Lecture 31

Boltzman Equation
& FP equation

Collisions

To the Vlasov Equation we need to add the effects of collisions.

$$\frac{d_V f}{dt} \equiv \frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} f + \underset{\sim}{F} \cdot \frac{\partial f}{\partial \underset{\sim}{p}}$$

= Collisional Rate in & out of phase space + Sources + radiation + Charge Exchange + Ionization

(Latter "kitchen" physics that make a discharge work)

For the most part we concentrate on the first terms, the effect of binary elastic collision due to the Coulomb interaction

We will even see that the Coulomb interaction is dominated by small angle collisions, which have far higher cross sections that large angle collisions, when one asks, which type of collisions dominate the scattering of particles, it turns out the the multiple scattering of small angle collisions, is larger by a factor $\log(n\lambda_D^3)$ from the large angle collisions

We start with the Boltzman description of collisions.

It is assumed that collisions occur on time scales short compared to the time scale of fundamental plasma processes, $t_c < (\omega_{pe}^{-1}, \omega_{ce}^{-1})$

$$\frac{d f d\Gamma}{dt} = -\text{Rate particles Leave Phase Space Volume } d\Gamma$$

$$+ \text{Rate particles Enter Phase Space Volume } \delta\Gamma$$

Remember $\frac{d\Gamma}{dt} = 0$,

Further on the spatial scale of the variation of ~~the~~ macroscopic distribution, the distribution is constant in space over the range of interaction of the collisions

In a binary elastic collision between two particles, ~~interacting~~ interacting via a central potential, energy and momentum conservation leads to the following relations between initial and final states of particle one $(m_1, \underset{\sim}{V_{1i}})$ and particle two $(m_2, \underline{V_{2i}})$ with final velocities $\underset{\sim}{V_{1f}}$ and $\underline{V_{2f}}$ (2)

$$\underset{\sim}{V}_{1f} = \bar{\underset{\sim}{V}} - \frac{m_2}{m_1+m_2}(\underset{\sim}{V}_{1i} - \underset{\sim}{V}_{2i}) \quad , \quad \bar{\underset{\sim}{V}} = \frac{m_1 \underset{\sim}{V}_1 + m_2 \underset{\sim}{V}_2}{m_1 + m_2}$$

$$\underset{\sim}{V}_{2f} = \bar{\underset{\sim}{V}} + \frac{m_1}{m_1+m_2}(\underset{\sim}{V}_{1i} - \underset{\sim}{V}_{2i})$$

$$|\underset{\sim}{V}_{1i} - \underset{\sim}{V}_{2i}| = |\underset{\sim}{V}_{1f} - \underset{\sim}{V}_{2f}| \equiv |\underset{\sim}{g}| \qquad (\underset{\sim}{g} = \underset{\sim}{V}_1 - \underset{\sim}{V}_2)$$

The change $\underset{\sim}{g}_f - \underset{\sim}{g}_i \equiv \Delta \underset{\sim}{g}$

It follows from elementary dynamics

that $\dfrac{\partial(\underset{\sim}{V}_1, \underset{\sim}{V}_2)}{\partial(\underset{\sim}{V}_{1f}, \underset{\sim}{V}_{2f})} = 1$ (i dropped)

The rate of colliding out of a phase space volume ($\Delta \underset{\sim}{r}$ can be ignored and collisions occur at the same macroscopic point of space)

$$\Delta \underset{\sim}{r}_1 \, d\underset{\sim}{P}_1 \int d\underset{\sim}{P}_2 \, f_1(\underset{\sim}{r}_1, \underset{\sim}{P}_1) \, f_2(\underset{\sim}{r}_2, \underset{\sim}{P}_2) \int \frac{d\sigma}{d\Omega}(|\underset{\sim}{g}|, \theta, \phi) \, |\underset{\sim}{g}| \, d\Omega$$

The rate of scattering into a volume element $\Delta \underset{\sim}{r}_1 \, \Delta P_1$ is

$$\Delta r_1 \, d\underset{\sim}{P}_{1f} \int d\underset{\sim}{P}_{2f} \, f_1(\underset{\sim}{r}_1 P_{1f}) \, f_2(\underset{\sim}{r}_2, P_{2f}) \int \frac{d\sigma}{d\Omega}(|g|, \theta, \phi) \, |g| \, d\Omega$$

(note the same cross-sections due to time reversal invariance of this process)

But $\quad dp_{1f}\, dp_{2f} = dp_1\, dp_2$

$\mathbf{p}_{1f} = M\bar{\mathbf{V}} - \mu \mathbf{g}$ $\qquad \mu = \dfrac{m_1 m_2}{m_1 + m_2} \equiv$ reduced mass

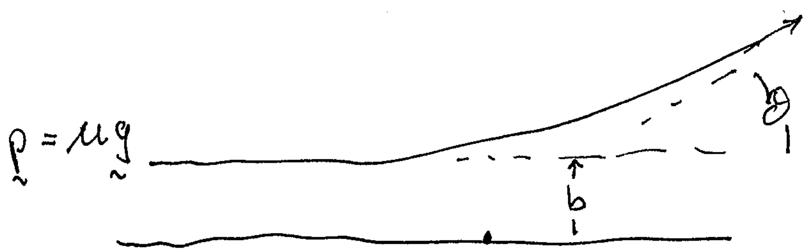
$\mathbf{p}_{2f} = M\bar{\mathbf{V}} + \mu \mathbf{g}$ $\qquad \bar{p} = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2$

This leads to the relation ("1" unsubscripted)
$1 \to$ species i, 2 species j ("2" prime)

$$\frac{df_i}{dt} = \sum_j \int d^3v' \left[-f_i(\mathbf{r}, \mathbf{v})\, f_j(\mathbf{r}, \mathbf{v}') + f_i(\mathbf{r}, \bar{V} - \tfrac{\mu}{m_1}\mathbf{g})\, f_j(\mathbf{r}, \bar{V} + \tfrac{\mu}{m_2}\mathbf{g}) \right]$$
$$\int d\Omega\, \frac{d\sigma}{d\Omega}(\mathbf{g}; \phi, \theta)\; g\,.$$

(define $g = |\mathbf{g}|$)

For coulomb collision



$$d\sigma = d\phi\, b\, db \qquad \boxed{b = b_0 \cot\frac{\theta}{2}, \quad b_0 = \frac{e_1 e_2}{\mu g^2}}$$

$$= \frac{b_0^2\, d\Omega}{4 \sin^4(\theta/2)} \qquad (b_0 \equiv \text{impact parameter for } 90^\circ \text{ scattering})$$

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

note for small scattering angles ($\sin\theta \simeq \theta$)

$$d\sigma = \frac{4b_0^2}{\theta^3}\, d\theta\, d\phi = \left(\frac{e_1 e_2}{g^2}\right)^2 \frac{d\theta\, d\phi}{\theta^3}$$

Why does small angle dominate scattering?

To illustrate,

Calculate for a test particle

$\frac{d|\Delta q_\perp|^2}{dt}$, where scattering particles are infinitely massive compared to test particle, of speed v,

$|\Delta v_\perp|^2 = v^2\sin^2\theta \simeq v^2\theta^2$

$$\frac{d|\Delta q_\perp|^2}{dt} = \frac{d|\Delta v_\perp|^2}{dt} = n v \int |\Delta v_\perp|^2 \frac{d\sigma}{d\Omega}\, d\Omega$$

$$\simeq 4 n v^3 b_0^2 \int_{\theta_{min}}^{\theta_{90^\circ}} 2\pi \frac{\theta^2\, d\theta}{\theta^3}$$

$\theta_{min} \simeq \frac{2b_0}{b_{max}} \simeq \frac{2b_0}{\lambda_D}$ (coulomb potential shield by plasma for $b > b_{max}$)

$\simeq \frac{1}{n\lambda_D^3}$ ($\lambda_D \equiv$ debye length)

$$\therefore \frac{d|\Delta v_\perp|^2}{dt} \simeq 8\pi v^3 b_0^2 \ln\left(\frac{\theta_{max}}{\theta_{min}}\right) = 8\pi v^3 b_0^2 \ln\Lambda$$

$\ln\Lambda \simeq 10 - 20 \rightarrow$ this factor significant

We will primarily deal with the Fokker-Planck equation, which is the Boltzman equation when only small angle deflections are taken into account (first derived by Landau)

However, the Boltzman equation has several basic properties that are retained by the Fokker Planck Equation derived from it.

(1) Conservation of particles

$$\frac{df}{dt} = -\hat{\underset{\sim}{\nu}} f \qquad |\hat{\nu}| \simeq \frac{\omega_{pe}}{(n\lambda_D^3)} \ln\Lambda \begin{pmatrix} 1 \\ \left(\frac{m}{M}\right)^{1/2} \\ \frac{m}{M} \end{pmatrix}$$

or

$$\frac{\partial f}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla} f + \frac{\underset{\sim}{F}}{m}\cdot\underset{\sim}{\nabla_v} f = -\hat{\nu} f$$

$$\int d^3v \frac{df}{dt} = \int d^3v \left(\frac{\partial f}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla} f + \frac{\underset{\sim}{F}}{m}\cdot\nabla_v f\right) = -\int \hat{\nu} f d^3v = 0$$

$\int \hat{\nu} f \, d^3\underset{\sim}{v} = 0$ because collisions conserve particles

$$\int d^3v \frac{\partial f_j}{\partial t} = \frac{\partial n_j}{\partial t}; \quad \int d^3v \underset{\sim}{v}\cdot\underset{\sim}{\nabla} f_j = \underset{\sim}{\nabla}\cdot\int d^3v \underset{\sim}{v} f_j = \underset{\sim}{\nabla}\cdot(n_j \underset{\sim}{u}_j)$$

$$\frac{\partial n_j}{\partial t} + \underset{\sim}{\nabla}\cdot(n_j \underset{\sim}{u}_j) = 0 \quad \text{(continuity equation)}$$

(2) momentum conservation

$$\sum_j \hat{\nu} \int m_j \, v_j f_j \, d^3v = 0$$

This condition leads fluid momentum equation, explicitly independent of collision operator

Let ρ_m be mass density

$$\rho_m \bar{u} = \sum_j m_j \int d^3v \, v \, f_j(v)$$

$$\rho_m \frac{\partial u}{\partial t} + \rho_m \, u \cdot \nabla u = \text{External Force Density} - \nabla \cdot \mathbb{P}$$

$\mathbb{P} \equiv$ momentum flux (or pressure tensor)

$$\equiv \sum_j \int d^3v \, m_j (v - u_j)(v - u_j) f$$

(3) energy conservation

$$\sum_j \hat{\nu} \int m_j \nu_j \, f_j \frac{v_j^2}{2} = 0$$

leads to relation between mean energy density and heat flux.

These properties the basis of fluid models

Most significant the H- theorem emerges from the Boltzman Equation

$H = - S/\varkappa$ $\varkappa \equiv$ Boltzman's constant $(=1)$

$S =$ entropy

$H = \langle \ln f(1) \rangle$ (take 1 species colliding with itself)

$= \int d^3 v_1 \, f(1) \ln f(1)$

(locally spatially homogeneous)

$$\frac{dH}{dt} = \int d^3 v_1 \frac{\partial f(1)}{\partial t} \left[\ln f(1) + 1 \right]$$

$$= - \int d^3 v_1 \left[\ln f(1) + 1 \right] \hat{\nu} f$$

$$= - \int d^3 v_1 \, d^3 v_2 \, d\Omega \ln f(v_1) \left[f(1) f(2) - f(1_f) f(2_f) \right] g \frac{d\sigma}{d\Omega}$$

Property of Boltzman Equation for arbitrary $\psi(\underset{\sim}{v})$

$$\int d^3 v \, \psi(v) \; \hat{\nu} f(\underset{\sim}{v}) =$$

$$= \frac{-1}{4} \int d^3 v \int d^3 v' \, d\Omega \left[\psi(\underset{\sim}{v}) + \psi(\underset{\sim}{v}') - \psi(\underset{\sim}{v}_f) - \psi(\underset{\sim}{v}_f') \right]$$

$$\cdot \frac{d\sigma}{d\Omega} g \left[f(\underset{\sim}{v}) f(\underset{\sim}{v}') - f(\underset{\sim}{v}_f) f(\underset{\sim}{v}_f') \right]$$

$$\int d^3v \; \psi(\underset{\sim}{v}) \; \hat{\nu} f(\underset{\sim}{v})$$

$$= -\frac{1}{4} \int d^3\underset{\sim}{v} \, d^3\underset{\sim}{v}' \, d\Omega \left[\psi(\underset{\sim}{v}) + \psi(\underset{\sim}{v}') - \psi(\underset{\sim}{v}_f) \, \psi(\underset{\sim}{v}_f') \right]$$

$$\cdot \left[f(\underset{\sim}{v}_1) f(\underset{\sim}{v}_2) - f(\underset{\sim}{v}_1') f(\underset{\sim}{v}_2') \right] \frac{d\sigma}{d\Omega}(g; \theta, \phi) \, g$$

Therefore:

$$\frac{dH}{dt} = -\frac{1}{4} \int d^3\underset{\sim}{v} \, d^3\underset{\sim}{v}' \, d\Omega \left[\ln \left[\frac{f(\underset{\sim}{v}) \; f(\underset{\sim}{v}')}{f(\underset{\sim}{v}_f) \; f(\underset{\sim}{v}_f')} \right] \left(f(1) f(1') - f(1_f) f(1_f') \right) \right.$$

$$\cdot \frac{d\sigma}{d\Omega}(g; \theta, \phi) \; g$$

Now: $\ln \left[\dfrac{f(\underset{\sim}{v}) \, f(\underset{\sim}{v}')}{f(\underset{\sim}{v}_f) \, f(\underset{\sim}{v}_f')} \right\} \left[f(\underset{\sim}{v}_1) f(v_2) - f(\underset{\sim}{v}_{1f}) f(v_{2f}) \right\} \geq 0$

for any $f(v) \geq 0$

$$\therefore \quad \frac{dH}{dt} \leq 0 \qquad \left(\text{or } \frac{dS}{dt} \geq 0 \right)$$

Hence; Boltzman collision operator has the property that as the system evolves the entropy always increases!

If we ask for what type of distribution can have $\hat{\nu} \hat{f} = 0$, which implies

$\frac{dH}{dt} = 0$; we minimize H

subject to the contraint of particle, momentum and energy conservation, and we find that only flowing Maxwell distributions can lead to

$$\frac{dH}{dt} = 0$$

$$f = \frac{n_0 \exp\left(-\frac{m(\underset{\sim}{v}-\underset{\sim}{u})^2}{T}\right)}{\left(\frac{2\pi T}{m}\right)^{3/2}} \equiv f_M$$

Further consequence:

If we impose a small thermodymic force (TF) on a Maxwellian distribution ($TF's \equiv e\underset{\sim}{E}$, $\frac{\partial \underset{\sim}{n}}{\partial \underset{\sim}{r}}$, $\frac{\partial T}{\partial \underset{\sim}{r}}$, etc)

the resulting stationary distribution is the one that maximizes the entropy production, in the presence of (10) these forces. The transport coefficients can be expressed as a variational principle, and this variational principle is useful in calculating transport coefficients and demonstrates the Onsager relations (e.g. if electric fields produces heat fluxes, then temperature gradients produce electrical current flows, with a common coupling factor for these "non-diagonal" processes)

The above is a nice background of basic near thermo equilibrium processes. In plasmas we usually simplify the Boltzman equation to small angle collisions (for reasons cited) and this allows us an easier (but not easy) ~~form to calculate~~ kinetic evolution and transport processes.