Lecture #29

Δ' calculation and electron drift wave

The form of the tearing mode equation is $\phi = \psi/r$

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \frac{m^2}{r}\phi - \frac{r\frac{dj_z(r)}{dr}\,\phi(r)}{B_\theta\left(1 - q(r)/q(r_s)\right)} = 0$$

It was established that

$$\Delta' \propto -\delta W_R$$

$$\Delta' = \lim_{\epsilon \to 0} \frac{\frac{d\phi(r_s+\epsilon)}{dr} - \frac{d\phi(r_s-\epsilon)}{dr}}{\phi(r_s)}$$

$\Delta' < 0$ stability

$\Delta' > 0$ instability

For homework: $m = 2$

Now $q(r)$ and $j(r)$ related

$$j(r) = j_0\left(1 - \frac{r^2}{a^2}\right)^2$$

$$2\pi r B_\theta = \pi\int_0^r j_0\, r\, dr\left(1 - \frac{r^2}{a^2}\right)^2$$

$$B_\theta(r) = \frac{j_0}{r}\int_0^r d\frac{r^2}{2}\left(1 - \frac{2r^2}{a^2} + \frac{r^4}{a^4}\right)$$

$$= \frac{j_0}{4r}\left[r^2 - \frac{r^4}{a^2} + \frac{r^6}{3a^4}\right] = \frac{j_0 r}{4}\left[1 - \frac{r^2}{a^2} + \frac{r^4}{3a^4}\right] \quad (1)$$

$$r\frac{dj}{dr} = -4j_0\frac{r^2}{a^2}\left(1-\frac{r^2}{a^2}\right) \; ; \qquad q(r_s) = \frac{m}{n}$$

$$\frac{B_\theta}{r}\left(1-\frac{q(r)}{q(r_s)}\right) = \frac{B_\theta(r)}{r} - \frac{B_\theta(r_s)}{r_s}$$

$$= \frac{j_0}{4}\left[1 - \frac{r^2}{a^2} + \frac{r^4}{3a^4} - 1 + \frac{r_s^2}{a^2} - \frac{r_s^4}{3a^4}\right\}$$

$$= \frac{j_0}{4}\left[\frac{r_s^2 - r^2}{a^2} + \frac{(r^4 - r_s^4)}{3a^2}\right]$$

$$= \frac{j_0}{4}\frac{(r_s^2 - r^2)}{a^2}\left[1 - \frac{(r^2 + r_s^2)}{3a^2}\right]$$

define $z \equiv (r/a)^2$, $\quad 2\frac{r\,dr}{a^2} = dz$

$$\frac{r\frac{dj}{dr}}{B_\theta\left(1-\frac{q(r)}{q(r_s)}\right)} = -\frac{16}{a}\,\frac{(1-z)\,z^{1/2}}{(z-z_s)\left[1-(z+z_s)/3\right]}$$

$$\boxed{\frac{d}{dz}z\frac{d\phi}{dz} - \frac{\phi}{z} + 4\frac{\phi\,(1-z)}{(z-z_s)\left[1-(z+z_s)/3\right]} = 0}$$

$0 \leq z < 1$

$$r_s\Delta' = \lim_{\delta\to0}\frac{\left(\frac{d\phi(r_s+\delta)}{dr} - \frac{d\phi(r_s-\delta)}{dr}\right)r_s}{\phi(r_s)}$$

$$= \lim_{\delta\to0}\frac{2\left(\frac{d\phi(z_s+\delta)}{dz} - \frac{d\phi(r_s-\delta)}{dz}\right)z_s}{\phi(z_s)}$$

Properties of this equation

1. Two independent solutions

$$\phi(z) = A\,\phi_1(z) + B\,\phi_2(z)$$

2. Singular character at origin
3. Singular character at $z = z_s$

At $z \simeq z_s$ there is a solution that goes as a constant, and its correction has a logarithmic character

We can write the equation near $z = z_s$, which is exactly:

$$z\frac{d^2\phi}{dz^2} + \frac{d\phi}{dz} - \frac{\phi}{z} + \frac{4\phi(1-z)}{(z-z_s)\left[1-\frac{(z+z_s)}{3}\right]} = 0$$

approximately as:

$$\boxed{1-\frac{2}{3}z_s} \qquad \frac{d^2\phi}{dz^2} + \frac{4(1-z_s)\phi}{\left(1-\frac{2z_s}{3}\right)(z-z_s)\,z_s} = -\frac{d\phi(z_s)}{z\,dz} + \text{extra terms}$$

where we have expanded about $z = z_s$, and near $z = z_s$ we look for a balance of the two terms on the left.

The other terms on the right can be accounted for with higher order iteration:

Even the left hand side alone has a trancendental solution (in terms of some Bessel functions...) However, it is ~~str~~more transparent to generate a solutions systematically

We have the equation

$$\overset{(1)}{\frac{d^2\phi}{dZ^2}} + \overset{(2)}{\frac{\alpha}{(Z-Z_s)}\phi} = \overset{(3)}{\frac{d\phi(Z_s)}{Z_s\, dZ}} \; ; \; \alpha = \frac{4(1-Z_s)}{\left(1-\frac{2Z_s}{3}\right)Z_s}$$

There is one solution that goes as $Z \to Z_s$ as a constant, which we take as unity

$\phi_{1,0} = 1$ as $z \to Z_s$; we then iterate:

$\phi_1 = \phi_{1,0} + \delta\phi_1$, using $\phi_{1,0} =$ in terms (2) & (3).

$$\frac{d^2\delta\phi_1}{dZ^2} = -\frac{\alpha}{(Z-Z_s)} \; ;$$

$$\therefore \quad \frac{d\delta\phi_1}{dZ} = -\alpha \ln(z-Z_s)$$

$$\delta\phi_1 = -\alpha(Z-Z_s)\left[\ln(Z-Z_s) - 1\right]$$

It is easy to check that other terms we neglected, go to zero, as $z \to Z_s$, faster that $(z-Z_s)$ (typically $\mathcal{O}((z-Z_s)^2\ln(z-Z_s))$)

Another solution to ϕ, near $z = Z_s$ goes as $\phi_2 = (Z-Z_s)$

Then iterating with this solution
$\phi_2 = \phi_{2,0} + \phi_{2,1} + \phi_{2,2} + \dots$

$$\frac{d^2\phi_{2,1}}{dz^2} = -\alpha\frac{\phi_{2,0}}{(z-z_s)} - \frac{1}{z_s}\frac{d\phi_{2,0}(z_s)}{dz}, \quad \phi_{20} = z - z_0$$

$$= -\alpha - \frac{1}{z_s}$$

$$\therefore \phi_2 = -\left(\alpha + \frac{1}{z_s}\right)\frac{(z-z_g)^2}{2}$$

(clearly smaller than $\phi_{1,0}$ as $z - z_g \to 0$

Thus we find around $z = z_0$, $\alpha = \frac{4}{z_0}(1-z_s)/(1-2z_s/3)$, $z = r^2$, $q(r_s) = \frac{m}{n}$

$$\boxed{\phi^+ \simeq A^+\left[1 - \alpha\left(\ln(z-z_g) - 1\right)(z-z_s) + \dots + \lambda^+(z-z_s) + \dots\right]} \quad z - z_s > 0 \quad \text{I}$$

where $\lambda^\pm$ needs to be determined

&

$z - z_s < 0$

$$\boxed{\phi^- = A^-\left\{1 - \alpha\left(\ln(z_s-z) - 1\right)(z-z_s) + \dots + \lambda^-(z-z_s)\right\}} \quad \text{II}$$

(We will see that $\lambda^+ - \lambda^-$ is essentially Δ')

Now suppose the boundary conditions of the equation are $\phi(z) = z$ as $z \to 0$*

and $\phi(z=1) = 0$

*Referring to boundary conditions at $z=0$. Note that the equation

$$\frac{\partial}{\partial z} z \frac{\partial \phi}{\partial z} - \frac{\phi}{z} = 0 + \text{negligible terms as } z \to 0$$

has a solution, $\phi = z^s$, with $s^2 = 1$. $\therefore$ $s = \pm 1$, and thus

$\phi = C_1 z + C_2/z$ is the form of the general solution. Excluding the singular solution, we set $C_2 = 0$, and for convenience we take $C_1 = 1$

We can integrate the interior solution $0 < z < z_s - \delta$ numerically and generate a solution that goes as Eq. (II) with A^- implicitly determined from the choices $C_2 = 0$, $C_1 = 1$.

We can then integrate the region $1 > z > z_s + \delta$ numerically, and we will find the form of the solution in the neighborhood of $z = z_s + \delta$. A convenient

choice of solution starting from $z=1$, is to have $\phi(1) = 0$, $\phi'(1)=1$.
Then A^+ is implicitly determined.

We can the obtain the value of

$$r_s \Delta' = \lim_{\delta\to 0} 2 Z_s \left(\frac{\frac{d\phi^+(z_s+\delta)}{dz}}{\phi^+(z_s+\delta)} - \frac{\frac{d\phi^-(r_s-\delta)}{dz}}{\phi^-(r_s-\delta)} \right)$$

$$= \lim_{\delta\to 0} 2 Z_s \left[\frac{\frac{d\phi_N^+(z_s+\delta)}{dz}}{\phi^+(z_s+\delta)} - \frac{\frac{d\phi_N^-(r_s-\delta)}{dz}}{\phi^-(r_s-\delta)} \right]$$

Note: the constants A^+ and A^- cancel out and we need not attempt to evaluate them. In particular, the terms

$A^\pm \frac{\alpha}{\delta}$ cancel out of the difference between $\frac{\phi'^\pm(z_s+\delta)}{\phi^\pm(z_s+\delta)} - \frac{\phi'^\pm(z_s-\delta)}{\phi^\pm(z_s-\delta)} = \lambda^+ + \lambda^-$

For take home for

the final:

Evaluate and plot $r_s \Delta'(r_s)$ for

$0 < r < a$ of

$$j_z(r) = j_0 \left(1 - \frac{r^2}{a^2}\right)^2$$

with boundary condition $\phi(a) = 0$

for the equation, with $m = 2$

$$\frac{d}{dr}\left(r \frac{d\phi}{dr}\right) - \frac{m^2}{r}\phi - \frac{r \frac{dj_z(r)}{dr} \phi(r)}{B_\theta\left(1 - q(r)/q(r_s)\right)} = 0$$

In previous lecture we discussed drift waves from kinetic point of view.

It is insightful to consider drift waves from fluid viewpoint

We take electrons as Maxwell Boltzman response

$$n_e = n_0(x)\, e^{+e\phi/T} \simeq n_0(x)\left(1 + \frac{e\phi}{T}\right) \quad (e > 0)$$

If Ions are treated as having only cross field dynamics $\left(\frac{\omega}{k_\parallel v_{thi}} >> 1\right)$ as described by $\underset{\sim}{E} \times \underset{\sim}{B}$ motion

$$\frac{\partial n_i}{\partial t} = -\underset{\sim}{\nabla}\cdot(n_i \underset{\sim}{v}) \ ; \quad \underset{\sim}{v} = \underset{\sim}{v}_E = -\underset{\sim}{\nabla}\phi \times \underset{\sim}{b}c/B$$

In homogeneous field $\underset{\sim}{\nabla}\cdot\underset{\sim}{v}_E = \underset{\sim}{\nabla}\cdot\left(\frac{\underset{\sim}{\nabla}\phi\times\underset{\sim}{b}c}{B}\right) = 0$

$$\overset{\shortparallel}{(\nabla\times\nabla\phi)}\cdot\underset{\sim}{b}/B$$

$$\therefore \quad \frac{\partial n_i}{\partial t} = c\,\frac{\partial n_0}{\partial x}\,\frac{b\times\nabla\phi}{B}\cdot\hat{x}$$

If $\phi = \hat{\phi}\exp(i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega t)$, $\delta n_i \propto e^{-i\omega t}$

$$-i\omega\,\delta n_i = -i\,\frac{k_y c}{B}\,\phi\,\frac{\partial n_0}{\partial x}$$

$$\delta n_i = \frac{k_y c\phi}{\omega B}\,\frac{\partial n_0}{\partial x} \ , \quad \delta n_e = +\frac{e\phi}{T_e}\,n_0$$

Quasi-neutrality condition is

$$\delta n_i = \delta n_e$$

$$\frac{k_y c \phi}{\omega B}\frac{\partial n_0}{\partial x} = +\frac{e\phi}{T_e} n_0$$

or $$\omega = \frac{k_y T_e c}{eB n_0}\frac{\partial n_0}{\partial x} = \omega_e^*$$

$$\omega_e^* = -\frac{k_y T_e}{\omega_{ce} m_e}\frac{\partial n_0}{n_0 \partial x} \equiv -\frac{k_y v_{the}^2}{\omega_{ce}}\frac{\partial n_0}{n_0 \partial x}$$

$$\simeq k_y \rho_e \frac{v_{the}}{L_p} \simeq k_y \rho_s \frac{v_{ths}}{L_p}$$

$$L_p^{-1} = \frac{\partial n_0}{n_0 \partial x}, \quad \rho_s = \rho_i \left(\frac{T_e}{T_i}\right)^{1/2}, \quad v_{ths} = v_{thi}\left(\frac{T_e}{T_i}\right)^{1/2}$$

We saw that when we treated the kinetic problem, we obtained the dispersion relation (when $k_\perp \rho_i \ll 1$, $k_z v_{thi}/\omega \ll 1$, $k_z v_{the}/\omega \gg 1$; $k\lambda_{De} \ll 1$) $F_{0j} = F_{Mj}(T_j)$

$$0 = \left(\overset{\Downarrow}{1} + i\pi \frac{(\omega - \omega_e^*(1-\eta_e))}{|k_\parallel| v_{the}}\right) + k_\perp^2 \rho_s^2 - \overset{\downarrow}{\frac{\omega_e^*}{\omega}}$$

$$- \frac{T_e}{T_i}\int dv_z \left[\frac{k_z v_z\left(1 - \frac{\omega_i^*}{\omega}\left(1 + \frac{\eta_i}{2}\left(\frac{v_z^2}{v_{thi}^2} - 1\right)\right)\right)}{\omega - k_z v_z} \times \frac{\exp(-v_z^2/2v_{thi}^2)}{(2\pi v_{thi}^2)^{1/2}}\right]$$

$$\rho_s^2 = \rho_i^2 \frac{T_e}{T_i}$$

The drift wave frequency emerges from balance of two "arrowed" terms