Lecture #32

Discussion of Midterm
exam

Mid term Exam

I made an error in problem 4d. I meant for you to obtain the ratio of the poloidal field at the plasma edge (separatrix) and the vertical field, with respect to the toroidal field on the magnetic axis in the $\rho = \rho_0$ (not $\rho = 0$ as written in the question). The field at the axis occurs where $\psi(\rho, \theta)$ is a maximum (Hence $B_\theta \propto \frac{\partial \psi}{\partial \rho}(\rho_0) = 0$)

poloidal field

Because of the linearity of the equation $\psi(\rho, \theta)$ can be taken with $\alpha = 1$.

//

$$\psi(\theta,\rho) = (1-\mu^2)\sqrt{\rho}\; J_{3/2}(\rho)\sqrt{\pi/2}$$

$\therefore$ $\mu = \cos\theta$

$$\psi(\theta,\rho) = \sin^2\theta\left[\frac{\sin\rho}{\rho} - \cos\rho\right]$$

The Toroidal Field is given by

$$\rho\sin\theta\, B_\phi = F = \psi = \sin^2\theta\left[\frac{\sin\rho}{\rho} - \cos\rho\right]$$

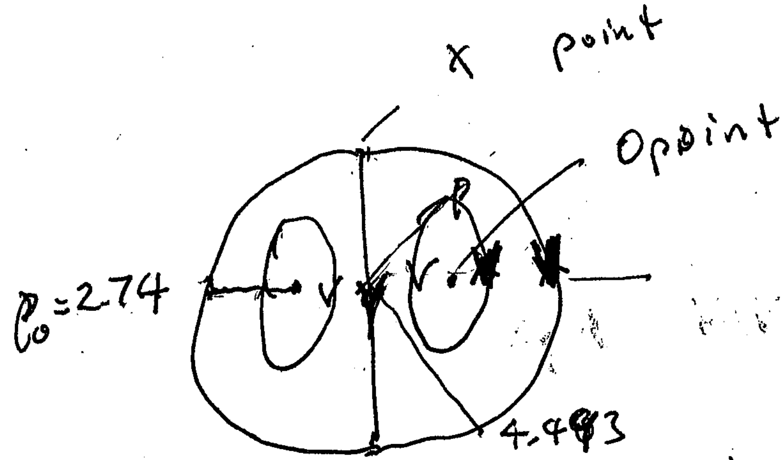
$$B_\phi = \sin\theta\left[\frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho}\right]$$

The poloidal field is $(\underset{\sim}{\nabla}\phi = \hat{\phi}/(\rho\sin\theta))$

$$\underset{\sim}{B} = \underset{\sim}{\nabla}\phi \times \underset{\sim}{\nabla}\psi$$

$$= \frac{1}{\rho\sin\theta}\,\hat{\phi}\times\left[\frac{2\sin\theta\cos\theta}{\rho}\,\underset{\sim}{\hat{\theta}}\left[\frac{\sin\rho}{\rho} - \cos\rho\right] + \sin^2\theta\left(\frac{\cos\rho}{\rho} - \sin\rho\left(\frac{1}{\rho^2}-1\right)\right)\underset{\sim}{\hat{\rho}}\right]$$

$$\boxed{\begin{aligned}\underset{\sim}{B}_\perp = &-\underset{\sim}{\hat{\rho}}\; 2\cos\theta\left[\frac{\sin\rho}{\rho^3} - \frac{\cos\rho}{\rho^2}\right] \\ &+ \underset{\sim}{\hat{\theta}}\,\sin\theta\left[\frac{\cos\rho}{\rho^2} + \frac{\sin\rho}{\rho}\left(1-\frac{1}{\rho^2}\right)\right]\end{aligned}}$$



Flux shape of spheromak

at $\theta = \frac{\pi}{2}$ there is an "O" point

$\rho = \rho_0$, where $B_\theta = 0$

Thus: $\frac{\cos \rho_0}{\rho_0} + \sin \rho_0 \left(1 - \frac{1}{\rho_0^2}\right) = 0$

$$\tan \rho_0 = \frac{-\rho_0}{\rho_0^2 - 1}$$

root is $\rho_0 = 2.743$

At this point

$$B_\phi(\rho_0) = \frac{\psi(\pi/2, \rho_0)}{\rho_0} = \frac{\sin \rho_0}{\rho_0^2} - \frac{\cos \rho_0}{\rho_0} = .388$$

Now B_θ at separatrix, ρ_s, where $(\psi \propto B_\phi = 0)$

is given by

$$\underset{\sim}{B}_\theta(\theta, \rho_s) = \underset{\sim}{\hat{\theta}} \sin\theta \left[\frac{\cos\rho_s}{\rho_s^2} + \frac{\sin\rho_s}{\rho_s}\left(1 - \frac{1}{\rho_s^2}\right)\right]$$

The position ρ_s satisfies

$\psi(\rho_s) = 0 \quad \therefore \quad \tan\rho_s = \rho_s$

$\rho_s = 4.493$

Then

$$B_\theta\left(\frac{\pi}{2}, \rho_s\right) = \left[\frac{\cos\rho_s}{\rho_s^2} + \frac{\sin\rho_s}{\rho_s}\left(1 - \frac{1}{\rho_s^2}\right)\right] = -0.217$$

If we now calculate the external field, we note that ψ that satisfies, $\underset{\sim}{\nabla} \cdot \frac{1}{R^2} \underset{\sim}{\nabla} \psi = 0$ in vacuum

$$\psi_V = \frac{B_V}{2}\left(\rho^2 - \frac{\rho_s^3}{\rho}\right)\sin^2\theta, \quad \text{where}$$

B_V is the magnitude of the constant field, and $\psi(\rho = \rho_s) = 0$, which matches boundary condition of plasma ψ

Note $\underset{\sim}{\nabla}\psi_V = \underset{\sim}{\nabla}\frac{B_V}{2}\left(\rho^2 - \frac{\rho_s^3}{\rho}\right)\sin^2\theta$

$$= B_V\,\hat{\rho}\left(\rho + \frac{1}{2}\frac{\rho_s^3}{\rho^2}\right)\sin^2\theta +$$

$$B_V\,\underset{\sim}{\hat{\theta}}\,\sin\theta\cos\theta\left(\rho - \frac{\rho_s^3}{\rho^2}\right)$$

For $\rho \gg \rho_s$

$$\underset{\sim}{\nabla}\psi \to \hat{\rho}B_V\sin^2\theta\,\rho + B_V\sin\theta\cos\theta\,\rho\,\hat{\theta}$$

$$\underset{\sim}{\nabla}\phi \times \underset{\sim}{\nabla}\psi = \frac{B_V\,\rho}{\rho\sin\theta}\left[\hat{\phi}\times\hat{\rho}\,\sin^2\theta + \hat{\phi}\times\hat{\theta}\,\sin\theta\cos\theta\right]$$

$$= B_V\left[-\hat{\rho}\cos\theta + \hat{\theta}\sin\theta\right]$$

This is a downward vertical field. To find B_V we match $\frac{\partial\psi}{\partial\rho}$ at $\rho = \rho_s$

In vacuum $\frac{\partial\psi(\rho_s)}{\partial\rho} = B_V\left(\rho_s + \rho_s/2\right)\sin^2\theta$

$= \frac{\partial\psi}{\partial\rho}(\rho_s)$ ~~for~~ plasma $= -\frac{\sin\rho_s}{\rho_s^2} + \sin\rho_s + \frac{\cos\rho_s}{\rho_s} \simeq -.879$

Hence:

$$\frac{3}{2} B_r \rho_s = -.879$$

$$B_r = -\frac{2}{3} \cdot \frac{.879}{\rho_s} = -.130 \qquad (\rho_s = 4.49)$$

Thus

$$\frac{B_r}{B_\varphi(\rho_0)} = \frac{-.130}{.388} = -.335 \approx 1/3$$

$$\frac{B_\theta(\theta = \frac{\pi}{2}, \rho_s)}{B_\varphi(\rho_0)} = \frac{-.217}{.388}$$

$$= -.559$$

Thus vertical field is downward in z-direction

To the extent $\frac{B_\varphi^2}{2}$ is like pressure

The "magnetic" beta of this system is

$$\frac{B_\varphi^2(\rho_0)}{B_r^2} = \frac{(.388)^2}{(.130)^2} = 8.91$$

This is the beta in which a vertical field can "contain" a toroidal field (6)

#2 Find T_e/T_i where damping from electrons & ion are equal

$$f_e = \frac{\exp\left(-\frac{m_e v^2}{2T_e}\right)}{(2\pi T_e/m_e)^{1/2}}, \quad f_i = \frac{\exp\left(-\frac{M_i v^2}{2T_i}\right)}{(2\pi T_i/M_i)^{1/2}}$$

$$T_e = m_e V_{the}^2 \;;\quad T_i = M_i V_{thi}^2$$

Dispersion Relation

$$0 = k^2 + \frac{\omega_{pe}^2}{(2\pi V_{the}^2)^{1/2}} \int_{-\infty}^{\infty} dv \frac{\frac{\partial}{\partial v} e^{-v^2/2V_{the}^2}}{\left(\frac{\omega}{k} - v\right)} + \frac{\omega_{pi}^2}{(2\pi V_{thi}^2)^{1/2}} \int_{-\infty}^{\infty} \frac{dv \frac{\partial}{\partial v} e^{-v^2/2V_{thi}^2}}{\left(\frac{\omega}{k} - v\right)}$$

Real part in limit $k^2\lambda_{De}^2 \ll 1$, $\frac{k^2 V_{the}^2}{\omega^2} \gg 1$, $\frac{k^2 V_{thi}^2}{\omega^2}$ ~~$\ll 1$~~, $\left(T_i/T_e\right) \ll 1$, leads to:

$$\frac{\omega^2}{k^2 V_{thi}^2} = \frac{T_e}{T_i} + 3 \equiv \tau + 3 \;;\; \tau = \frac{T_e}{T_i}$$

Imaginary part is

$$\left(\frac{\pi}{2}\right)^{1/2} \frac{\omega_{pi}^2}{V_{thi}^2} \left[\left(\frac{m_e}{m_i}\right)^{1/2} \left(\frac{T_i}{T_e}\right)^{3/2} \frac{\omega}{kV_{thi}} e^{-\frac{\omega^2 m/M}{2k^2V_{thi}^2\tau}} + \frac{\omega}{kV_{thi}} e^{-\omega^2/2k^2V_{thi}^2}\right]$$

Take $\exp\left(\frac{m}{2M}\frac{\omega^2}{\tau k^2 V_{thi}^2}\right) \doteq 1$

Two damping contributions equal when

$$\left(\frac{m_e}{M_i}\right)^{1/2} \simeq \tau^{3/2} e^{-\frac{1}{2}(\tau+3)}$$

For $\dfrac{m_e}{M_i} = \dfrac{1}{1836}$

$$2.334 \times 10^{-2} = \tau^{3/2} e^{-\left(\frac{\tau}{2}+3/2\right)} \quad , \tau = 11.9$$

(b) At this temperature, what is the critical relative velocity, between e & i for the onset of acoustic instability?

$$f_e = \frac{1}{(2\pi V_{the}^2)^{1/2}} \exp\left(-(v-u)^2/2V_{the}^2\right)$$

$$\operatorname{Im} D = \left(\frac{\pi}{2}\right)^{1/2} \frac{\omega_{pi}^2}{k^2 V_{thi}^2}\left[\left(\frac{m_e}{M_i}\right)^{1/2} \tau_0^{-3/2}\left(\frac{\omega}{kV_{thi}} - \frac{u}{V_{thi}}\right) + \frac{\omega}{kV_{thi}} e^{-\frac{\omega^2}{2k^2V_{thi}^2}}\right]$$

Because $\left(\frac{m_e}{M}\right)^{1/2} \frac{1}{\tau^{3/2}} = e^{-\omega^2/2k^2V_{thi}^2}$

$$= \left(\frac{\pi}{2}\right)^{1/2} \frac{\omega_p^2}{kV_{thi}^2}\left[\left(\frac{m_e}{M_i}\right)^{1/2} \frac{1}{\tau^{3/2}}\left(2\frac{\omega}{kV_{thi}} - \frac{u}{V_{thi}}\right)\right] = 0$$

$$\therefore \frac{u}{V_{thi}} = 2\left(\frac{\omega}{kV_{thi}}\right) = 2\left[\tau+3\right]^{1/2} = 7.7$$

$$\left(\frac{\omega}{kV_{thi}}\right)^2 = \tau + 3$$

Prob. #3

Plasma Oscillations to fourth order

$$k^2 + \omega_p^2 k \int \frac{dv \; \frac{\partial f(v)}{\partial v}}{\omega - kv} = 0$$

$$f = e^{-v^2/2v_{th}^2} / (2\pi v_{the}^2)^{3/2} = f_{max}$$

$$k^2 + \frac{\omega_p^2 k}{\omega} \int dv \left(\underset{0}{\underset{\shortparallel}{1}} + \frac{kv}{\omega} + \underset{0}{\underset{\shortparallel}{\left(\frac{kv}{\omega}\right)^2}} + \left(\frac{kv}{\omega}\right)^3 + \underset{0}{\underset{\shortparallel}{\left(\frac{kv}{\omega}\right)^4}} + \left(\frac{kv}{\omega}\right)^5 \right) \frac{\partial f}{\partial v}$$

(even $\left(\frac{kv}{\omega}\right)^6$ can be kept as integral vanishes)

$$= k^2 + \frac{\omega_p^2 k^2}{\omega^2} \int dv \left[v + \frac{k^2 v^3}{\omega^2} + \frac{k^4 v^5}{\omega^4} \right] \frac{\partial f}{\partial v}$$

$$= k^2 \left\{ 1 - \frac{\omega_p^2}{\omega^2} \left[1 + 3 \int_{-\infty}^{\infty} \frac{k^2 v^2 f}{\omega^2} dv + 5 \int_{-\infty}^{\infty} dv \, \frac{k^4 v^4 f}{\omega^4} \right\} \right.$$

$f = f_{max}$

$$0 = 1 - \frac{\omega_p^2}{\omega^2} \left[1 + \frac{3 k^2 v_{th}^2}{\omega^2} + 15 \frac{k^4 v_{th}^4}{\omega^4} \right]$$

Then

$$\omega^2 = \omega_p^2 \left[1 + \frac{3k^2 v_{th}^2}{\omega^2} + 15 \frac{k^4 v_{th}^4}{\omega^4} \right]$$

In last term $\omega^2 = \omega_p^2$ accurate enough

In middle term $\omega^2 = \omega_p^2 \left[1 + 3\frac{k^2 v_{th}^2}{\omega_p^2} \right]$

needed to keep 4th order

accuracy:

$$\therefore \frac{\omega^2}{\omega_p^2} = \left[1 + \frac{3k^2 v_{th}^2}{\omega_p^2 + 3\frac{k^2 v_{th}^2}{\omega_p^2}} + 15 \frac{k^4 v_{th}^4}{\omega_p^4} \right]$$

$$\frac{1}{\omega_p^2 + 3k^2 v_{th}^2} = \frac{1}{\omega_p^2}\left(1 - \frac{3k^2 v_{th}^2}{\omega_p^2}\right) \text{— expand}$$

$$= 1 + \frac{3k^2 v_{th}^2}{\omega_p^2} + 6\frac{k^4 v_{th}^4}{\omega_p^4}$$

$$\boxed{\frac{\omega^2}{\omega_p^2} = 1 + \frac{3k^2 v_{th}^2}{\omega_p^2} + 6\frac{k^4 v_{th}^4}{\omega_p^4}}$$

Landau

Fokker Planck ~~E~~quation from

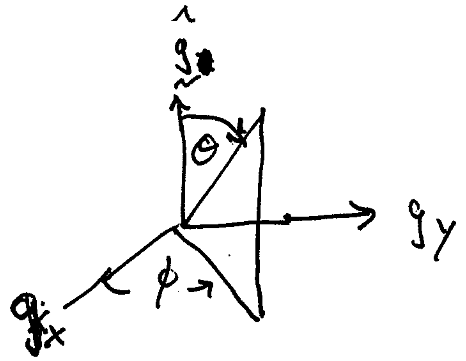
Boltzman Equation (take collisions ~~betw~~ same species)

$$\frac{d\sigma}{d\Omega} = \left(\frac{e_1^2 e_2^2}{\cancel{4}\mu^2}\right) 1/(g \sin\theta/2)^4$$

$$\frac{\partial f}{\partial t} = \left(\frac{e^2}{m}\right)^2 \int d^3v' \int d\Omega \frac{g}{g^4 \sin^4\theta/2} \left[f(\underset{\sim}{v} - \frac{\Delta g}{2}) f(\underset{\sim}{v}' + \frac{\Delta g}{2}) - f(\underset{\sim}{v}) f(\underset{\sim}{v}') \right]$$

$g = |\underset{\sim}{v} - \underset{\sim}{v}'|$

$$\underset{\sim}{\Delta g} = \underset{\sim}{g}' - \underset{\sim}{g}$$



$$\underset{\sim}{g}' = \underset{\sim}{g} \cos\theta \, \hat{z} + g \sin\theta \cos\phi \, \hat{x} + g \sin\theta \cos\phi \, \hat{y}$$

$$\sin\theta = 2\cos\frac{\theta}{2} \sin\theta/2 \quad , 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

$$\underset{\sim}{\Delta g} = 2 g \sin\frac{\theta}{2} \left[-\hat{g} \sin\frac{\theta}{2} + \hat{x} \cos\frac{\theta}{2} \cos\phi + \hat{y} \cos\frac{\theta}{2} \sin\phi \right]$$

Now expand:

$$f(\underset{\sim}{v}' + \frac{\Delta g}{2}) f(\underset{\sim}{v} - \frac{\Delta g}{2}) = f(\underset{\sim}{v}') f(\underset{\sim}{v})$$

$$+ \left[\frac{\partial f(\underset{\sim}{v}')}{\partial v'} f(\underset{\sim}{v}) - \frac{\partial f(v)}{\partial \underset{\sim}{v}} f(\underset{\sim}{v}') \right] \cdot \underset{\sim}{\Delta g}/2$$

$$+ \frac{1}{2} \left[f(v) \frac{\partial^2 f(v')}{\partial \underset{\sim}{v}' \partial \underset{\sim}{v}'} + 2 \frac{\partial f(\underset{\sim}{v})}{\partial \underset{\sim}{v}} \frac{\partial f(\underset{\sim}{v}')}{\partial \underset{\sim}{v}'} + \frac{\partial^2 f(v) f(\underset{\sim}{v}')}{\partial \underset{\sim}{v} \partial \underset{\sim}{v}} \right] : \frac{\underset{\sim}{\Delta g} \, \underset{\sim}{\Delta g}}{4} /g$$

$$\frac{\partial f}{\partial t} = (2\pi)\left(\frac{e^2}{m}\right)^2 \ln\Lambda \frac{\partial}{\partial \underset{\sim}{v}} \cdot \int d^3\underset{\sim}{v}' \left(f(v')\frac{\partial f}{\partial \underset{\sim}{v}} - f(\underset{\sim}{v})\frac{\partial f(v')}{\partial \underset{\sim}{v}'} \right) \cdot \frac{(\underset{\approx}{I} - \hat{\underset{\sim}{g}}\hat{\underset{\sim}{g}})}{g^3}$$

For arbitrary species the FP equation is given by:

$$\frac{\partial f_i}{\partial \underset{\sim}{t}} = \sum_j 2\pi \frac{(e_i e_j)^2}{m_i} \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left[\int d^3\underset{\sim}{v}' \frac{(\underset{\approx}{I} - \hat{\underset{\sim}{g}}\hat{\underset{\sim}{g}})}{g^3} \left[f_j(\underset{\sim}{v}')\frac{\partial f_i(\underset{\sim}{v})}{\partial \underset{\sim}{v}\, m_i} - \frac{\partial f_j(\underset{\sim}{v}')}{m_j \partial \underset{\sim}{v}'} f_i(\underset{\sim}{v}) \right] \right]$$

This F. P. equation has similiar properties of Boltzman equation

(a) particle conservation

(b) momentum " "

(c) energy "con.... "

(d) H- theorem leading to thermo & variational principles

Now we perform ϕ integration and term linear in $\cos\phi$ or $\sin\phi$ vanish, leaving

$$\frac{\partial f}{\partial t} = 8\pi \left(\frac{e^2}{m}\right)^2 \int d^3v' \int_{\theta_{min}}^{\theta_{max}} d\frac{\theta}{2} \cos\theta/2$$

$$\times \left[-\left(f(\underset{\sim}{v})\frac{\partial f(v')}{\partial \underset{\sim}{V}'} - f(\underset{\sim}{v'})\frac{\partial f(\underset{\sim}{v})}{\partial \underset{\sim}{V}}\right) \cdot \frac{\underset{\sim}{g}}{g^3 \sin\theta/2} \right.$$

$$+ \frac{1}{2}\left[f(\underset{\sim}{v})\frac{\partial^2 f(v')}{\partial \underset{\sim}{V}'\partial \underset{\sim}{V}'} + 2\frac{\partial f(v)}{\partial \underset{\sim}{V}}\frac{\partial f(v')}{\partial \underset{\sim}{V}'} + f(\underset{\sim}{v}')\frac{\partial^2 f(v)}{\partial \underset{\sim}{V}\partial \underset{\sim}{V}}\right]$$

$$\left. : \left(\frac{1}{2g}\left(\underset{\approx}{I} - \hat{\underset{\sim}{g}}\hat{\underset{\sim}{g}}\right)\frac{1}{\sin\frac{\theta}{2}}\right) + \mathcal{O}(\sin\theta/2) \right]$$

Now $\displaystyle\int_{\theta_{min}}^{\theta_{max}} d\frac{\theta}{2}\,\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \simeq \int_{\theta_{min}}^{\theta_{max}} \frac{d\theta}{\theta} = \ln\left(\frac{\theta_{max}}{\theta_{min}}\right) = \ln\Lambda$

$$\Lambda \simeq n\lambda_D^3 \approx 10^{10} - 10^{20}$$

If we use $\displaystyle \frac{\partial}{\partial \underset{\sim}{g}} \cdot \left(\frac{\left(\underset{\approx}{I} - \hat{g}\hat{g}\right)}{g}\right) = -2\,\underset{\sim}{g}/g$

we find:

$$\frac{\partial f}{\partial t} = \frac{2\pi e^2}{m} \ln\Lambda \frac{\partial}{\partial \underset{\sim}{v}} \cdot \int d^3v' \frac{(\underset{\sim}{I} - \hat{g}\hat{g})}{g^3} \cdot \left(f(\underset{\sim}{v}') \frac{\partial f(v)}{\partial \underset{\sim}{v}} - f(\underset{\sim}{v}) \frac{\partial f(v')}{\partial \underset{\sim}{v}'} \right)$$

For collisions with all species for species, j, the FP collision operator is found to be for $f_i(\underset{\sim}{v})$

$$\frac{\partial f_i}{\partial t} = \frac{2\pi e^2 \ln\Lambda}{m_i} \frac{\partial}{\partial \underset{\sim}{v}} \sum_j \int d^3v' \frac{(\underset{\approx}{I} - \hat{\underset{\sim}{g}}\hat{\underset{\sim}{g}})}{g^3} \cdot$$

$$\left[\frac{f_j(\underset{\sim}{v}')}{m_i} \frac{\partial f_i(\underset{\sim}{v})}{\partial \underset{\sim}{v}} - \frac{f_i(\underset{\sim}{v})}{m_j} \frac{\partial f(v')}{\partial \underset{\sim}{v}'} \right]$$

This Fokker-Planck operators retains the follows properties of the full Boltzman Eq

(1) particle conservation (2) momentum conservation

(3) energy conservation (4) H-theorem

Some applications will be discussed in the forthcoming lecture