Lecture # 29

$\Delta^1$ calculation and electron drift wave
The form of the tearing mode equation is \( \phi = \frac{4}{\omega} \)

\[
\frac{d}{dr} \left( r \frac{d\phi}{dr} \right) - \frac{m^2}{r} \phi = \frac{r \phi(0) \phi_0(r)}{B_0 \left( 1 - \frac{q(r)}{q(r_0)} \right)} = 0
\]

It was established that

\[ \Delta' \propto -6 \frac{d}{dr} \rho \]

\[ \Delta' = \lim_{\varepsilon \to 0} \frac{\phi(5+\varepsilon) - \phi(5-\varepsilon)}{\phi(5)} \]

\[ \Delta' < 0 \quad \text{stability} \]

\[ \Delta' > 0 \quad \text{instability} \]

For homework: \( m = 2 \)

Now \( q(r) \) and \( j(r) \) related

\[ j(r) = j_0 \left( 1 - \frac{r_a^2}{a^2} \right)^2 \]

\[ 2\pi r B_0 = \pi j_0 r \, dr \left( 1 - \frac{r_a^2}{a^2} \right)^2 \]

\[ B_0(r) = \frac{j_0}{2\pi} \int_0^r \frac{dr'}{2\pi} \left( 1 - \frac{r_a^2}{a^2} + \frac{r_a^4}{a^4} \right) \]

\[ = \frac{j_0}{4\pi} \left[ r^2 - \frac{r_a^4}{a^2} + \frac{r_a^6}{3a^4} \right] = \frac{j_0}{4\pi} \left[ 1 - \frac{r_a^2}{a^2} + \frac{r_a^4}{3a^4} \right] \]
\[ r \frac{d}{dr} \left( \frac{1}{q(r)} \right) = -4i \frac{\phi(z)}{a^2} \left( 1 - \frac{r^2}{a^2} \right) \]

\[ q(r) = \frac{m}{n} \]

\[ \frac{B_0(1 - \frac{q(r)}{q_0(r)})}{r} = \frac{B_0(r_0) - B_0(r)}{r} \]

\[ = \frac{j_0}{q} \left[ 1 - \frac{r^2}{a^2} + \frac{r^4}{3a^4} - 1 + \frac{r^4}{a^4} - \frac{r^8}{3a^8} \right] \]

\[ = \frac{j_0}{q} \left[ \frac{r^2 - r^4}{a^2} + \frac{(r^4 - r^8)}{3a^8} \right] \]

\[ = \frac{j_0}{q} \frac{(r^2 - r^4)}{a^2} \left[ 1 - \frac{(r^4 + r^8)}{3a^8} \right] \]

Define \( z = \frac{r^2}{a^2} \), \( 2 \frac{dz}{dr} = \frac{dz}{dz} \)

\[ r \frac{d}{dr} \frac{1}{q(r)} = \frac{-16}{a} \frac{\phi(z)}{a^2} \left[ (1 - z) z^{1/2} \right] \]

\[ \left[ (Z - Z_S) \left[ 1 - \frac{(Z + Z_S)}{3} \right] \right] \]

\[ \frac{d}{dz} \frac{Z}{Z_S} - \frac{\phi(z)}{Z} + 4 \frac{\phi(z)}{Z_S} \left( 1 - Z \right) \left[ \frac{Z - Z_S}{1 - \frac{(Z + Z_S)}{3}} \right] = 0 \]

\[ 0 < z < 1 \]

\[ p^2 \Delta' = \lim_{\rho \to 0} \left( \frac{d}{dr} \frac{\phi(z_S + \delta)}{\phi(z_S)} - \frac{d}{dr} \frac{\phi(z - \delta)}{\phi(z)} \right) \]

\[ \phi(z) \]

\[ \lim_{\rho \to 0} \left( \frac{-2}{\phi(z_S)} \right) \left( \frac{\phi(z_S + \delta)}{\phi(z_S)} - \frac{\phi(z - \delta)}{\phi(z)} \right) \]

\[ \phi(z_S) \]

\[ \lim_{\rho \to 0} \]

\[ \lim_{\rho \to 0} \left( \frac{d}{d\zeta} \frac{\phi(z_S + \delta)}{\phi(z_S)} - \frac{d}{d\zeta} \frac{\phi(z - \delta)}{\phi(z)} \right) \]

\[ \phi(z_S) \]
Properties of this equation

1. Two independent solutions
   \[ \phi(z) = A \phi_1(z) + B \phi_2(z) \]

2. Singular character at origin

3. Singular character at \( z = z_0 \)

At \( z = z_0 \) there is a solution that goes as a constant, and its correction has a logarithmic character.

We can write the equation near \( z = z_0 \), which is exactly:

\[ \frac{d^2 \phi}{dz^2} + \frac{d\phi}{dz} \frac{1}{z} + \frac{4 \phi(1-z)}{(z-z_0)(1-\frac{(z-z_0)}{3})} = 0 \]

approximately as:

\[ \frac{d^2 \phi}{dz^2} + \frac{4 \phi(1-z)}{(1 \frac{z-z_0}{3})(z-z_0)} = -\frac{d\phi(z)}{zdz} + \text{extra terms} \]

where we have expanded about \( z = z_0 \), and near \( z = z_0 \) we look for a balance of the two terms on the left.

The other terms on the right can be accounted for with higher order iteration.
Even the left hand side alone has a transcendental solution (in terms of some Bessel functions.) However, it is more transparent to generate a solution systematically.

We have the equation

\[
\frac{d^2 \phi}{dz^2} + \alpha \phi = \frac{d\phi(z)}{dz}; \quad \alpha = \frac{4\left(1 - z_s\right)}{(1 - \frac{z_s}{3}) z_s}
\]

There is one solution that goes as \( z \to z_s \) as a constant, which we take as unity

\( \phi_0 = 1 \) as \( z \to z_s \); we then iterate

\( \phi = \phi_0 + \delta \phi \), using \( \phi_0 = 1 \) in terms (2) + (3).

\[
\frac{d^2 \delta \phi}{dz^2} = -\frac{\alpha}{z - z_s};
\]

\[
\delta \phi = -\alpha \ln(z - z_s)
\]

\[
\delta \phi = -\alpha(z - z_s) \left[ \ln(z - z_s) - 1 \right]
\]

It is easy to check that other terms we neglected go to zero as \( z \to z_s \), faster than \( (z - z_s)^{1/2} \).

Another solution to \( \phi \) near \( z = z_s \) goes as \( \phi_2 = (z - z_s) \).
Then iterating with this solution \( \phi_2 = \phi_{2,0} + \phi_{2,1} + \phi_{2,2} + \cdots \)

\[
\frac{d^2 \phi_{2,0}}{dz^2} = -\alpha \frac{\phi_{2,0}}{(z-z_0)} - \frac{1}{z} \frac{d}{dz} \phi_{2,0}(z), \quad \phi_{2,0} = z-z_0
\]

\[
\phi_2 = -(\alpha + \frac{\lambda}{z_0}) \left( \frac{z-z_0}{z} \right)^2 \quad \text{clearly smaller than} \quad \phi_{2,0}
\]

Thus we find around \( z = z_0 \):

\[
\phi_+ = A_+ \left[ -\alpha \left( \ln (z-z_0) + 1 \right) (z-z_0) + \ldots \right]
\]

\[
+ \lambda (z-z_0) + \ldots
\]

where \( \lambda^+ \) needs to be determined

\[
\phi^- = A^- \left[ 1 - \alpha \left( \ln (z_0-z) - 1 \right) (z-z_0) + \lambda^- (z-z_0) \right]
\]

(We will see that \( \lambda^+ - \lambda^- \) is essentially \( A \).)
Now suppose the boundary conditions of the equation are \( \phi(z) = 0 \) as \( z \to 0 \)*

and \( \phi(z=1) = 0 \)

*Referring to boundary conditions at \( z=0 \).

Note that the equation

\[
\frac{2}{z} \frac{d\phi}{dz} + \frac{\phi}{z} = \text{negligible terms as } z \to 0
\]

has a solution, \( \phi \propto z^S \), with

\[ S^2 = 1, \quad \therefore \quad S = \pm 1 \]

and thus

\( \phi = C_1 z + C_2/z \)

is the form of the general solution. Excluding the singular solution, we set \( C_2 = 0 \), and for convenience we take \( C_1 = 1 \).

We can integrate the interior solution numerically and generate a solution that goes as \( \frac{z^2}{z}$ \) (II) with \( A \) determined from the choices \( C_2 = 0, \ C_1 = 1 \).

We can then integrate the region \( 1 > z > z_5 + 8 \) numerically, and we will find the form of the solution in the neighborhood of \( z = z_5 + 8 \). A convenient
choice of solution starting from $\xi=1$, is to have $\phi(1) = 0$, $\phi'(1) = 1$. Then $A^+$ is implicitly determined.

We can then obtain the value of

$$\xi = \lim_{\delta \to 0} \frac{2 \mathbf{Z}_s}{2 \mathbf{Z}_s} \left( \frac{d}{d \mathbf{Z}} \frac{\phi^+(\mathbf{Z}_s + \delta)}{\phi^+(\mathbf{Z}_s + \delta)} - \frac{d}{d \mathbf{Z}} \frac{\phi^-(\mathbf{Z}_s - \delta)}{\phi^-(\mathbf{Z}_s - \delta)} \right)$$

$$= \lim_{\delta \to 0} \frac{2 \mathbf{Z}_s}{2 \mathbf{Z}_s} \left[ \frac{d}{d \mathbf{Z}} \frac{\phi^+(\mathbf{Z}_s + \delta)}{\phi^+(\mathbf{Z}_s + \delta)} - \frac{d}{d \mathbf{Z}} \frac{\phi^-(\mathbf{Z}_s - \delta)}{\phi^-(\mathbf{Z}_s - \delta)} \right]$$

Note: the $A^+$ and $A^-$ cancel out and we need not attempt to evaluate them. In particular, the terms $A^+ \xi$ cancel out of the difference between

$$\left( \frac{\phi^+(\mathbf{Z}_s + \delta)}{\phi^+(\mathbf{Z}_s + \delta)} - \frac{\phi^-(\mathbf{Z}_s - \delta)}{\phi^-(\mathbf{Z}_s - \delta)} \right) = \lambda^+ \lambda^-$$
For the final, plot

\[ j(r) = j_0 \left( 1 - \frac{r^2}{a^2} \right)^2 \]

with boundary condition \( \phi(a) = 0 \) for the equation with \( m=2 \)

\[
\frac{d}{dr} \left( r \frac{d\phi}{dr} \right) - m^2 \phi - \frac{r \frac{d\phi}{dr}}{B_0 (1 - \frac{q(r)}{q_{03}})} = 0
\]
In previous lecture we discussed drift waves from kinetic point of view.

It is insightful to consider drift waves from fluid viewpoint.

We take electrons as Maxwell Boltzmann response

\[ n_e = n_0(x) e^{\frac{e\phi}{T}} = n_0(x) \left(1 + \frac{e\phi}{T}\right) \quad (e > 0) \]

If ions are treated as having only cross field dynamics \( \frac{\omega}{k_BT_i} \gg 1 \) as described by \( \mathbf{v} \times \mathbf{B} \) motion

\[ \frac{\partial n_i}{\partial t} = -\nabla \cdot (n_i \mathbf{v}_i) \quad \mathbf{v}_i = \mathbf{V}_E = -\frac{e\phi}{B} \frac{\mathbf{B}}{B} \]

In homogeneous field \( \nabla \cdot \mathbf{V}_E = \nabla \cdot \left(\frac{e\phi \mathbf{B}}{B^2}\right) = 0 \)

\( (\mathbf{v} \times \nabla) \cdot \mathbf{B} \)

\[ \frac{\partial n_i}{\partial t} = \frac{e\phi}{B} \frac{\partial n_0}{\partial x} \mathbf{B} \]

If \( \phi = \phi \exp (ik \cdot \mathbf{r} - i\omega t) \), \( n_i \propto e^{-i\omega t} \)

\[ n_i = -i \frac{kx}{B} \frac{e\phi}{B} \frac{\partial n_0}{\partial x} \]

\[ n_i = \frac{kx e\phi}{\omega B} \frac{\partial n_0}{\partial x}, \quad n_e = \frac{e\phi n_0}{Te} \]

(1)
Quasi-neutrality condition is

\[
\frac{k_y \phi_e \omega_0}{\omega B} \frac{\partial n_e}{\partial x} + \frac{e \phi}{T_e} = 0
\]

or

\[
\frac{k_y T_e \omega_0 n_0}{e B n_0} \frac{\partial n_e}{\partial x} = \omega_e^* = \frac{k_y T_e}{\omega_e m_e n_0} \frac{\partial n_0}{\partial x} = -\frac{k_y}{\omega_e} \frac{n_0}{n_0 j_x}
\]

\[
\omega_e^* = \frac{k_y T_e}{\omega_e} \frac{V_{the}}{L_0} = \frac{k_y \rho_s}{L_0} \frac{V_{ths}}{L_p}
\]

\[
L_p = \frac{n_0 j_x}{V_{the}}, \quad \rho_s = \rho_i \left( \frac{T_e}{T_i} \right), \quad V_{ths} = V_{thi} \left( \frac{T_e}{T_i} \right) \frac{v_2}{v_z}
\]

We saw that when we treated the kinetic problem, we obtained the dispersion relation (when \( k_e \rho_i << 1 \), \( k_e V_{thi}/c << 1 \), \( k_e V_{the}/c >> 1 \), \( k_e \omega c << 1 \))

\[
\omega = \left( 1 + i \tau \frac{(\omega - \omega^*)}{\omega} \right) + k_y \rho_s^2 \frac{\omega^*}{\omega} \frac{1}{k_y v_{the}} \left( \int dV_z V_z \left( 1 + \frac{n_i}{n_0} \frac{V_z^2}{V_{thi}^2} - 1 \right) \right)
\]

\[
\rho_s^2 = \rho_i \frac{T_e}{T_i}
\]

The drift wave frequency emerges from balance of two "arrowed" terms