Lecture # 26

Plasma Oscillations in large magnetic Field;

Discussion of Δ' numerical solution

$$F = F_0 + f \; ; \qquad F_0 \simeq F_0(v_\parallel, v_\perp, X_g = x - v_y/\omega_c)$$

$$\frac{Df}{Dt} \simeq \left(\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} + \omega_c \underset{\sim}{v} \times \underset{\sim}{b} \cdot \frac{\partial}{\partial \underset{\sim}{v}}\right) f$$

$$= \frac{e}{m}\left[\underset{\sim}{v_\perp} \cdot \underset{\sim}{\nabla}\phi \frac{\partial F_0}{\partial v_\perp^2/2} + \frac{\partial \phi}{\partial z} v_\parallel \frac{\partial F_0}{\partial v_\parallel} - \frac{1}{\omega_c}\frac{\partial F}{\partial X_g}\frac{\partial \phi}{\partial y}\right]$$

$$\phi \simeq \exp\left(i\int_{r_0}^{r} \underset{\sim}{k} \cdot d\underset{\sim}{r}' - i\omega t\right)$$

$$\underset{\sim}{k} = k_z \hat{z} + k_y \hat{y} + k_x(x)\hat{\underset{\sim}{x}}$$

$$k_x L_p \gg 1, \quad \frac{dk_x(x)}{k_x^2 dx} \ll 1$$

$$L_p \simeq F_0/\partial F_0/\partial x$$

$$\underset{\sim}{v_\perp} \cdot \nabla\phi = \frac{D}{Dt}\phi - \left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z}\right)\phi$$

$$f = \frac{e}{m}\phi \frac{\partial F_0}{\partial v_\perp^2/2} + g \quad , \quad (v_\parallel \equiv v_z, \; k_\parallel \equiv k_z)$$

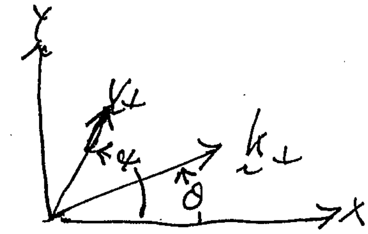
$$\frac{Dg}{Dt} = \frac{e}{m}\left[\left(-\frac{\partial \phi}{\partial t} - v_z \frac{\partial \phi}{\partial z}\right)\frac{\partial F}{\partial v_\perp^2/2} + \frac{\partial \phi}{\partial z}\frac{\partial F}{\partial v_z} - \frac{\partial F/\partial X_g}{\omega_c}\frac{\partial \phi}{\partial y}\right]$$

If $\phi = \phi_{k\omega} \exp\left[-i\omega t + i\int \underset{\sim}{k} \cdot d\underset{\sim}{r}\right]$

$$= \frac{ie}{m}\left[(\omega - k_z v_z)\frac{\partial F}{\partial v_\perp^2/2} + k_z \frac{\partial F}{\partial v_z} - \frac{k_y}{\omega_c}\frac{\partial F}{\partial X_g}\right]\phi_{k\omega}\exp\left[-i\omega t + i\underset{\sim}{k} \cdot \underset{\sim}{r}\right]$$

$$\frac{Dg}{Dt} = e^{-i\omega t}\frac{ie}{m}\left[(\omega - k_z v_z)\frac{\partial F}{\partial v_\perp^2/2} + k_z \frac{\partial F}{\partial v_z} - \frac{k_y}{\omega_c}\frac{\partial F}{\partial X_g}\right]\phi_{k,\omega}\exp\left[i\underset{\sim}{k} \cdot \underset{\sim}{R_g} + \frac{\underset{\sim}{b} \times \underset{\sim}{v} \cdot \underset{\sim}{k}}{\omega_c}\right]$$

(1)



$$\underset{\sim}{k} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\underset{\sim}{v} = \cos\psi\,\hat{x} + \sin\psi\,\hat{y}$$

$$\underset{\sim}{b}\cdot v_\perp \times \underset{\sim}{k_\perp} = k_\perp v_\perp \sin(\psi-\theta)$$

$$\underset{\sim}{k}\cdot \underset{\sim}{b}\times\underset{\sim}{v} = \underset{\sim}{b}\cdot(v\times \underset{\sim}{k})$$

$$= k_z v_z + k_\perp v_\perp \sin(\psi-\theta)$$

$$g(\underset{\sim}{v}, t) = \frac{ie}{m}\left[(\omega - k_z v_z)\frac{\partial}{\partial v_\perp^2/2} + k_z\frac{\partial}{\partial v_z} - \frac{k_y}{\omega_c}\frac{\partial}{\partial X_g}\right] F_0(v_\perp^2, v_z, X_g)$$

$$\times e^{-i\omega t}\int_{-\infty}^{t} dt' \exp\left[-i\omega(t'-t) + i\,\underset{\sim}{k}\cdot \underset{\sim}{R_g}(t') + i\frac{k_\perp v_\perp}{\omega_c}\sin\left(\psi - \omega_c(t'-t) - \theta\right)\right]$$

$$R_g(t') = \underset{\sim}{r} + \underset{\sim}{\hat{z}}\, v_\parallel(t'-t) - \frac{\underset{\sim}{b}\times\underset{\sim}{v}(t')}{\omega_c}$$

$$= \frac{ie}{m} e^{i\underset{\sim}{k}\cdot r - i\omega t}\left[(\omega - k_z v_z)\frac{\partial}{\partial v_\perp^2/2} + k_z\frac{\partial}{\partial v_z} - \frac{k_y}{\omega_c}\frac{\partial}{\partial X_g}\right] F_0(v_\perp^2, v_\parallel, X_g)$$

$$\int_{-\infty}^{\infty} d\tau \left[\exp\left[-i(\omega - k_z v_z)\tau - i\frac{k v_\perp}{\omega_c}\overset{\sin}{(}\psi - \omega_c \tau - \theta) + i\frac{k_\perp v_\perp}{\omega_c}\sin(\psi-\theta)\right]\right]$$

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\theta}$$

$$g\, e^{-i\underline{k}\cdot\underline{r} + i\omega t} = \frac{ie}{m}\left[(\omega - k_z v_z)\frac{\partial}{\partial v_\perp^2/2} + k_z\frac{\partial}{\partial v_z} - \frac{k_y}{\omega_c}\frac{\partial}{\partial X_g}\right]F_0$$

$$\times\, \phi_k \sum_{p,p'=-\infty}^{\infty} \int^{0} d\tau\, \exp\left[-i(\omega - k_\parallel v_\parallel)\tau + ip(\omega_c\tau - \psi + \theta) - ip'(\theta - \psi)\right]$$

$$J_p\left(\frac{k_\perp v_\perp}{\omega_c}\right) J_{p'}(k v_\perp/\omega_c)$$

$$= -\phi_k \frac{e}{m}\left[(\omega - k_z v_z)\frac{\partial}{\partial v_\perp^2/2} + k_z\frac{\partial}{\partial v_z} - \frac{k_y}{\omega_c}\frac{\partial}{\partial x_g}\right]F_0$$

$$\sum_{p,p'} \frac{J_p\left(\frac{k_\perp v_\perp}{\omega_c}\right) J_{p'}\left(\frac{k_\perp v_\perp}{\omega_c}\right)\exp\left[-i(p-p')(\psi-\theta)\right]}{(\omega - p\omega_c - k_z v_z)}$$

Therefore g can be placed in Poisson's equation.

As $\int d^3v = \int dv_\perp\, v_\perp\, d\psi\, dv_z$

The ψ integration will demand $p = p'$! Hence

$-\nabla^2\phi = 4\pi \sum_{\substack{spec\\ j}} e_j \int d^3v\, f_j$, becomes

$$k^2 \phi_{k,\omega} = \phi_{k\omega} \sum_j \omega_{pj}^2 \int d^3v \, \frac{\partial F_j}{\partial v_\perp^2/2}$$

$$- \phi_{k,\omega} \sum_j \omega_{pj}^2 \int d^3v \left[\frac{\partial F_j}{\partial v_\perp^2}(\omega - k_z v_z) + \frac{\partial F_j}{\partial v_\parallel} k_z - \frac{k_y}{\omega_c} \frac{\partial F}{\partial X_g} \right]$$

$$\cdot \sum_p \frac{J_p^2(k v_\perp/\omega_c)}{\omega - k_z v_z - p\omega_c}$$

Dispersion Relation in a Magnetic Field

$$k^2 - \sum_j \left[\omega_{pj}^2 \int d^3v \, \frac{\partial F_j}{\partial v_\perp^2/2} - \sum_p \omega_{pj}^2 \int d^3v \frac{\left(\frac{\partial F_j}{\partial v_\perp^2/2} (\omega - k_z v_z) + k_z \frac{\partial F_j}{\partial v_z} - \frac{k_y}{\omega_c} \frac{\partial F}{\partial X_g} \right) J_p^2\left(\frac{k v_\perp}{\omega_c}\right)}{\omega - k_z v_z - p\omega_c} \right]$$

$$= D(k_z, \underset{\sim}{k_\perp}, \omega)$$

If we wish to study low frequency, long wave length waves, $(\omega \ll \omega_{cj}\,,\; k_\perp \rho_i \ll 1)$

$$J_p^2\left(\frac{k_\perp v_\perp}{\omega_c}\right) \propto \left(\frac{k_\perp v_\perp}{\omega_c}\right)^{2p} \,, \quad \frac{\omega}{\omega_c} \ll 1$$

Only $p = 0$ term important

$$0 = k^2 - \sum_j \omega_{pj}^2 \int d^3v \left[\frac{\partial F_j}{\partial \frac{v_\perp^2}{2}} \left(1 - J_0^2\left(\frac{k_\perp v_\perp}{\omega_c}\right)\right) + \frac{k_z \frac{\partial F}{\partial v_z}}{\omega - k_z v_z} + \frac{k_y}{\omega_c} \frac{\partial F}{\partial x_g} \frac{1}{\omega - k_z v_z} \right]$$

$$J_0 = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{ix\sin\theta} \approx \int_0^{2\pi} \frac{d\theta}{2\pi} \left[1 + ix\sin\theta - x^2 \frac{\sin^2\theta}{2}\right]$$

$$= 1 - \frac{x^2}{4}$$

$$J_0^2 \simeq 1 - \frac{x^2}{2}$$

$$0 = k^2 - \sum_j \omega_{pj}^2 \int \left(\frac{\partial F_0}{\partial v_\perp^2/2} \frac{k_\perp^2 v_\perp^2}{2\omega_c^2} + \frac{k_z \frac{\partial F}{\partial v_z} + \frac{k_y}{\omega_c} \frac{\partial F}{\partial x_g}}{\omega - k_z v_z} \right) d^3v$$

Let's look at electron plasma oscillations in a magnetic field, with the assumption that the ions are rigid.

Also neglect $\frac{k_y}{\omega_c}\frac{\partial F}{\partial X_g}$ for now

$$k^2 - \omega_{pe}^2 \int d^3v \, \frac{\partial F_0}{\partial v_\perp^2/2} \, \frac{k_\perp^2 v_\perp^2}{2\omega_c^2} + k_z \int d^3v \, \frac{\frac{\partial F}{\partial v_z}}{(\omega - k_z v_z)} = 0$$

$$\overset{\shortparallel}{-k_\perp^2/\omega_c^2}$$

$$k^2 + \frac{\omega_{pe}^2}{\omega_{ce}^2} k_\perp^2 + \omega_{pe}^2 \int dv_z \, \frac{\frac{\partial F_\parallel(v_z)}{\partial v_z} k_z}{\omega - k_z v_z} = 0$$

$$F_\parallel(v_z) = \int dv_\perp v_\perp 2\pi \, F(v_\perp, v_z)$$

Looks a lot like old electron plasma oscillation, except that both $k_\perp^2$ and k_z^2 appear. Typically $\frac{\omega_{pe}^2}{\omega_{ce}^2} < 1$

$(n = 3\times10^{13},\ B = 1T,\ \frac{\omega_{pe}^2}{\omega_{ce}^2} = 1/4)$

suppress "t" in $F_{\parallel}(v_{\parallel})$

$$\left(1+\frac{\omega_{pe}^2}{\omega_{ce}^2}\right)k_\perp^2 + k_z^2 + \omega_{pe}^2\, k_z \int \frac{dv_z \,\frac{\partial F(v_\parallel)}{\partial v_\parallel}}{\omega - k_z v_z} = 0$$

If $\omega \gg k v_{the}$

$$\frac{1}{\omega - k_z v_z} = \frac{1}{\omega}\left(1 + \frac{k_z v_z}{\omega} + \cdots\right)$$

$$\left(1+\frac{\omega_{pe}^2}{\omega_{ce}^2}\right)k_\perp + k_z^2 + \frac{\omega_{pe}^2}{\omega^2} k_z^2 \int dv_z \frac{\partial F}{\partial v_z} v_z \left(1 + \frac{k^2 v_z^2}{\omega^2} + \cdots\right)$$

assumed $F(v_z) = F(-v_z)$

$$\left(1+\frac{\omega_{pe}^2}{\omega_c^2}\right)k_\perp^2 + k_z^2 - \frac{\omega_{pe}^2}{\omega^2} k_\parallel^2 \left(1 + 3\frac{k_\parallel^2 v_{th}^2}{\omega^2} + \cdots\right) = 0$$

$$\boxed{\omega^2 = \frac{\omega_{pe}^2 k_\parallel^2}{\left(1+\frac{\omega_{pe}^2}{\omega_{ce}^2}\right)k_\perp^2 + k_\parallel^2}\left(1 + \frac{3\left[\left(1+\frac{\omega_{pe}^2}{\omega_{ce}^2}\right)k_\perp^2 + k_\parallel^2\right]v_{th}^2}{\omega_{pe}^2}\right)}$$

$$v_{th}^2 = \int d^3v\, F v^2$$

valid if $\left[k_\perp^2\left(1+\frac{\omega_{pe}^2}{\omega_c^2}\right)v_{th}^2 + k_\parallel^2 v_{th}^2\right]/\omega_{pe}^2 \ll 1$

or typically $k\lambda_D \ll 1$

If $k_\perp \gg k_\parallel$ $\quad \omega \simeq \omega_{pe}\frac{k_\parallel}{k_\perp} \ll \omega_{pe}$

$\omega_{pe} < \omega_{ce}$