Lecture 3

Orbit Theory

Particle Orbits in a $\underset{\sim}{E}$ & $\underset{\sim}{B}$ - field

$E << B$ (cgs units)

$$\frac{d\underset{\sim}{v}}{dt} = \frac{e}{m}\left(\underset{\sim}{E}(\underset{\sim}{r},t) + \frac{\underset{\sim}{v}}{c}\times\underset{\sim}{B}(\underset{\sim}{r},t)\right)$$

$$\frac{d\underset{\sim}{r}}{dt} = \underset{\sim}{v}$$

Suppose $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are uniform in space, $\underset{\sim}{E}\cdot\underset{\sim}{b} = 0$ and time independent

$$\boxed{\frac{d\underset{\sim}{v}}{dt} = \frac{e}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{v}}{c}\times\underset{\sim}{B}\right)}$$

We can go to frame where $\underset{\sim}{E}_\perp = 0$

$$\boxed{\underset{\sim}{v} = \underset{\sim}{u} + c\underset{\sim}{E}\times\underset{\sim}{b}/B} = \underset{\sim}{u} + \underset{\sim}{v}_E\,; \quad \underset{\sim}{v}_E = \frac{c\underset{\sim}{E}\times\underset{\sim}{b}}{B}$$

$$\frac{d\underset{\sim}{v}}{dt} = \frac{d\underset{\sim}{u}}{dt} = \frac{e}{m}\left(\underset{\sim}{E} + c\frac{(\underset{\sim}{E}\times\underset{\sim}{b})}{B}\times\underset{\sim}{B}/c + \underset{\sim}{u}\times\underset{\sim}{B}/c\right)$$

$$(E \times b)\times B = b\,\underset{0}{\underbrace{E\cdot B}} - E_\perp |B|$$

$\therefore$

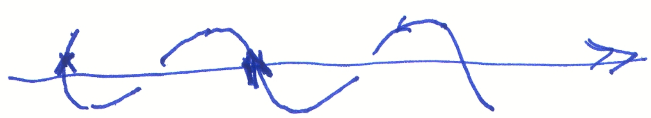
$$\frac{du}{dt} = \frac{e}{m}\left[E - \frac{\cancel{c}}{\cancel{B}} E_\perp + \frac{u}{c}\times B\right]$$

$$= \frac{e}{m}\,\frac{u \times B}{c}$$

Thus in frame $v = c\,\frac{E \times b}{B} \equiv V_E$ electric field does not appear

$$\boxed{\frac{du}{dt} = \frac{e}{mc}\, u \times b}$$

We know that motion is 'helical' around magnetic field



B, $B \parallel \hat{z}$

$u\cdot b = u_\parallel = \text{const.}$

$\rho_L = \frac{u_\perp}{\omega_c}$

$\omega_c = \frac{eB}{mc}$

gyration clockwise for positive charge particle, as B-field points to us

$$u_\perp = u_{\perp 0}\left[\cos(\phi - \omega_c t)\,\hat{x} + \sin(\phi - \omega_c t)\,\hat{y}\right]$$

$$\rho_L(t) = \rho_{L0}\left[-\sin(\phi - \omega_c t)\,\hat{x} + \cos(\phi - \omega_c t)\,\hat{y}\right]$$

$u_\parallel \equiv u\cdot b \equiv \text{constant}$; $\quad z = z_0 + u_\parallel t$

$$\frac{d\underset{\sim}{u}}{d\underset{\sim}{t}} = \omega_c \, \underset{\sim}{u} \times \underset{\sim}{b}$$

Particle gyrate about a guiding center

$$\boxed{R_g(t) = \underset{\sim}{r} - \frac{\underset{\sim}{b} \times \underset{\sim}{u}}{\omega_c}}$$

$$\frac{dR_g}{dt} = \frac{d\underset{\sim}{r}}{dt} - \frac{\underset{\sim}{b} \times \frac{d\underset{\sim}{u}}{dt}}{\omega_c}$$

$$= \frac{d\underset{\sim}{r}}{dt} - \frac{\underset{\sim}{b} \times \omega_c(\underset{\sim}{u} \times \underset{\sim}{b})}{\omega_c}$$

$$= \underset{\sim}{V} - b \times (\underset{\sim}{u} \times \underset{\sim}{b})$$

$$= \underbrace{c\frac{\underset{\sim}{E} \times \underset{\sim}{b}}{B} + \underset{\sim}{u}}_{=} - (\underset{\sim}{u} - u_\parallel b)$$

$$\boxed{\frac{d\underset{\sim}{R}_g}{dt} = c\,\frac{\underset{\sim}{E} \times \underset{\sim}{b}}{B} + u_\parallel \underset{\sim}{b} = \underset{\sim}{V}_E + u_\parallel b}$$

$$\underset{\sim}{R}_g = c\,\frac{E \times b}{B}\,t + u_\parallel \underset{\sim}{b}\,t$$

$\underset{\sim}{R}_g$ moves without gyration

$$\frac{d\underset{\sim}{v}}{dt} = \frac{e}{m}\left[\underset{\sim}{E}(\underset{\sim}{r},t) + \frac{\underset{\sim}{v}}{c}\times\underset{\sim}{B}(\underset{\sim}{r},t)\right]$$

$$\underset{\sim}{v} = \underset{\sim}{u} + c\,\underset{\sim}{E}(\underset{\sim}{r},t)\times\underset{\sim}{b}/B(\underset{\sim}{r},t)$$

$$\frac{d\underset{\sim}{v}}{dt} = \frac{d\underset{\sim}{u}}{dt} + c\frac{d}{dt}\left(\frac{\underset{\sim}{E}(\underset{\sim}{r},t)\times b}{B(r,t)}\right) = \frac{e}{m}\left(\frac{\underset{\sim}{u}}{c}\times B(\underset{\sim}{r},t) + E_{\parallel}\,\underset{\sim}{b}\right)$$

$$R_g = \underset{\sim}{r} - \underset{\sim}{b}\times\frac{\underset{\sim}{u}}{\omega_c(\underset{\sim}{r},t)}$$

$$\frac{d\underset{\sim}{R}_g}{dt} = \underset{\sim}{v} - \frac{\underset{\sim}{b}}{\omega_c}\times\frac{d\underset{\sim}{u}}{dt} + \underset{\sim}{u}\times\frac{d}{dt}\left(\frac{b(\underset{\sim}{r},t)}{\omega_c(\underset{\sim}{r},t)}\right)$$

substitute $\frac{d\underset{\sim}{u}}{dt} = \frac{e}{m}\left(\frac{\underset{\sim}{u}\times\underset{\sim}{B}}{c} + E_{\parallel}\,\underset{\sim}{b}\right) - \frac{d}{dt}\left(\frac{\underset{\sim}{E}\times\underset{\sim}{b}}{B}\right)$

$$\boxed{\frac{d\underset{\sim}{R}_g}{dt} = u_{\parallel}\underset{\sim}{b} + \underbrace{c\frac{\underset{\sim}{E}\times\underset{\sim}{b}}{B}}_{=\underset{\sim}{V}_E} + \frac{\underset{\sim}{b}}{\omega_c}\times\frac{d}{dt}\left(\frac{\underset{\sim}{E}\times\underset{\sim}{b}}{B}\right)c + \underset{\sim}{u}\times\frac{d}{dt}\left(\frac{\underset{\sim}{b}(r,t)}{\omega_c(\underset{\sim}{r},t)}\right)}$$

Expand about guiding center; $\frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla}$

$$\frac{dR_g}{dt} \approx u_{\parallel}\underset{\sim}{b}(R_g) + c\frac{\underset{\sim}{E}(R_g)\times\underset{\sim}{b}(R_g)}{B(R_g)} + c\frac{b\times\underset{\sim}{u}}{\omega_c}\cdot\underset{\sim}{\nabla}\left(\frac{\underset{\sim}{E}(R_g)\times\underset{\sim}{b}(R_g)}{B(R_g,t)}\right) + u_{\parallel}\frac{b\times\underset{\sim}{u}}{\omega_c}\cdot\underset{\sim}{\nabla} b(R_g)$$

$$+ \frac{\underset{\sim}{b}}{\omega_c}\times\left(\frac{\partial}{\partial t} + u_{\parallel}\underset{\sim}{b}\cdot\underset{\sim}{\nabla} + \underset{\sim}{V}_E\cdot\underset{\sim}{\nabla} + \underset{\sim}{u}_{\perp}\cdot\underset{\sim}{\nabla}\right)\left(\frac{\underset{\sim}{E}\times\underset{\sim}{b}}{B}\right)\Bigg|_{\underset{\sim}{r}=R_g}$$

$$+ (\underset{\sim}{u}_{\perp} + u_{\parallel}\underset{\sim}{b})\times\left(\frac{\partial}{\partial t} + u_{\parallel}(b\cdot\underset{\sim}{\nabla}) + \underset{\sim}{V}_E\cdot\underset{\sim}{\nabla} + \underset{\sim}{u}_{\perp}\cdot\underset{\sim}{\nabla}\right)\left(\frac{\underset{\sim}{b}(R_g,t)}{\omega_c(R_g,t)}\right)$$

Now average $\frac{dR_g}{dt}$ over a gyration time, to remove rapid oscillations from the drifts

$$\overline{\frac{d\underset{\sim}{R_g}}{dt}} = u_{\parallel}\, \underset{\sim}{b}(R_g,t) + c\,\frac{\underset{\sim}{E}(R_g,t)\times \underset{\sim}{b}(R_g,t)}{B(R_g,t)}$$

$$+ \overline{c\,\frac{b\times u}{\omega_c}\cdot\nabla\left(\frac{\underset{\sim}{E}(R_g,t)\times b(\underset{\sim}{R_g},t)}{B(R_g,t)}\right) + u_{\parallel}\,\frac{\underset{\sim}{b}\times \underset{\sim}{u}}{\omega_c}\cdot\underset{\sim}{\nabla}\,\underset{\sim}{b}(R_g)}$$

$\overset{\shortparallel}{0}$ as $\overline{b\times \underset{\sim}{u}} = 0$ (both terms

$$+ \quad \overline{\text{More}}$$

$$\text{More}_1 \equiv \overline{\frac{\underset{\sim}{b}}{\omega_c}\times\left(\frac{\partial}{\partial t} + u_{\parallel}\,\underset{\sim}{b}\cdot\underset{\sim}{\nabla} + \underset{\sim}{V_E}\cdot\underset{\sim}{\nabla} + \underset{\sim}{u_\perp}\cdot\underset{\sim}{\nabla}\right)c\,\frac{\underset{\sim}{E}\times b}{B}}\Bigg|_{\underset{\sim}{r}=\underset{\sim}{R_g}}$$

$$= \frac{c\,\partial(\underset{\sim}{E_\perp}/B)}{\omega_c\,\partial t} + \frac{c\,u_{\parallel}}{\omega_c}\,\underset{\sim}{b}\cdot\underset{\sim}{\nabla}\left(\frac{\underset{\sim}{E_\perp}}{B}\right) + \frac{c}{\omega_c}\,\underset{\sim}{V_E}\cdot\underset{\sim}{\nabla}\left(\frac{\underset{\sim}{E_\perp}}{B}\right)$$

we used $\overline{\frac{\underset{\sim}{b}}{\omega_c}\times\left(\underset{\sim}{u_\perp}\cdot\underset{\sim}{\nabla}\; c\,\frac{\underset{\sim}{E}\times\underset{\sim}{b}}{B}\right)} = 0$

$$\underset{\sim}{b}\cdot\frac{\partial \underset{\sim}{b}}{\partial t} = u_{\parallel}\underset{\sim}{b}\cdot(\underset{\sim}{b}\cdot\nabla)\,\underset{\sim}{b} = \cdots = 0$$

Also define guiding center convective derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{V_E}\cdot\underset{\sim}{\nabla}_{R_g} + u_{\parallel}\,\underset{\sim}{b}\cdot\underset{\sim}{\nabla}_{R_g}$

$$\boxed{\text{More}_1 = \frac{1}{\omega_c(R_g,t)}\,\frac{D}{Dt}\left(\frac{c\,\underset{\sim}{E_\perp}(R_g,t)}{B(R_g,t)}\right) \equiv \text{Polarization Flow Velocity} \equiv \underset{\sim}{V_p}}$$

$$\text{More } z \equiv \overline{(\underset{\sim}{u}_\perp + u_\parallel \underset{\sim}{b}) \times \left(\frac{\partial}{\partial t} + u_\parallel (\underset{\sim}{b}\cdot\underset{\sim}{\nabla}) + \underset{\sim}{V}_E \cdot \underset{\sim}{\nabla} + \underset{\sim}{u}_\perp \cdot \nabla\right) \left(\frac{\underset{\sim}{b}}{\omega_c}\right)_{\underset{\sim}{r} = \underset{\sim}{R}_g}}$$

(odd powers of $\underset{\sim}{u}_\perp$ vanish from averging)

$$= \frac{u_\parallel}{\omega_c} \underset{\sim}{b} \times \left(\frac{\partial}{\partial t} + \underset{\sim}{V}_E \cdot \underset{\sim}{\nabla}\right) \underset{\sim}{b} \qquad \boxed{\frac{D'}{D't} = \frac{\partial}{\partial t} + V_E \cdot \nabla}$$

$$+ \frac{u_\parallel^2}{\omega_c} \underset{\sim}{b} \times \underset{\sim}{b} \cdot \underset{\sim}{\nabla}\, b + \overline{\underset{\sim}{u}_\perp \times (\underset{\sim}{u}_\perp \cdot \underset{\sim}{\nabla}) \left(\frac{b}{\omega_c}\right)}$$

$$= \frac{u_\parallel}{\omega_c} \underset{\sim}{b} \times \frac{D' \underset{\sim}{b}}{D't} + \underset{\sim}{V}_\varkappa + \overline{\underset{\sim}{u}_\perp \times (\underset{\sim}{u}_\perp \cdot \underset{\sim}{\nabla}) \left(\frac{\vec{b}}{\omega_c}\right)}$$

$$\underset{\sim}{V}_\varkappa = \frac{u_\parallel^2}{\omega_c} \underset{\sim}{b} \times (\underset{\sim}{b} \cdot \underset{\sim}{\nabla})\, b \equiv \text{curvature drift.}$$

$$\overline{u_\perp \times (u_\perp \cdot \nabla) \left(\frac{b}{\omega_c}\right)} \approx \frac{u_\perp^2}{2} \frac{\underset{\sim}{b} \times \underset{\sim}{\nabla} B}{\omega_c B} = \underset{\sim}{V}_B \equiv \text{grad } B \text{ drift}$$

(will discuss below)

Then

$$\frac{d\underset{\sim}{R}_g}{dt} = u_\parallel \underset{\sim}{b} + \underset{\sim}{V}_E + \underset{\sim}{V}_p + \underset{\sim}{V}_\varkappa + \underset{\sim}{V}_B + \frac{u_\parallel}{\omega_c} \underset{\sim}{b} \times \frac{D\underset{\sim}{b}}{Dt}$$

$$\underset{\sim}{V}_E = c\, \underset{\sim}{E} \times \underset{\sim}{b}/B, \quad \underset{\sim}{V}_p = \frac{c}{\omega_c} \frac{d}{dt} \frac{\underset{\sim}{E}}{B}$$

$$\underset{\sim}{V}_\varkappa = \frac{u_\parallel^2}{\omega_c} \underset{\sim}{b} \times \varkappa, \quad \underset{\sim}{V}_B = \frac{u_\perp^2}{2} \frac{\underset{\sim}{b} \times \underset{\sim}{\nabla} |B|}{|B|}$$

More accurately

$$\boxed{\frac{d\underset{\sim}{R}_g}{dt} = \underset{\sim}{b} \left(u_\parallel + \frac{4\pi J_\parallel}{2\omega_c} \frac{u_\perp^2}{Bc}\right) + \underset{\sim}{V}_E + \underset{\sim}{V}_p + \underset{\sim}{V}_\varkappa + \underset{\sim}{V}_B + u_\parallel \frac{\underset{\sim}{b}}{\omega_c} \times \frac{D'\underset{\sim}{b}}{D't}}$$

$$\underset{\sim}{\alpha} = \overline{\underset{\sim}{u}_\perp \times (\underset{\sim}{u}_\perp \cdot \nabla)\left(\frac{\underset{\sim}{b}}{\omega_c}\right)} \overset{?}{\simeq} \frac{u_\perp^2}{2} \frac{\underset{\sim}{b} \times \nabla |B|}{\omega_c B}$$

$$= -\frac{\underset{\sim}{b}'}{\omega_c'} \times \overline{\underset{\sim}{u}_\perp \underset{\sim}{u}_\perp \cdot \overleftarrow{\nabla}'} = -\frac{\underset{\sim}{b}'}{\omega_c'} \times \frac{u_\perp^2}{2}\left(\underset{\approx}{I} - \underset{\sim}{b}\underset{\sim}{b}\right) \cdot \overleftarrow{\nabla}'$$

[used $\overline{\underset{\sim}{u}_\perp \underset{\sim}{u}_\perp} = \frac{u_\perp^2}{2}\left(\underset{\approx}{I} - \underset{\sim}{b}\underset{\sim}{b}\right)$] [demonstrate for HW]

$$= -\frac{u_\perp^2}{2} \nabla \times \left(\frac{\underset{\sim}{b}}{\omega_c}\right) + \frac{u_\perp^2}{2\omega_c} \underset{\sim}{b} \times (\underset{\sim}{b} \cdot \nabla) \underset{\sim}{b}$$

$$= \frac{u_\perp^2}{2} \frac{\underset{\sim}{b} \times \nabla B}{\omega_c B} - \frac{u_\perp^2}{2\omega_c}\left[\nabla \times \underset{\sim}{b} - \underset{\sim}{b} \times (\underset{\sim}{b} \cdot \nabla)\underset{\sim}{b}\right]$$

Now $\underset{\sim}{b} \times (\nabla \times \underset{\sim}{b}) = \underbrace{\nabla\left(\frac{\underset{\sim}{b}\cdot\underset{\sim}{b}}{2}\right)}_{=0} - (\underset{\sim}{b}\cdot\nabla)\underset{\sim}{b} = -(\underset{\sim}{b}\cdot\nabla)\underset{\sim}{b} = -\underset{\sim}{\kappa}$

$$\therefore (\nabla \times \underset{\sim}{b}) = \underset{\sim}{b} \times \underset{\sim}{\kappa} + (\nabla \times \underset{\sim}{b}) \cdot \underset{\sim}{b} \; \underset{\sim}{b}$$

$$\therefore \underset{\sim}{\bar{\alpha}} = \underset{\sim}{V}_B - \frac{u_\perp^2}{2\omega_c} \underset{\sim}{b} \cdot (\nabla \times \underset{\sim}{b}) \; \underset{\sim}{b}$$

but as: $\underset{\sim}{b} \cdot \nabla \times \underset{\sim}{B} = \underset{\sim}{b} \cdot \nabla \times B \underset{\sim}{b} = B \underset{\sim}{b} \cdot \nabla \times \underset{\sim}{b} = \frac{4\pi J_\parallel}{c}$

$$\vec{\alpha} = \underset{\sim}{V}_B - \frac{u_\perp^2}{2\omega_c} \frac{4\pi J_\parallel}{cB} \underset{\sim}{b}$$

Non-Oscillatory Drift Motion

$$\mathbf{V}_d = \mathbf{V}_E + \mathbf{V}_p + \mathbf{V}_B + \mathbf{V}_\kappa + u_\parallel' \mathbf{b} + \mathbf{V}_R$$

$$= c\frac{\mathbf{E}\times\mathbf{b}}{B} + \frac{c}{\omega_c}\frac{D}{Dt}\mathbf{E}/|\mathbf{B}| + \frac{v_\perp^2}{2}\frac{\mathbf{b}\times\nabla B}{\omega_c} + u_\parallel^2\frac{\mathbf{b}\times\boldsymbol{\kappa}}{\omega_c}$$

$$+ \underbrace{\left(u_\parallel \mathbf{b} - \frac{4\pi J_\parallel}{2\omega_c c}\frac{u_\perp^2}{B}\right)\cdot\mathbf{b}}_{u_\parallel'} + \underbrace{\frac{v_\parallel}{2\omega_c}\,\mathbf{b}\times\frac{D\mathbf{b}}{Dt}}_{\mathbf{V}_R}$$

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{V}_E\cdot\nabla + u_\parallel \mathbf{b}\cdot\nabla\right)$$

One important observation

The $c\frac{\mathbf{E}\times\mathbf{b}}{B} \equiv \mathbf{V}_E$ is the same for electrons and ions!

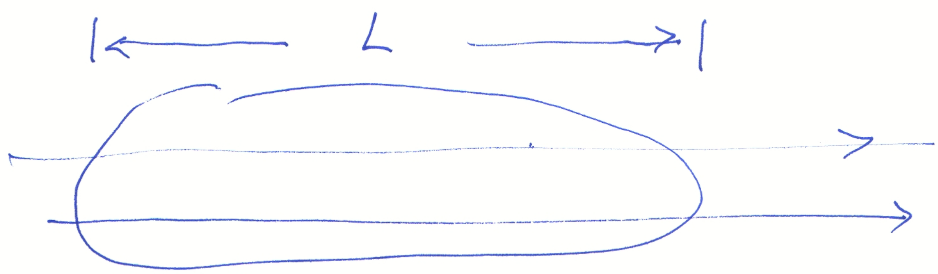
Thus this motion does cause charge separation or currents.

Essentially $\mathbf{V}_E$ is the bulk plasma motion across field lines. It is the corrections to this motion that produce electric fields, and thus we need these corrections.

$V_{\parallel} \underset{\sim}{b}$ $\equiv$ Motion along the field line

When, $T_e \sim T_i$ $\langle |V_e| \rangle \gg \langle |V_{\parallel i}| \rangle$,
and electrons move rapidly
along field lines.

Consider an isolated blob
of plasma, in uniform B-field



Will electrons escape system
in a time $T \simeq L/V_{the}$?

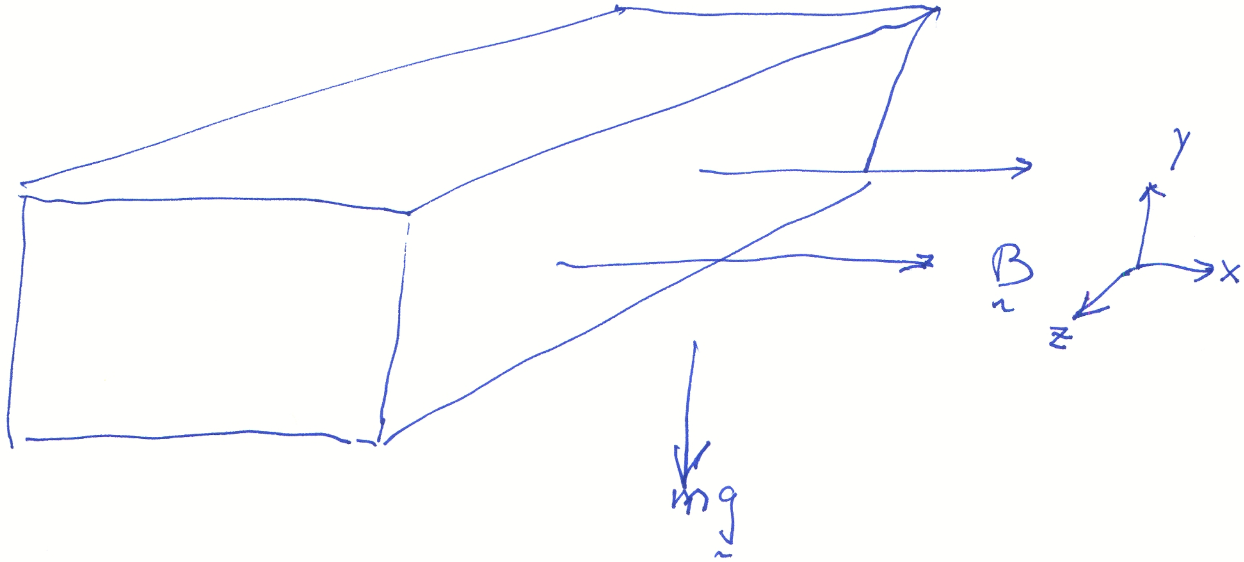
The characteristic time for
spreading out is ?

The polarization drift direction

$$\frac{c}{\omega_c}\frac{D(\underset{\sim}{E}_\perp/B)}{Dt} \equiv \frac{c^2 m}{eB}\frac{D}{Dt}\left(\frac{\underset{\sim}{E}_\perp}{B}\right)$$

depends on charge, and proportional to mass. Thus this drift produces currents that accumulates charge, that then produces electric fields.

Let us consider the following problem



Given a plasma in a magnetic field, $\mathbf{B}$ and gravitational field, $m\mathbf{g}$.

Does the plasma fall, or is it suspended and contained in space?

Note, the equation,

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{g} + e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

leads to a drift

$$\mathbf{V}_g = \frac{mc}{e}\mathbf{g}\times\frac{\mathbf{B}}{|B|} \qquad \left(\frac{m\mathbf{g}}{e} \text{ plays same role as } \mathbf{E}\right)$$

$$= \frac{g\,\hat{\mathbf{z}}}{c\,\omega_c} \qquad (\text{not in direction of fall})$$

Note that the gravitational drift causes charge flow, and is due essentially to ions ($\underset{\sim}{V}_g(e) = \frac{m}{M}\underset{\sim}{V}_g(i)$ and can be neglected)

The charge accumulation at the plasma edge will cause large electric fields, that will continue to increase, unless a counter-acting drift sets up. This drift is polarization drift, which allows the current flow to stop.

$$V_{total\ z} = \underset{\sim}{V}_g + \underset{\sim}{V}_p = \frac{mc\,\underset{\sim}{g} \times \underset{\sim}{b}}{e|B|} + \frac{mc^2}{e|B|}\frac{D(\underset{\sim}{E}/B)}{Dt} = 0$$

Thus

$$\frac{D}{Dt}c(\underset{\sim}{E}/B) = -\underset{\sim}{g} \times \underset{\sim}{b} \; ; \quad \text{or} \quad \frac{c\underset{\sim}{E}}{|\underset{\sim}{B}|} = -\underset{\sim}{g} \times \underset{\sim}{b}\, t = -g\,t\,\hat{\underset{\sim}{z}}$$

But electric field has $\underset{\sim}{V}_E$ drift

$$\underset{\sim}{V}_E = \frac{c\,\underset{\sim}{E} \times \underset{\sim}{b}}{B} = -g\,t\,\hat{\underset{\sim}{y}}$$

(Plasma, (electron & ions) accelerate just as if $\underset{\sim}{B}$-field absent)

13