Lecture # 23

Kinetic

Damping and Growth
Dispersion Relation for 1-d Homogeneous Plasma (No external field)

\[ k^2 + \sum_j \int \frac{dV}{n_{0j}} \frac{\partial F_j}{\partial V_j} \frac{\omega_p^2}{\omega - kv} = 0 \]

\[ F_j = n_i \frac{\exp \left( -\frac{V_i^2}{2 V_{thi}^2} \right)}{\left( 2\pi V_{thi}^2 \right)^{3/2}} \]

For a plasma wave
\[ \omega^2 = \omega_{pe}^2 + 3 k^2 V_{th}^2 \quad \gamma = 3 \]

For sound wave
\[ k V_{th} \gg \omega \quad k V_{thi} \ll \omega \]

For electrons:
\[ \int_{-\infty}^{\infty} dV \frac{k \frac{\partial F_e}{\partial V}}{\omega - kv} = - \int_{-\infty}^{\infty} dV \frac{\partial F_e}{\partial V} \]

\[ \frac{\partial F_e}{\partial V_e} = -\frac{e}{V_{th}} F_e \quad (\text{for Maxwellian}) \]

For ions
\[ k V \ll \omega \]
\[ \int_{-\infty}^{\infty} dV \frac{k \frac{\partial F_i}{\partial V}}{\omega - kv} \]
\[ \approx \int_{-\infty}^{\infty} dV \frac{\partial F_i}{\partial V} \left( 1 + \frac{kv}{\omega} + \frac{k^2 V_e^2}{\omega^2} + \frac{k^3 V_e^4}{\omega^3} + \cdots \right) \]

\[ \approx -\frac{k^2}{\omega^2} \left( 1 + \frac{3 k^2 V_{thi}^2}{\omega^2} \right) \quad (1) \]
Sound Wave Dispersion Relation

\[ k^2 + \frac{\omega \dot{\nu}_e}{V_{\text{the}}} - k^2 \frac{\omega \dot{\nu}_i}{V_{\text{the}}} \left( 1 + 3 \frac{k^2 V_{\text{thi}}}{\omega^2} \right) = 0 \]

\[ \omega^2 \frac{\rho \dot{\nu}_e}{V_{\text{the}}} = \frac{\omega \dot{\nu}_i}{V_{\text{thi}}} \frac{T_i}{T_e} \]

\[ \omega^2 = \frac{k^2 C_s^2 \left( 1 + 3 \frac{k^2 V_{\text{thi}}}{\omega^2} \right)}{1 + k^2 \lambda_{\text{De}}^2}, \quad C_s^2 = \frac{V_{\text{thi}}}{T_i} \]

If \( \frac{k^2 V_{\text{thi}}}{\omega^2} \ll 1 \)

\[ \omega^2 = \frac{k^2 C_s \left( 1 + 3 \frac{T_i}{T_e} \right)}{1 + k^2 \lambda_{\text{De}}^2} \]

\[ \rightarrow \begin{cases} 
\omega^2 \dot{\nu}_i \left( 1 + 3 \frac{T_i}{T_e} \right), & \left( k^2 \lambda_{\text{De}}^2 \ll 1 \right) \\
\omega \dot{\nu}_i, & \left( k^2 \lambda_{\text{De}}^2 \gg 1, \frac{3T_i}{T_e} \ll 1 \right) 
\end{cases} \]

Looking at sound wave we have, when comparing with fluid result

\( \gamma_i = 3, \quad \gamma_e = 1 \)

\[ \omega^2 = \gamma_i k^2 V_{\text{thi}}^2 + \gamma_e k^2 V_{\text{the}}^2 \quad (\text{fluid result}) \]
Caution about acoustic mode

We assumed \( \frac{k V_{thi}}{w} \ll 1 \). Is this the case?

\[
\frac{\omega^2}{k V_{thi}} = \frac{k^2 (V_{thi}^2 T_e/T_i + 3 V_{thi}^2)}{k^2 V_{thi}} = \frac{T_e}{T_i} + 3
\]

(3 is somewhat large, but not really)

\( \frac{T_e}{T_i} \) needs to be large to have unequivocal validity of acoustic wave solution

When \( \frac{k V_{thi}}{w} = 1 \) we have to treat dispersion relation more precisely
Let's look at electron plasma oscillations.

\[ D(\omega, k) = 1 + \frac{\omega^2}{\omega_0^2} \int_{\Delta v} \frac{2 \frac{E}{e}}{\omega - kv} = 0 \]

How should one treat denominator that has zero \( \frac{1}{\omega - kv} \).

For this we need to remember that in the initial value problem, with \( \phi(\nu, \tau = 0) \) given

\[ \phi(k, t) = \int_{C} d\omega e^{-i\omega t} \int \frac{d\nu \frac{\delta \phi(\nu, \tau = 0)}{\omega - kv}}{D(\omega, k)} \]

With the contour \( C \) required to be in the upper half plane.

\[ \omega \uparrow \]
\[ \omega \]
\[ \omega_0 \]
\[ \omega_\tau \]
\[ D(\omega_0, k) = 0 \]
Note \( \Phi(w, h) \) and \( \int \frac{dv}{\omega - hv} \) are analytic in upper half plane when \( \text{Im} \omega > 0 \), if \( \text{Im} \omega \) sufficiently large.

We note that for \( t < 0 \), \( \Phi_k(t) = 0 \) which is obtained by closing contour in upper half plane.

\[ e^{-iwt} \rightarrow 0 \]
if \( \text{Im} \omega > 0 \)
\( t < 0 \)

However, integral cannot be closed in upper half plane if \( t > 0 \).

We can close in lower half plane, but then there are singularities to pick up, especially zero of \( D(w, h) \), common to any initial perturbation.
When we consider the dispersion function

\[ \int_{-\infty}^{\infty} \frac{dF_k(v)}{\partial \nu} \]

The contour integral is \( w \) in the upper half plane. As we lower contour we hit poles of the real axis. If we lower \( w \) into lower half plane we either have to pick up a continuum of poles, \( \omega = kV \), or \( \frac{dF(v)}{\partial \nu} \) is analytic on the real \( \nu \)-axis, we can distort \( \nu \)-contour into the complex plane so that \( \text{Im} \frac{w}{\nu} > \text{Im} V \), and no pole need to be picked up.

This analytic continuation procedure must be employed in order to evaluate the dispersion relation.
$$D(v, k) = 1 + \frac{W}{k^2 \omega} \int \frac{\partial F}{\partial \nu} \frac{\omega}{k - \nu}$$

When \( \frac{\omega}{k} \rightarrow 1 \)

Let extract imaginary part of \( D \) when \( \omega \) is real

$$\text{Im} = \frac{W}{k^2} \int \frac{\partial F}{\partial \nu} \frac{\omega}{k - \nu}$$

\( v \)-plane

\( \text{Im} \omega > 0 \)

Distortion of \( v \)-contour

leads to \( \frac{\partial}{\partial \nu} \int \frac{\omega}{k - \nu} = \frac{1}{\omega - \nu} \)

Take

\( V = \frac{\omega}{k} e^{i \theta} \)

\( dV = i \frac{\omega}{k} e^{i \theta} \ d\theta \)

\( \int dV \frac{1}{1 + e^{i \theta}} = -i \pi \)
Now if \( \frac{\omega}{\hbar v_{\text{the}}} \ll 1 \)

\[
\downarrow (\omega, k) \approx 1 - \frac{\omega_{pe}}{\omega} (1 + 3 \frac{\hbar^2 v_{\text{the}}^2}{\omega_{pe}^2}) - i \pi \frac{\omega_{pe}^2}{\hbar^2} \frac{2F(\frac{\omega}{\omega_{pe}})}{\omega_{pe}} \approx 0
\]

\[
D_R (\omega_0, k) = 0 \quad ; \quad D = D_R (\omega) + i D_I (\omega)
\]

\( \omega = \omega_0 + \delta \omega \)

\[
D_R (\omega_0 + \delta \omega, k) + i D_I (\omega_0, k) \approx 0
\]

\[
\delta \omega \frac{\partial \Omega_{\text{m}}}{\partial \omega} = - i D_I (\omega_0, k)
\]

\[
\frac{\partial \Omega_{\text{m}}}{\partial \omega} = \frac{2 \omega_{pe}}{\omega^3} (1 + 3 \frac{\hbar^2 v_{\text{the}}^2}{\omega_{pe}^2})
\]

\[
D_I (\omega, h) \approx \frac{\omega_{pe}^2}{\hbar^2} \frac{2F(\frac{\omega}{\omega_{pe}})}{\omega_{pe}}
\]

\[
\delta \omega = - \left( \frac{\hbar^2 v_{\text{the}}^2}{2 \omega_{pe}^2} \right) \exp \left( - \frac{\omega_0}{2 \hbar^2 v_{\text{the}}^2} \right) \frac{2}{\omega} \frac{\partial \Omega_{\text{m}}}{\partial \omega} (\omega_0)
\]

\[
\omega + h \approx \omega_{pe} (1 + 3 \frac{\hbar^2 v_{\text{the}}^2}{\omega_{pe}^2})
\]

\[
\frac{\partial D(\omega_0)}{\partial \omega} \approx \frac{2 \omega_{pe}}{\omega^3} + 12 \frac{\omega_{pe}^2 \hbar^2 v_{\text{the}}^2}{\omega^5} \frac{2}{\omega_{pe}}
\]