# Lecture #27

# Bernstein Waves

# Solution of Vlasov Equation

"Slab" geometry

$$\underset{\sim}{B} = B_0 \hat{z} \,, \qquad \begin{matrix} n = \\ T = \end{matrix} \left\{ \begin{matrix} n(x) \\ T(x) \end{matrix} \right.$$

$F_0$ a function of constants of motion

in general $E$, $\mu$, $\underset{\sim}{R}_g = -\dfrac{\underset{\sim}{b} \times \underset{\sim}{v}}{\omega_c} + \underset{\sim}{r}$

For homogeneous slab, it is convenient to take $F = F_0\left(v_z, \frac{v_\perp^2}{2}, x + v_y/\omega_c\right)$

$v_z = \underset{\sim}{v} \cdot \underset{\sim}{b} \equiv v_\parallel$; $X_g = x + v_y/\omega_c$

$$f = F_0\left(v_z, \frac{v_\perp^2}{2}, x + v_y/\omega_c\right) + \delta f(\underset{\sim}{r}, \underset{\sim}{v}, t)$$

For short wave lengths

$$\delta f = \phi_k \exp(-i\omega t + i\underset{\sim}{k} \cdot \underset{\sim}{r})\, g(\underset{\sim}{v})$$

can be exactly chosen for $\underset{\sim}{k} = k_z \hat{z} + k_y \hat{y}$

The x-componenent an approximation if $k_x L_p \gg 1$ and $\dfrac{1}{k_x^2}\dfrac{dk_x}{dx} \ll 1$

$$L_p = n \Big/ \frac{dn}{dx}$$

We found that

$$\frac{D\delta f}{Dt} = \frac{\partial F_0}{\partial \frac{v_\perp^2}{2}} \frac{D}{Dt} \frac{e}{m}\phi(\underset{\sim}{r},t)$$

$$+ \frac{e}{m} \left[ \frac{\partial F}{\partial v_\perp^2/2}\left(-\frac{\partial\phi}{\partial t} - v_z\frac{\partial}{\partial z}\phi\right) + \frac{\partial F}{\partial v_z} v_z \frac{\partial\phi}{\partial z} + \frac{1}{\omega_c}\frac{\partial F}{\partial X_g} \frac{\partial\phi}{\partial y} \right]$$

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z} + \underset{\sim}{v_\perp}\cdot\underset{\sim}{\nabla} + \omega_c\, \underset{\sim}{v}\times\underset{\sim}{b}\,\frac{\partial}{\partial \underset{\sim}{v}}\right)$$

Use $\phi(\underset{\sim}{r},t) = \hat{\phi}_k \exp(i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega t)$

$\therefore \quad \frac{\partial\phi}{\partial t} = -i\omega\phi \; ; \quad v_z\frac{\partial\phi}{\partial z} = i k_z v_z \phi, \quad \frac{\partial\phi}{\partial y} = i k_y \phi$

Integrate by method of characteristics:

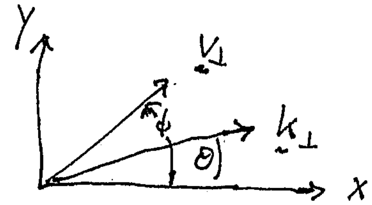
$F_0(v_z, v_\perp^2/2, X_g)$ constant along characteristic

$$g(\underset{\sim}{v}) \equiv \delta f(\underset{\sim}{r},\underset{\sim}{v},t)\, e^{i\omega t - i\underset{\sim}{k}\cdot\underset{\sim}{r}} = \frac{e}{m}\hat{\phi}_k \frac{\partial F(v_z, v_\perp^2/2, X_g)}{\partial v_\perp^2/2}$$

$$+ \frac{ie\hat{\phi}_k}{m}\left[(\omega - k_z v_z)\frac{\partial F}{\partial v_\perp^2/2} + k_z\frac{\partial F}{\partial v_z} + \frac{k_y}{\omega_c}\frac{\partial F}{\partial X_g}\right]$$

$$\cdot \int_{-\infty}^{0} d\tau\, \exp[-i\omega\tau + i k_z v_z \tau]\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{b}\times(\underset{\sim}{v}(\tau) - \underset{\sim}{v}(0))/\omega_c}$$

$$\exp(ia\sin\theta) = \sum_{p=-\infty}^{\infty} J_p(a)\exp(ip\theta)$$




$$\underset{\sim}{V_\perp}(\tau) = V_\perp\left(\cos(\phi - \omega_c\tau)\,\hat{x} + \sin(\phi - \omega_c\tau)\,\hat{y}\right)$$

$$\frac{\underset{\sim}{k}\cdot\underset{\sim}{b}\times\underset{\sim}{V}(\tau)}{\omega_c} = \frac{k_\perp V_\perp}{\omega_c}\left[\sin(\phi-\theta-\omega_c\tau)\right]$$

$$q \equiv \int_{-\infty}^{0} d\tau\,\exp[-i\omega\tau + ik_z V_z\tau]\,e^{\frac{i}{\omega_c}\underset{\sim}{k}\times\underset{\sim}{b}\cdot[\underset{\sim}{V}(\tau)-\underset{\sim}{V}(0)]}$$

$\underset{\sim}{V}(0) \equiv \underset{\sim}{V}$

$$= \sum_{p,p'} J_p\left(\frac{k_\perp V_\perp}{\omega_c}\right) J_{p'}\left(\frac{k_\perp V_\perp}{\omega_c}\right) e^{+i(p-p')(\phi-\theta)}$$

$$\cdot \int_{-\infty}^{\tau} d\tau\,\exp\left[-i(\omega - k_z V_z - p\omega_c)\tau\right]$$

$$= \frac{i}{\omega - k_z V_z - p\omega_c}$$

For perturbed charge density, in Poisson's equation, we only need:

$$\langle q\rangle = \int_0^{2\pi}\frac{d\phi}{2\pi}\,q \qquad (p = p')$$

$$= i\sum_p J_p^2\left(\frac{k_\perp V_\perp}{\omega_c}\right)\frac{1}{\omega - k_z V_z - p\omega_c}$$

Then from Poisson Equation we obtain dispersion relation

$$\nabla^2 \phi + \sum_j 4\pi e_j \int d^3v \, \delta f = 0$$

leads to:

$$0 = \left[ k^2 - \sum_j \omega_{pj}^2 \int d^3v \frac{\partial F_j(v_z, v_\perp^2, X_{gj})}{\partial v_\perp^2/2} / n_{0j} \right.$$
$$\left. + \sum_j \omega_{pj}^2 \sum_p \int d^3v \left( \left[ (\omega - k_z v_z) \frac{\partial F}{\partial v_\perp^2/2} + k_z \frac{\partial F}{\partial v_z} + \frac{k_y}{\omega_c} \frac{\partial F(v_z, v_\perp^2, X_g)}{\partial X_g} \right] \cdot \frac{J_p^2(k_\perp v_\perp/\omega_c)}{\omega - k_z v_z - p\omega_c} \frac{1}{n_{0j}} \right) \right]$$

$$\equiv D(\omega, k)$$

(a) Electron plasma oscillations (neglect $\frac{\partial F}{\partial X_g}$)

$\frac{k_\perp v_\perp}{\omega_c} \ll 1$, ~~$\omega \ll \omega_{ce}$~~, ions immobile

only $p=0$ significant

$$k^2 + k_\perp^2 \frac{\omega_{pe}^2}{\omega_{ce}^2} + k_z \omega_{pe}^2 \int dv_z \frac{\frac{\partial G(v_z)}{\partial v_z}}{(\omega - k_z v_z)} = 0$$

$$n_{0e} G(v_z) \equiv 2\pi \int dv_\perp v_\perp \, F(v_z, v_\perp^2).$$

If $k_z v_z/\omega \ll 1$

$$\frac{1}{\omega - k_z v_z} = \frac{1}{\omega} \left( 1 + \frac{k_z v_z}{\omega} + \frac{k^2 v_z^2}{\omega^2} + \dots \right)$$

$$k^2 + \frac{k_\perp^2 \omega_{pe}^2}{\omega_{ce}^2} - \frac{k_z^2 \omega_{pe}^2}{\omega^2}\left(1 + 3\frac{k^2 v_{th}^2}{\omega^2} + \dots\right) \simeq 0$$

$$\omega^2 \approx \frac{k_z^2 \omega_{pe}^2}{k^2 + \frac{k_\perp^2 \omega_{pe}^2}{\omega_{ce}^2}}\left(1 + \frac{3 k_z^2 v_{the}^2}{\omega_0^2}\right)$$

If $\frac{\omega_{pe}^2}{\omega_{ce}^2} \ll 1$ , $k_z^2 v_{the}^2/\omega_0^2 \ll 1$

$$\boxed{\omega^2 \approx \frac{k_z^2}{k^2} \omega_{pe}^2} \ll \omega_{pe}^2 \quad \left(\text{if } \frac{k_z^2}{k^2} \ll 1\right)$$

Landau damping $\left(\frac{\omega}{k v_{the}} \gg 1\right)$

$$F_0 = n_0 \frac{\exp(-v^2/2v_{the}^2)}{(2\pi v_{the}^2)^{3/2}} \; ; \quad m v_{the}^2 = T_e$$

$$G(v_z) = \frac{\exp(-v_z^2/2v_{the}^2)}{(2\pi v_{the}^2)^{1/2}}$$

$$\frac{1}{\omega - k_z v_z} = \frac{P}{(\omega - k_z v_z)} - i\pi\,\delta(\omega - k_z v_z)$$

$$0 = k^2 - \frac{\omega_{pe}^2 k_z^2}{\omega^2} + \frac{i\pi k_z \omega_{pe}^2}{|k_z|} \frac{\partial G}{\partial v_z}\left(\frac{\omega}{k}\right)$$

$$\omega = \omega_0 + \delta\omega \; ; \qquad \frac{1}{\omega^2} = \frac{1}{(\omega_0+\delta\omega)^2} = \frac{1}{\omega_0^2} - \frac{2\delta\omega}{\omega_0^3}$$

$$0 = \underbrace{k^2 - \frac{\omega_{pe}^2}{\omega_0^2}k_z^2}_{0} + \frac{2\delta\omega}{\omega_0^3}\omega_{pe}^2 k_z^2 + \frac{i\pi k_z \omega_{pe}^2 \omega_0}{|k_z|(2\pi)^{1/2} k_z^3 v_{the}^3}\, e^{-\frac{\omega_0^2}{2k_z^2 v_{the}^2}}$$

$$\therefore \quad \delta\omega = -i\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega_0^4}{k_z^3 v_{the}^3}\exp\left(-\frac{\omega_0^2}{2k_z^2 v_{the}^2}\right)$$

$$\omega_0 \simeq \omega_{pe}\, k_z/k$$

$$\frac{\delta\omega}{\omega_0} = -i\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega_{pe}^3}{k^3 v_{the}^3}\exp\left(-\frac{1}{2k^2\lambda_{pe}^2}\right)$$

$$\boxed{\frac{\delta\omega}{\omega_0} = -i\left(\frac{\pi}{2}\right)^{1/2}\frac{1}{(k\lambda_{pe})^3}\exp\left(-\frac{1}{2k^2\lambda_{De}^2}\right)}$$

assumptions fulfilled if $k^2\lambda_{De}^2 \ll 1$

i.e $\frac{\delta\omega}{\omega_0} \ll 1$.

# Bernstein Waves

Let us look at a Maxwellian electron distribution with $k_z = 0$ $\left(\frac{\partial F}{\partial v_\perp^2} = -\frac{F}{2 V_{the}^2}\right)$

$$k^2 + \frac{\omega_{pe}^2}{V_{the}^2} \int d^3v \; F_e/n_0$$

$$- \frac{\omega_{pe}^2/n_0}{V_{the}^2} \int d^3v \sum_p \frac{\omega F_e \; J_p^2\left(\frac{k_\perp v_\perp}{\omega_c}\right)}{\omega - p\omega_c} = 0$$

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$$\int d^3v \frac{F_e}{n_0} = 1$$

$$\int d v_\perp v_\perp F_e(v) J_p^2\left(\frac{k_\perp v_\perp}{\omega_c}\right) = \int_0^\infty dv_\perp \frac{v_\perp}{V_{the}^2} e^{-v_\perp^2/2V_{the}^2} J_p^2\left(\frac{k_\perp v_\perp}{\omega_{ce}}\right)$$

$$x = \frac{v_\perp}{V_{the}}$$

Now $\int_0^\infty e^{-x^2/2} dx \, x \, J_p^2\left(\frac{k_\perp x V_{th}}{\omega_c}\right) = I_p\left(\frac{k_\perp^2 V_{the}^2}{\omega_{ce}^2}\right) e^{-\frac{k_\perp^2 V_{the}^2}{\omega_{ce}^2}}$

$I_p(x) \equiv$ Bessel function of Imaginary argument

$$\frac{k_\perp^2 V_{the}^2}{\omega_{ce}^2} = k_\perp^2 \rho_e^2$$

$$k^2 + \frac{\omega_{pe}^2}{V_{the}^2} - \frac{\omega_{pe}^2}{V_{the}^2} \sum_p \frac{\omega \, I_p\left(\frac{k_\perp V_{the}}{\omega_c}\right) e^{-k_\perp^2 V_{the}^2/\omega_c^2}}{\omega - p\omega_{ce}} = 0$$

There is no Landau Damping in this case.

Oscillations are stable and do not damp. !

$$I_p(x)e^{-x} \rightarrow \begin{cases} \left(\frac{x}{2}\right)^p / p!, & x \ll 1 \\ \frac{1}{\sqrt{2\pi}\, x^{1/2}}, & x \to \infty \end{cases}$$

$$I_p(x)e^{-x} > 0 \quad ; \quad I_{-p}(x) = I_p(x)$$

Only stable, pure oscillatory roots exist (i.e. $\omega_{\text{Imaginary}} = 0$)

$(b \equiv k_\perp^2 V_{the}^2 / \omega_{ce}^2)$

$$k^2 \lambda_{De}^2 = -1 + \sum_p \frac{\omega}{\omega - p\omega_{ce}} I_p(b) e^{-b}$$

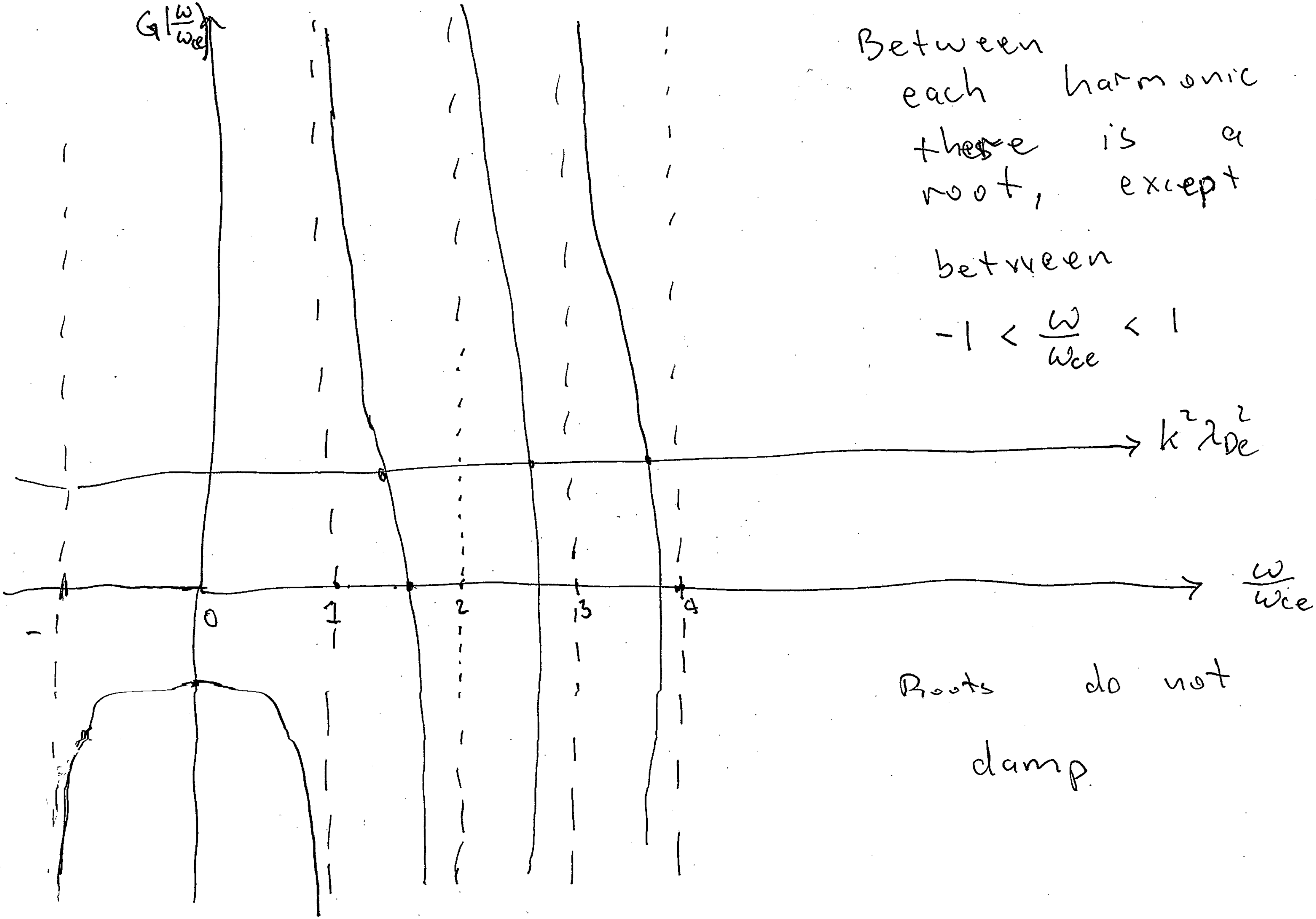
$$= -1 + \sum_p \frac{(\omega - p\omega_{ce}) + p\omega_{ce}}{\omega - p\omega_{ce}} I_p(b) e^{-b}$$

(one can show $\sum_p I_p(b) e^{-b} = 1$)

$$k^2 \lambda_{De}^2 = \sum_{p=-\infty}^{\infty} \frac{p \quad I_p(b) e^{-b}}{\frac{\omega}{\omega_{ce}} - p}$$

one can show that rhs is even function of $\frac{\omega}{\omega_{ce}}$

$$k^2\lambda_{De}^2 \equiv G\left(\frac{\omega}{\omega_{ce}}\right) \equiv \sum_{p=-\infty}^{\infty} \frac{p}{\left(\frac{\omega}{\omega_{ce}} - p\right)} I_p(b)e^{-b}$$




Between each harmonic ~~these~~ there is a root, except between $-1 < \frac{\omega}{\omega_{ce}} < 1$

Roots do not damp

Damping, with finite $k_\parallel$

Let $\frac{k_\parallel v_{the}}{p\omega_{ce}} \ll 1$, but not zero

$$0 = D(\omega) = 1 - \sum_{\substack{p=-\infty \\ p \neq p_0}}^{\infty}{}' \frac{p}{\frac{\omega}{\omega_{ce}} - p} I_p(b) e^{-b} \qquad x = v_\parallel / v_{the}$$

$$- p_0 \omega_{ce} I_{p}(b) e^{-b} \int \frac{dx \exp(-x^2/2)}{(2\pi)^{1/2} (\omega - k_{\cancel{z}} v_{the} x - p_0 \omega_{ce})}$$

$$= D(\omega_0 + \delta\omega) + i\left(\frac{\pi}{2}\right)^{1/2} p_0 \omega_{ce} I_p(b) e^{-b} \int_{-\infty}^{\infty} dx\, \delta(\omega - k_z v_{th} \cancel{x} - p\omega_{ce}) e^{-\frac{x^2}{2}}$$

$\therefore$

$$\frac{\delta\omega}{\omega_{ce}} = \frac{-i\omega_{ce}}{\omega_{ce} \frac{\partial D(\omega_0)}{\partial \omega}} \frac{I_{p_0}(b) e^{-b} \left(\frac{\pi}{2}\right)^{1/2} \exp\left(-\frac{(\omega - p_0\omega_{ce})^2}{2 k_z^2 v_{the}^2}\right)}{|k_z| v_{the}}$$

$$\omega_{ce} \frac{\partial D}{\partial \omega}(\omega_0) = \sum_p \frac{p}{\left(\frac{\omega_0}{\omega_{ce}} - p\right)^2} I_p(b) e^{-b}$$

$$\frac{\gamma_{LD}}{\omega_{ce}} = \frac{\left(\frac{\pi}{2}\right)^{1/2} I_{p_0}(b) e^{-b} \exp\left(-\frac{(\omega - p_0 \omega_{ce})^2}{2 k_z^2 V_{the}^2}\right)}{\sum_p \frac{p \; I_p(b)\, e^{-b}}{\left(\frac{\omega_0}{\omega_{ce}} - p\right)^2}}$$

Valid expression if

$$\frac{\omega - p_0 \omega_{ce}}{k_z V_{the}} \gg 1$$