Lecture # 24

Kinetic Instabilities

$\boxed{\text{Now if } \frac{\omega}{k v_{the}} \gg 1}$

$$D(\omega, k) \simeq 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{3k^2 v_{th}^2}{\omega^2}\right) - i\pi \frac{\omega_p^2}{k^2} \frac{\partial F_0\left(\frac{\omega}{k}\right)}{\partial v} = 0$$

$$D_R(\omega_0, k) = 0 \quad ; \quad D(\omega) = D_R(\omega) + i D_I(\omega)$$

$$\omega = \omega_0 + \delta\omega \quad ; \quad \omega_0^2 = \omega_{pe}^2\left(1 + \frac{3k^2 v_{the}^2}{\omega_{pe}^2}\right)$$

$$D_R(\omega_0 + \delta\omega) + i D_I(\omega_0, k) \simeq 0$$

↑ expand

$$D_R(\omega_0) + \delta\omega \frac{\partial D_R(\omega_0)}{\partial \omega} + i D_I(\omega_0) = 0$$

$$\overset{\shortparallel}{0}$$

$$\delta\omega = -i D_I(\omega_0) \Big/ \frac{\partial D_R(\omega_0)}{\partial \omega}$$

For $\quad F_0\left(\frac{\omega}{k}\right) = \frac{1}{(2\pi v_{th}^2)^{1/2}} \exp\left(-\frac{v^2}{2 v_{th}^2}\right)\Big|_{v=\frac{\omega}{k}}$

$$D_I = \frac{\omega_0 \omega_{pe}^2}{k\, k^2 v_{the}^2} \exp\left(-\frac{\omega_0^2}{2k^2 v_{th}}\right)\left(\frac{\pi}{2}\right)^{1/2}$$

$$\therefore \quad \delta\omega = -i\left(\frac{\pi}{2}\right)^{1/2} \omega_p^2 \frac{\omega_0}{k^3 v_{th}^2} \frac{\exp\left(-\omega_0^2/2k^2 v_{th}^2\right)}{\frac{\partial D_R(\omega_0)}{\partial \omega}}$$

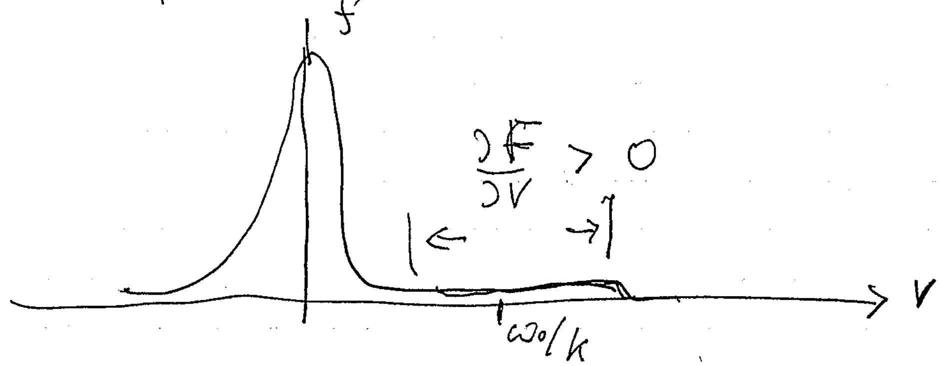
$$\frac{\partial D_R}{\partial \omega}(\omega_0, k) = \frac{2}{\omega_{pe}}\left(1 + \frac{3k^2 v_{the}^2}{2\omega_{pe}^2}\right) \quad \left(\begin{array}{l}\text{obtained after} \\ \text{some algebra}\end{array}\right)$$

$$\delta\omega \simeq -i\left(\frac{\pi}{2}\right)^{1/2} \frac{\omega_{pe}^3}{2|k|\, k^2 v_{the}^2}\Bigg)\left(1 + O\left(\frac{k^2 v_{th}^2}{\omega_{pe}^2}\right)\right) \exp\left[-\frac{\omega_{pe}^2}{2k^2 v_{the}^2} - \frac{3}{2}\right]$$

$$\equiv -i\,\gamma_{damping}$$

Bump-on-tail instability

Suppose we added a mild bump to the distribution function



example

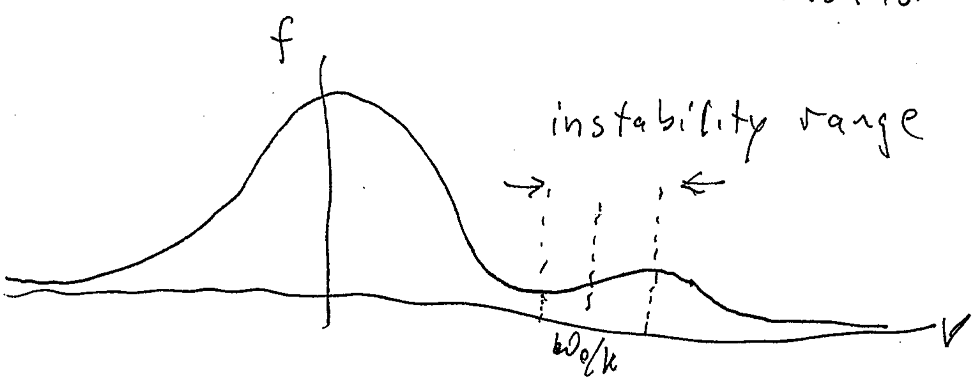
$$F = \frac{(1-\epsilon)}{(2\pi)^{1/2}} \left(\frac{\exp\left(-\frac{2V^2}{V_{the}^2}\right)}{V_{the}} \right) + \frac{\epsilon}{(2\pi V_b^2)^{1/2}} \exp\left(-\frac{(V-V_0)^2}{V_b^2}\right)$$

$V_0 \gg V_{the}$

If ϵ is large enough, a double humped distribution emerges

$$\gamma \simeq -i\left(\frac{\pi}{2}\right)^{1/2} \frac{\omega_{pe}^3}{2|k|k^2 V_{the}^2} \left[\exp\left(-\frac{\omega_0^2}{2k^2 V_{th}^2}\right) + \left(\frac{V_{the}}{V_b}\right)^3 \frac{\left(\frac{\omega_0}{k} - V_0\right)\epsilon}{\omega_{pe}/k} \cdot \exp\left(-\frac{\left(\frac{\omega_0}{k} - V_0\right)^2}{V_b^2}\right) \right]$$

If $V_0 > \frac{\omega_0}{k}$, possibility of instability (ϵ large enough)



Sound Waves

$$\frac{\partial f_{e,i}}{\partial t} + v\frac{\partial f_{e,i}}{\partial x} = \frac{Ze}{m}\nabla\phi\,\frac{\partial F_{e,i}(v)}{\partial v}, \qquad Z_e=-1 \text{ electron}, \quad Z_i = 1 \text{ ions}$$

$$\nabla^2\phi = 4\pi e\left[\int d^3v\, f_e - \int d^3v\, f_i\right]$$

As before, if we look

solution going as

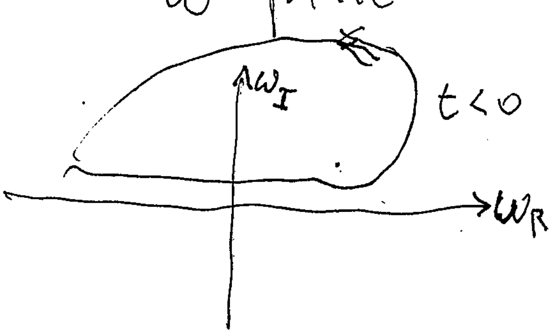
$$f_j = \int_C f_j(\omega,k)\,\exp(ikx - i\omega t)\,\frac{d\omega}{2\pi}$$

$$f_e = +\frac{e}{m}\,k\phi_k\,\frac{\partial F_e(v)}{\partial v}\Big/(\omega - kv)$$

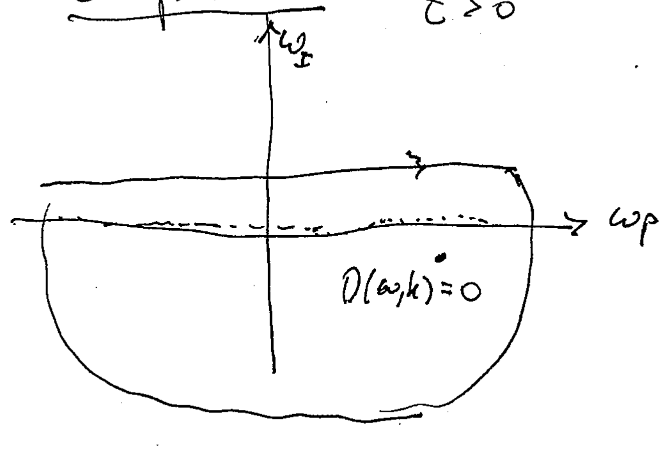
$$f_i = -\frac{e}{m}\,k\,\phi_k\,\frac{\partial F_i}{\partial v}\Big/\,\omega - kv$$

$$\phi_k(t) = \int_{C(\omega)}\frac{d\omega}{2\pi}\,e^{-i\omega t}\sum_j\left[\frac{4\pi e_j\int dv\,\delta f_{jk}(v,t=0)/(\omega-kv)}{k^2 D(\omega,k)}\right]$$

ω - plane



ω - plane t>0



$$D(\omega,k) \equiv 1 + \frac{1}{k^2}\left(\omega_{pe}^2\int \frac{dv\, k\frac{\partial F_e(v)}{\partial v}}{\omega - kv} + \omega_{pi}^2\int \frac{dv\frac{\partial F_i}{\partial v}k}{(\omega-kv)}\right)$$

We will assume

$$\frac{\omega}{k V_{thi}} \gg 1 \; ; \quad \frac{\omega}{k V_{the}} \ll 1$$

Natural oscillation frequencies arise whe $D(\omega_0, k) = 0$

Take $F_e(v) = \frac{1}{(2\pi V_{the}^2)^{1/2}} \exp\left(-\frac{v^2}{2V_{the}^2}\right)$

$$F_i(v) = \frac{1}{(2\pi V_{thi}^2)^{1/2}} \exp\left(-\frac{v^2}{2V_{thi}^2}\right)$$

Ion contribution to dispersion is just like electrons were in electron plasmas wave

$$\frac{\omega_{pi}^2}{k^2}\int dv \frac{\frac{\partial F_i}{\partial v}k}{\omega - kv} \simeq \frac{-\omega_{pi}^2}{\omega^2}\left(1 + \frac{3k^2V_{thi}^2}{\omega^2} + \cdots\right) - \frac{i}{k|k|}\frac{\omega_{pi}^2\pi}{V_{thi}}\frac{\partial F_i}{\partial v}\left(\frac{\omega}{k}\right)$$

electrons in limit $\frac{\omega}{k v_{the}} \ll 1$

$$-\frac{\omega_{pe}^2}{k^2} \int dv \frac{k \frac{\partial F_e}{\partial v}}{kv} - \frac{i\omega_{pe}^2 \pi}{|k| k} \frac{\partial F_e}{\partial v}\left(\frac{\omega}{k}\right)$$

$$\left[\text{use} \quad F_e = \frac{1}{(2\pi)^{1/2}} \frac{\exp\left(-\frac{v^2}{2v_{the}^2}\right)}{v_{the}}\right]$$

$$\simeq \frac{\omega_{pe}^2}{k^2 v_{the}^2} + \frac{i\pi^{1/2}\omega_{pe}^2}{\sqrt{2}\,k^2 |k|} \frac{\omega}{v_{the}^3}$$

Thus: for sound wave

$$D(\omega, k) = 1 - \frac{\omega_{pi}^2}{\omega^2}\left(1 + 3\frac{k^2 v_{thi}^2}{\omega^2}\right) + \frac{\omega_{pe}^2}{k^2 v_{the}^2}$$

$$+ \frac{i\pi^{1/2}\omega_{pe}^2\, \omega}{\sqrt{2}\, k^2 |k|\, v_{the}^3} + \frac{i\pi^{1/2}\omega\, \omega_{pi}^2 e^{-\omega^2/k^2 v_{thi}^2}}{\sqrt{2}\, k^2 |k|\, v_{thi}^3}$$

$$\approx D_R(\omega_0 + \delta\omega) + i D_I(\omega_0)$$

$$\simeq \delta\omega \frac{\partial D_R}{\partial \omega}(\omega_0) + i D_I(\omega_0)$$

$$\delta\omega = -i D_I(\omega_0) \Big/ \frac{\partial D_R}{\partial \omega}(\omega_0)$$

$k^2 \lambda_{De}^2 << 1$

with some algebra one obtains (double check for accuracy) $\left(\tau = \frac{T_e}{T_i}\right)$

$$\frac{\delta\omega}{\omega_0} = -i \frac{\left(\frac{\pi}{2}\right)^{1/2}}{2} \left(1 + \frac{3}{\tau}\right)^{3/2} \left[\left(\frac{m}{M}\right)^{1/2} + \left(\frac{T_e}{T_i}\right)^{3/2} \exp\left(-\frac{(\tau+3)}{2}\right) \right]$$

$$= -i \frac{\left(\frac{\pi}{2}\right)^{1/2}}{2} (\tau + 3)^{3/2} \left[\left(\frac{m}{M\tau}\right)^{1/2} + \exp\left(-\frac{(\tau+3)}{2}\right) \right]$$

electron contribution (intrinsically small)

ion contribution (small only if $\tau >> 1$)

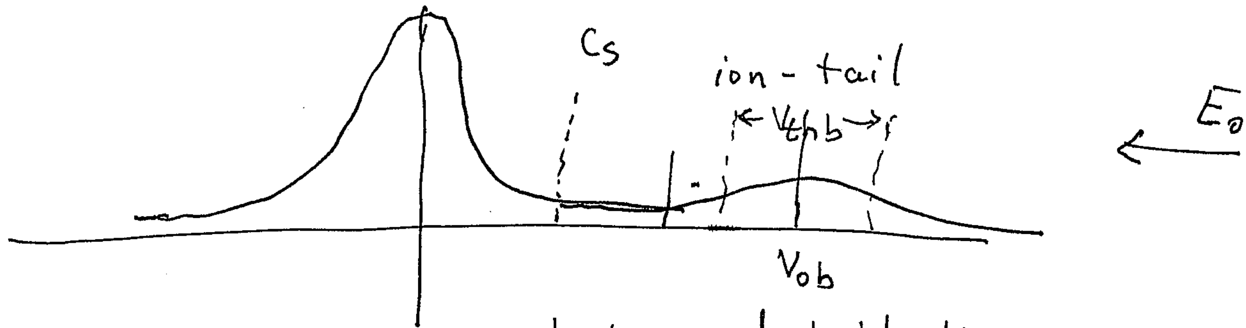
$$\omega_0 \simeq k c_s \left(1 + 3/\tau\right)^{1/2}$$

How can we get instabilities?

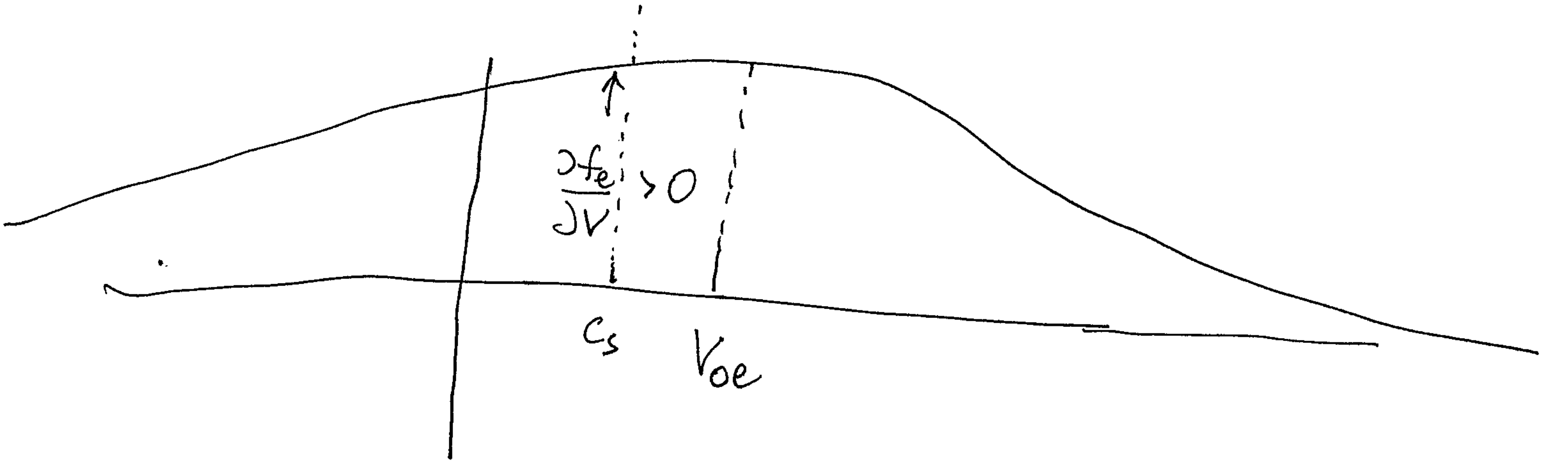
(1) Prepare ions with a tail

(2) Move electrons with respect to ions

Ion Distribution



electron distribution



$$f_i(v) = \frac{(1-\epsilon)}{(2\pi)^{1/2} V_{thi}} \exp\left(-\frac{v^2}{2V_{thi}^2}\right) + \frac{\epsilon}{V_{thb}(2\pi)^{1/2}} \exp\left(-\frac{(v-V_{ob})^2}{V_{thb}^2}\right)$$

$$F_e(v) = \frac{\exp\left(-\frac{(V-V_{oe})^2}{2V_{the}^2}\right)}{(2\pi V_{the}^2)^{1/2}}$$

If $V_{oe} \ll V_{the}$, real part of $D(\omega, k)$ almost same as before

But imaginary part can change sign to give

$$D_{Ie}(\omega) \rightarrow D_{Ie}(\omega - k v_{0e})$$

$$\Downarrow$$

$$i\left(\frac{\pi}{2}\right)^{1/2} \frac{\omega\, \omega_{pe}^2}{k^2\, k\, v_{the}^3} \longrightarrow i\left(\frac{\pi}{2}\right)^{1/2} \frac{(\omega - k v_{0e})}{k^2\, k\, v_{the}^3}$$

If bump-on-ion-tail a Maxwellian

$$\frac{\epsilon}{(2\pi v_{thb}^2)^{1/2}} \exp\left(-\frac{(v - v_{0b})^2}{2 v_{thb}^2}\right)$$

The frequency shift due to these two distributions is

shifted electron drive

$$\delta\omega \simeq -i\,\frac{\left(\frac{\pi}{2}\right)^{1/2}}{2}(\tau+3)^{1/2}\left(\frac{m}{M\tau}\right)^{1/2}\left(1 - \frac{v_{0e}}{c_s(1+3/\tau)^{1/2}}\right)$$

bump on ion tail drive

$$-i\,\frac{\left(\frac{\pi}{2}\right)^{1/2}}{2}(\tau+3)^{1/2}\left[\exp\left(-\frac{(\tau+3)}{2}\right) - \epsilon\,\frac{\left(v_{0b} - c_s(1+3/\tau)^{1/2}\right) v_{thi}^3}{c_s(1+3/\tau)^{1/2}\, v_{thb}^3} \times \exp\left[-\frac{\left(v_{0b} - c_s(1+3/\tau)^{1/2}\right)^2}{2 v_{tb}^2}\right]\right]$$

$\text{Im}\,\omega > 0$ is possible from both shifted electrons and bump-on-tail.