Lecture #30

Z_J modes

In previous lecture we discussed drift waves from kinetic point of view.

It is insightful to consider drift waves from fluid viewpoint

We take electrons as Maxwell Boltzman response

$$n_e = n_0(x)\, e^{+e\phi/T} \simeq n_0(x)\left(1+\frac{e\phi}{T}\right) \quad (e>0)$$

If Ions are treated as having only cross field dynamics $\left(\frac{\omega}{k_\parallel v_{thi}} >> 1\right)$ as described by $\underset{\sim}{E}\times\underset{\sim}{B}$ motion

$$\frac{\partial n_i}{\partial t} = -\underset{\sim}{\nabla}\cdot(n_i \underset{\sim}{v}) \;;\quad \underset{\sim}{v} = \underset{\sim}{v_E} = -\underset{\sim}{\nabla}\phi\times\underset{\sim}{b}c/B - \frac{\partial}{\partial t}\frac{\nabla\phi}{\omega_{ci}B}$$

In homogeneous field $\underset{\sim}{\nabla}\cdot\underset{\sim}{v_E} = \underset{\sim}{\nabla}\cdot\left(\frac{\underset{\sim}{\nabla}\phi\times\underset{\sim}{b}c}{B}\right) = 0$

$$\parallel$$

$$(\nabla\times\nabla\phi)\cdot b/B$$

$$\therefore\ \frac{\partial n_i}{\partial t} = c\frac{\partial n_0}{\partial x}\frac{b\times\nabla\phi}{B}\cdot\hat{x} + c\nabla^2\frac{\partial\phi}{\partial t}\frac{n_0}{\omega_{ci}}$$

If $\phi = \hat{\phi}\exp(i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega t)$, $\delta n_i \propto e^{-i\omega t}$

$$-i\omega\delta n_i = -i\frac{k_y c}{B}\phi\frac{\partial n_0}{\partial x} + i k_\perp^2 n_0\frac{\omega}{\omega_{ci}}\phi$$

$$\delta n_i = \frac{c\phi}{\omega B}\left(\frac{\partial n_0}{\partial x}k_y - \frac{\omega k_\perp^2}{\omega_{ci}}\right), \quad \delta n_e = +\frac{e\phi}{T_e}n_0$$

Quasi-neutrality condition is

$$\delta n_i = \delta n_e$$

$$-\frac{k_\perp^2 n_0 \phi}{\omega_{ci} B} + \frac{k_y c \phi}{\omega B}\frac{\partial n_0}{\partial x} = +\frac{e\phi}{T_e} n_0 \;;\quad \therefore\; \omega = \frac{\omega_e^*}{(1 + k_\perp^2 \rho_s^2)}$$

$$\cancel{\text{or } \omega = \frac{T_e k_y}{eBn_0}\frac{\partial n_0}{\partial x} - \frac{\omega k_\perp^2 n_0 \rho_s^2}{\omega_{ci}} \quad \omega_e^*}$$

$$\omega_e^* = -\frac{k_y T_e}{\omega_{ce} m_e}\frac{\partial n_0}{n_0 \partial x} = -\frac{k_y v_{the}^2}{\omega_{ce}}\frac{\partial n_0}{n_0 \partial x}$$

$$\simeq k_y \rho_e \frac{v_{the}}{L_p} \simeq k_y \rho_s \frac{v_{ths}}{L_p}$$

$$L_p^{-1} = \frac{\partial n_0}{n_0 \partial x}, \quad \rho_s = \rho_i \left(\frac{T_e}{T_i}\right)^{1/2}, \quad v_{ths} = v_{thi}\left(\frac{T_e}{T_i}\right)^{1/2}$$

We saw that when we treated the kinetic problem, we obtained the dispersion relation (when $k_\perp \rho_i \ll 1$, $k_z v_{thi}/\omega \ll 1$, $k_z v_{the}/\omega \gg 1$; $k\lambda_{De} \ll 1$) $F_{0j} = F_{Mj}(T_j)$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$0 = \left(1 + i\pi \frac{(\omega - \omega_e^*(1-\eta_e))}{|k_\parallel| v_{the}}\right) + k_\perp^2 \rho_s^2 - \frac{\omega_e^*}{\omega}$$

$$- \frac{T_e}{T_i}\int dv_z \left[\frac{k_z v_z\left(1 - \frac{\omega_i^*}{\omega}\left(1 + \frac{\eta_i}{2}\left(\frac{v_z^2}{v_{thi}^2} - 1\right)\right)\right)}{\omega - k_z v_z} \times \frac{\exp(-v_z^2/2v_{thi}^2)}{(2\pi v_{thi}^2)^{1/2}}\right]$$

$$\rho_s^2 = \rho_i^2 \frac{T_e}{T_i}$$

The drift wave frequency emerges from balance of two "arrowed" terms

Simple Fluid Theory

$$\omega = -\frac{\omega_i^* \frac{T_e}{T_i}}{1+k_\perp^2\rho_s} = \omega_e^* / (1 + k_\perp^2 \rho_s^2)$$

Frequency downshifts due to $(k_\perp \rho_i)^2$ term, and upshifts due to finite k_z in ion term

In kinetic theory

$$\omega = \omega_e^* + \delta\omega^{(1)} + \delta\omega^{(2)}$$

$$\frac{\delta\omega^{(1)}}{\omega_e^*} = -k_\perp^2\rho_s^2 + \frac{k_z^2 V_{ths}^2}{\omega_e^{*2}}\left(1 + \frac{T_i}{T_e}(1+\eta_i)\right)$$

Further, frequency shift ~~due to~~ gives electron dissipative term

$$\frac{\delta\omega^{(2)}}{\omega_e^*} = \frac{-i\pi\left(\omega - \omega_e^*(1+\eta_e)\right)}{|k_\parallel| V_{the}} = \frac{i\pi\,\omega_e^*}{k_\parallel V_{the}}\left[k_\perp^2\rho_s^2 - \eta_e\right]$$

η_e stabilizes this drift wave mode!

But temperature gradient rears 'ugly' head in another mode.

We can see this effect in fluid theory

Let us look at fluid equation accounting for ion parallel motion, and assuming pressure is determined by adiabatic law

$$\frac{d}{dt}\left(p_i n_i^{-\gamma}\right) = 0 \qquad ; \; \gamma = 5/3$$

$$\frac{\partial}{\partial t}\left(p_i n^{-\gamma}\right) + \underset{\sim}{v}\cdot\nabla\left(p_{i0} n_0^{-\gamma}\right) = 0$$

$$\searrow \; T_{i0} n_0^{1-\gamma}$$

$$\underset{\sim}{v}_i = c\frac{\underset{\sim}{b}\times\nabla\phi}{B} + \frac{c\,\underset{\sim}{\nabla}\frac{\partial\phi}{\partial t}}{B\omega_{ci}} + v_\parallel \underset{\sim}{b}$$

$$\cancel{\underset{\sim}{V} = \underset{\sim}{V}_E + V_P = 0}$$

$\therefore$

$$-i\omega\left(\frac{\delta p_i}{p_i} - \frac{5}{3}\frac{\delta n_i}{n_0}\right) - i\frac{k_\theta\tilde{\phi}}{B}\left(\frac{dT_{i0}/dr}{T_i} - \frac{2}{3}\frac{dn_{i0}}{n_i\,dr}\right) = 0$$

We also consider parallel flow:

$$n m_i \frac{\partial v_\parallel}{\partial t} = -\underset{\sim}{\nabla}_\parallel p_i - en\nabla_\parallel\phi \qquad (e>0)$$

$$-i\omega v_\parallel = -ik_\parallel\left(e\phi + \frac{\delta p_\parallel}{n_0}\right)\frac{1}{m}$$

& ion continuity equation

$$\frac{\partial n_i}{\partial t} + \underset{\sim}{\nabla}\cdot\left(n\underset{\sim}{v}_i\right) = 0$$

$$-i\omega\,\delta n_i + ik_\parallel\,\delta v_i\,n_0 - \frac{ick_y\phi}{B}\frac{\partial n_0}{\partial x} + \nabla^2\frac{c\,n_0}{\omega_{ci}}\frac{\partial\phi}{\partial t} = 0$$

We again use that electrons are Maxwellian, so that

$$\frac{\tilde{n}_e}{n_0} = \frac{e\phi}{T_e}$$

We then find

$$\frac{\delta n_i}{n_0} = \frac{V_{\parallel}}{\omega/k_{\parallel}} + \frac{e\phi}{T_e}\left(\frac{\omega_e^*}{\omega} + k_\perp^2\rho_s^2\right); \quad \boxed{\frac{\delta n_e}{n_0} = \frac{e\phi}{T_e}}$$

$$V_{\parallel} = \frac{1}{m_i\,\omega/k_{\parallel}}\left(e\phi + \frac{\delta p_i}{n_0}\right); \quad \frac{\delta p_i}{p_{i0}} = \frac{5}{3}\frac{\delta n_i}{n_0}$$

$$= \frac{\omega_e^*}{\omega}\left(\eta_i - \frac{2}{3}\right)\frac{e\phi}{T_e}$$

Dispersion Relation found from quasi-neutrality

$$\delta n_i = \delta n_e$$

After some algebra we find dispersion relation ($k_\perp^2\rho_s^2 \ll 1$

$$-k_\perp^2\rho_s^2 + 1 - \frac{\omega_e^*}{\omega} - \frac{V_{thi}^2 k_{\parallel}^2}{\omega^2}\left[\frac{T_e}{T_i} + \frac{5}{3} + \frac{\omega_e^*}{\omega}\left(\eta_i - \frac{2}{3}\right)\right] = 0$$

This dispersion relation gives ($\frac{k_{\parallel}V_{thi}}{\omega} \ll 1$) drift wave without FLR effects.

$$\omega \simeq \omega_e^* + k_\parallel^2 C_s^2 + k_\parallel^2 V_{thi}^2 (\eta_i - \tfrac{2}{3}) - k_\perp^2 \rho_s^2 \omega_e^*$$

There is also a low frequency mode $\frac{\omega^*}{\omega} >> 1$

keeping just those large terms

$$-\frac{\omega_e^*}{\omega}\left[1 + \frac{k_\parallel^2 V_{thi}^2}{\omega^2}(\eta_i - \tfrac{2}{3})\right] \simeq 0$$

$$\omega^2 = -k_\parallel^2 V_{thi}^2 (\eta_i - \tfrac{2}{3})$$

Purely growing mode if $\eta_i > \frac{2}{3}$.

How does fluid theory compare with kinetic theory

First take kinetic theory under assumption $k_z V_{thi}/\omega << 1$.

In contrast $\left(k_\perp^2 \rho_i^2 < < 1\right)$ for kinetic theory

$$1 + i\pi \frac{(\omega - \omega_e^*)(1-\eta_e)}{|k_\parallel| v_{the}} + k_\perp^2 \rho_s^2 - \frac{\omega_e^*}{\omega}$$

$$+ \frac{T_e}{T_i} \int dv_z \frac{k_z v_z \left(1 - \frac{\omega_i^*}{\omega}\left(1 + \frac{\eta}{2}\left(\frac{v_z^2}{v_{thi}^2} - 1\right)\right)\right)}{\omega - k_z v_z} \frac{e^{-v_z^2/2v_{the}^2}}{(2\pi v_{the}^2)^{1/2}}$$

$$= 0$$

If we neglect dissipative terms and assume $\frac{k_z v_{the}}{\omega} << 1$, the last term becomes

$$-\frac{T_e}{T_i} \int dv_z \frac{k_z v_z}{\omega}\left(1 + \frac{k_z v_z}{\omega} + \left(\frac{k_z v_z}{\omega}\right)^2 + \left(\frac{k_z v_z}{\omega}\right)^3 + \cdots\right) \frac{e^{-v_z^2/2v_{thi}^2}}{(2\pi v_{the}^2)^{1/2}}$$

$$\cdot \left(1 - \frac{\omega_i^*}{\omega}\left(1 + \frac{\eta_i}{2}\left(\frac{v_z^2}{v_{th}^2} - 1\right)\right)\right)$$

$$= -\frac{T_e}{T_i}\left(\frac{k_z^2 v_{thi}^2}{\omega^2} + \frac{3 k_z^4 v_{thi}^4}{\omega^4}\right)\left(1 -\right.$$

$$= -\frac{k_z^2 v_{thi}^2}{\omega^2}\left(\frac{T_e}{T_i} + \frac{\omega_e^*}{\omega}(1 + \eta_i)\right)$$

$$- 3\frac{k_z^4 v_{thi}^4}{\omega^4}\left(\frac{T_e}{T_i} + \frac{\omega_e^*}{\omega}(1 + 2\eta_i)\right)$$

Fluid:

$$-k_\perp^2\rho_s^2 + 1 - \frac{\omega_e^*}{\omega} - \frac{k_z^2 V_{thi}^2}{\omega^2}\left[\frac{T_e}{T_i} + \frac{5}{3} + \frac{\omega_e^*}{\omega}\left(\eta_i - \frac{2}{3}\right)\right]$$

Kinetic (without imaginary term) $\left(\begin{matrix} k_\parallel V_{thi}/\omega \ll 1 \\ \text{assumed}\end{matrix}\right)$

$$-k_\perp^2\rho_s^2 + 1 - \frac{\omega_e^*}{\omega} - \frac{k_z^2 V_{thi}^2}{\omega^2}\left[\frac{T_e}{T_i} + \frac{\omega_e^*}{\omega}(1+\eta_i)\right]$$

$$-3\frac{k_z^4 V_{thi}^4}{\omega^4}\left[\frac{T_e}{T_i} + \frac{\omega_e^*}{\omega}(1+2\eta_i)\right] + \ldots = 0$$

(1) Kinetic term has additional $-k_\perp^2\rho_s^2$ that downshifts $\omega \simeq \omega_e^*$ mode, so does fluid if Polarization Drift Included

(2) The $3\frac{k_z^4 V_{thi}^4}{\omega^4}$ term only replicates $\gamma = 3$ if we can iterate in ω^2 (valid if $T_e/T_i \ggg 1$)

(3) η_i mode (coming from balance of $\frac{\omega_e^*}{\omega}$ terms) is $\omega^2 = -k_\parallel^2 V_{thi}^2\left(\eta_i - \frac{2}{3}\right)$ fluid

$\omega^2 = -k_\parallel^2 V_{thi}^2(\eta_i + 1)$ kinetic

However, $\eta_i > \frac{2}{3}$ required for instability in fluid, while no threshold, η_i, in kinetic expansion

$\gamma = \eta^{1/2} k_\parallel V_{the}$ agrees for large η.

However, kinetic expansion only valid if $\frac{k_\parallel V_{thi}}{\omega} \gg 1$

Hence threshold condition requires analysis without expansion.

For $k_\perp \rho_i \ll 1$ we find, $\left(\frac{\omega_i^*}{\omega} \gg 1\right)$

$$1 + \int \frac{dv_z\, e^{-v_z^2/2v_{thi}^2}}{(2\pi v_{thi}^2)^{1/2}} \frac{\left(1 + \frac{\eta_i}{2}\left(\frac{v_z^2}{v_{thi}^2} - 1\right)\right) k_z v_z}{\omega - k_z v_z} = 0$$

This equation may admit a purely growing mode $\omega = i\gamma$

$$\frac{k_z v_z}{\omega - k_z v_z} = \frac{k_z v_z(-i\gamma + k_z v_z)}{(i\gamma - k_z v_z)(-i\gamma + k_z v_z)}$$

$$x = \frac{v_z}{v_{thi}} \qquad = - \frac{(k_z v_z)^2 - i\gamma k_z v_z}{\gamma^2 + k_z^2 v_z^2}$$

$\leftarrow$ odd will integrate to zero

$$1 - \int_{-\infty}^{\infty} \frac{dx}{(2\pi)^{1/2}} \frac{e^{-x^2/2}\, x^2}{\left(\frac{\gamma}{k_z v_{thi}}\right)^2 + x^2} \left(1 + \frac{\eta_i}{2}(x^2 - 1)\right) = 0$$

$$0 \simeq 1 - \int_{-\infty}^{\infty} \frac{dx}{(2\pi)^{1/2}} e^{-x^2/2} \frac{\left[\left(x^2 + \left(\frac{\gamma}{k v_{thi}}\right)^2\right) - \left(\frac{\gamma}{k v_{thi}}\right)^2\right]}{\left(\frac{\gamma}{k v_{thi}}\right)^2 + x^2}\left(1 + \frac{\eta_i}{2}(x^2-1)\right)$$

now:

$k \equiv k_z$

integrates to unity

$$\left(\frac{\gamma}{k v_{thi}}\right)^2 \int_{-\infty}^{\infty} \frac{dx}{(2\pi)^{1/2}} \frac{e^{-x^2/2}}{\left(\frac{\gamma}{k v_{thi}}\right)^2 + x^2} \left[1 + \frac{\eta_i}{2}(x^2 - 1)\right] = 0$$

cancels:

Let $\hat{\gamma} = \gamma / k v_{thi}$: We find approximately

$$\int_{-\infty}^{\infty} \frac{dx}{(2\pi)^{1/2}} \frac{1 - \frac{\eta_i}{2}}{\left(\frac{\gamma}{k v_{thi}}\right)^2 + x^2} + \frac{\eta_i}{2} \int_{-\infty}^{\infty} \frac{dx}{(2\pi)^{1/2}} e^{-x^2/2}$$

$$+ \int_{-\infty}^{\infty} \frac{dx}{(2\pi)^{1/2}} \frac{\left(e^{-x^2/2} - 1\right)}{x^2} (1 - \eta_i/2)$$

or for $\eta_i \approx 2$

$$\left(\frac{\pi}{2}\right)^{1/2} \frac{\left(1 - \frac{\eta_i}{2}\right)}{\hat{\gamma}} + \frac{\eta_i}{2} = 0$$

$$\boxed{\hat{\gamma} = \left(\frac{\pi}{2}\right)^{1/2} (\eta_i - 2)}$$

Thus for $k\rho_i \ll 1$, $\eta_i > 2$ gives instability
For finite $k\rho_i$ critical η_i is ≈ 1.5