$,

But even if $\hat{\nu} f$ is the Landau (and for that matter the Boltzman) collision operator, we do not have to invert the collision operator

We only have to perform an explicit integration!

But even more important, is to note that to get the flux we just have to take moment of the momentum on the collision operator

But there cannot be a momentum exchange between like particles, which implies that there is no diffusion due to collisions between like particles (at least proportional to $\frac{\partial n}{\partial x}$).

Furthermore, the momentum transfer between opposite charged species (electron and ions) are equal but opposite

Ih

Thus,

$$\Gamma_{\perp i} = \Gamma_{\perp e}$$

since $\frac{1}{e_j} \int m_j \underset{\sim}{v} \times \underset{\sim}{b} \, \hat{\nu}_{jk} f_j^{(1)}$

(k the oppositely charged species)

Thus, to this order, the collisional flux across a magnetic field is ambipolar!

Recall that the Landau Fokker-Planck equation takes the form

$$\hat{\nu}\left(f_i - f_m\right) = \sum_j \frac{\Gamma_{ij}}{2} \int d^3v' \frac{\partial}{\partial \underset{\sim}{v}'} \cdot \frac{g^2 \underset{\approx}{I} - \underset{\sim}{g}\underset{\sim}{g}}{g^3} \cdot \left(-f_i(v) \frac{\partial f_j(v')}{\partial \underset{\sim}{v}'} \frac{m_i}{m_j} + \frac{\partial f_i(v)}{\partial \underset{\sim}{v}} f_j(v') \right)$$

$$\Gamma_{ij} = \frac{4\pi e_i^2 e_j^2 \ln\Lambda}{m_i^2} \quad , \quad \underset{\sim}{g} = \underset{\sim}{v} - \underset{\sim}{v}' \quad , \quad g = |\underset{\sim}{g}|$$

Let us consider $i =$ electrons

A dominant characteristic for electrons colliding with ions is that

$$\underset{\sim}{g} \doteq (\underset{\sim}{v}_e - \underset{\sim}{v}_i) \simeq \underset{\sim}{v}_e \quad : \quad \text{Then}$$

$$\hat{\nu}\left(f_e - f_m\right) \simeq \frac{\Gamma_{ei} n_{io}}{2} \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left(\frac{v^2 \underset{\approx}{I} - \underset{\sim}{v}\underset{\sim}{v}}{v^3} \right) \frac{\partial f_e}{\partial \underset{\sim}{v}} + \text{ee collisions}$$

If $Z_{eff} >> 1$, e-e collision do not ever compete with e-i collision (although e-e collisions needed to achieve self-Maxwellization)

Thus we have

$$\frac{\partial}{\partial \underset{\sim}{v}} \cdot \left(\frac{v^2 \underset{\approx}{I} - \underset{\sim}{v}\underset{\sim}{v}}{v^3} \right) \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_e(v)$$

$$= \frac{1}{v}\left(\nabla_v^2 f_e - \frac{1}{v^2}\frac{\partial}{\partial v} v^2 \frac{\partial f_e}{\partial v}\right)$$

$$= \frac{1}{v^3}\left[\frac{\partial}{\sin\theta\,\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] f_e \equiv \frac{1}{v^3}\mathcal{L} f$$

Just the angular part of the Laplacian with eigenfunctions

$Y_{\ell m}(\theta, \phi)$ and eigen values

$-\ell(\ell+1)$

i.e. $\mathcal{L}\, Y_{\ell m}(\theta, \phi) = -\ell(\ell+1)\, Y_{\ell m}(\theta, \phi)$

This property allows us a fairly realistic way for modelling electron collisions.

Let us look at cross field transport

$$\underset{\sim}{v}\cdot\frac{\underset{\sim}{\nabla} n}{n} F_M + \omega_c \frac{\partial}{\partial \underset{\sim}{v}}\cdot\left(\underset{\sim}{v}\times\underset{\sim}{b} f^{(1)}\right) = -\hat{\nu} f$$

$$\hat{\nu}_e = -\frac{\nu_{oe}}{2}\frac{v_{the}^3}{v^3}\frac{\partial}{\partial \underset{\sim}{v}}\cdot\left(v^2 \underset{\approx}{I} - \underset{\sim}{v}\,\underset{\sim}{v}\right)\cdot\frac{\partial}{\partial \underset{\sim}{v}}$$

ordering

$$\underset{\sim}{v}\cdot\frac{\underset{\sim}{\nabla} n}{n} F_{Me} + \omega_{ce}\frac{\partial}{\partial \underset{\sim}{v}}\cdot\left(\underset{\sim}{v}\times\underset{\sim}{b}\, f_e^{(1,0)}\right) = 0$$

$$f_e^{(1,0)} = \frac{\underset{\sim}{v}\times\underset{\sim}{b}}{\omega_{ce}}\cdot\frac{\underset{\sim}{\nabla} n}{n} F_{Me}$$

$$\omega_{ce}\frac{\partial}{\partial \underset{\sim}{v}}\cdot\left(\underset{\sim}{v}\times\underset{\sim}{b}\, f_e^{(1,1)}\right) = -\hat{\nu}_e f_e^{(1,0)}$$

$$\frac{\partial}{\partial \underset{\sim}{v}}\cdot\left[(\underset{\sim}{v}\times b) f_e^{(1,1)}\right] = -\frac{m}{eB}\hat{\nu} f^{(1,0)}$$

Take $\underset{\sim}{v}\times\underset{\sim}{b}$ moment

$$\int d^3v\,(\underset{\sim}{v}\times\underset{\sim}{b})\left[\frac{\partial}{\partial \underset{\sim}{v}}\cdot(\underset{\sim}{v}\times\underset{\sim}{b}) f_e^{(1,1)}\right] = +\int d^3\underset{\sim}{v}\left(\underset{\approx}{I}\times\underset{\sim}{b}\right)\cdot(\underset{\sim}{v}\times\underset{\sim}{b}) f_e^{(1,1)}$$

$$= +\int d^3v\, \underset{\sim}{v}_\perp f_e^{(1,1)} \equiv \underset{\sim}{\Gamma}_{e\perp}^{(1)}$$

$$\equiv -\frac{m_e}{eB}\int\left(\hat{\nu} f_e^{(1,0)}\right) d^3\underset{\sim}{v}\,(\underset{\sim}{v}\times\underset{\sim}{b})$$

$$= -\frac{m}{eB}\int\hat{\nu}_e\left(\frac{\underset{\sim}{v}\times\underset{\sim}{b}}{\omega_c}\cdot\frac{\underset{\sim}{\nabla} n}{n} F_{Me}\right) d^3\underset{\sim}{v}\,(\underset{\sim}{v}\times\underset{\sim}{b})$$

Now $\frac{v \times b \cdot \nabla n}{n}$ is a

spherical Harmonic $\mathcal{L} Y_l = -l(l+1) Y_{lm}$ $(l=1)$

$$\hat{\nu}_e \, \underset{\sim}{v} \times \underset{\sim}{b} \cdot \frac{\nabla n}{n} = -\frac{\nu_{oe}}{2} \mathcal{L} \left(\underset{\sim}{v} \times \underset{\sim}{b} \cdot \frac{\nabla n}{n} \right)$$

$$= \nu_{oe} (\underset{\sim}{v} \times \underset{\sim}{b}) \cdot \frac{\underset{\sim}{\nabla} n}{n}$$

$\therefore$

$$\Gamma_{e\perp}^{(1)} = -\frac{\nu_{oe}}{\omega_{ce}^2} \int d^3 v \, F_M \, (\underset{\sim}{v} \times \underset{\sim}{b}) \, (\underset{\sim}{v} \times \underset{\sim}{b}) \cdot \frac{\nabla n}{n}$$

$$= -\frac{\nu_{oe}}{\omega_{ce}^2} V_{the}^2 \, \underset{\sim}{\nabla} n$$

From general momentum conservation relation

$$\Gamma_{e\perp}^{(1)} = \Gamma_{i\perp}^{(1)}$$