Lecture # 23

Kinetic Damping and Growth

Dispersion Relation for 1-d Homogeneous Plasma (No external fields)

$$k^2 + \sum_j \int \frac{dv}{n_{oj}} \frac{\frac{\partial F_j}{\partial v} k \;\; \omega_{pj}^2}{\omega - kv} = 0$$

$$F_j = \frac{n_j}{(2\pi V_{thj}^2)^{1/2}} \exp\left(-\frac{v^2}{2V_{thj}^2}\right) \; ; \; T_j = m V_{thj}^2$$

For electron plasma wave

$$\omega^2 = \omega_{pe}^2 + 3k^2 V_{the}^2 \; ; \; \gamma = 3$$

For sound wave

$$k V_{the} \gg \omega \; ; \quad k v_{thi} \ll \omega$$

For electrons: $\displaystyle\int_{-\infty}^{\infty} dv \frac{k \frac{\partial F_e}{\partial v}}{\omega - kv} \simeq -\int_{-\infty}^{\infty} \frac{dv \frac{\partial F_e}{\partial v}}{v} \simeq \int_{-\infty}^{\infty} \frac{dv}{V_{the}^2} \frac{F_e}{n_0} = \frac{1}{V_{the}^2}$

$$\frac{\partial F_e}{\partial V_e} = -\frac{v}{V_{the}^2} F_e \quad \text{(for Maxwellian)}$$

For Ions $kv \ll \omega$

$$\int_{-\infty}^{\infty} dv \frac{k \frac{\partial F_i}{\partial v}}{n_0(\omega - kv)} \simeq \int_{-\infty}^{\infty} \frac{dv \, k}{\omega} \frac{\partial F_i}{\partial v}\Big(\underset{0}{1} + \frac{kv}{\omega} + \underset{0}{\frac{k^2v^2}{\omega^2}} + \frac{k^3v^3}{\omega^3} + \dots\Big)$$

$$\simeq -\frac{k^2}{\omega^2}\left(1 + \frac{3k^2 V_{thi}^2}{\omega^2} + \dots\right)$$

Sound Wave Dispersion Relation

$$k^2 + \frac{\omega_{pe}^2}{V_{the}^2} - k^2\frac{\omega_{pi}^2}{\omega^2}\left(1 + 3\frac{k^2 V_{thi}^2}{\omega^2}\right) = 0$$

$$\therefore \quad \frac{\omega_{pe}^2}{V_{the}^2} = \frac{\omega_{pi}^2}{V_{thi}^2}\frac{T_i}{T_e}$$

$$\omega^2 = \frac{k^2 C_s^2\left(1 + \frac{3k^2V_{thi}^2}{\omega^2}\right)}{1 + k^2\lambda_{De}^2}, \quad C_s^2 = V_{thi}^2\frac{T_e}{T_i}$$

If $\frac{k^2 V_{thi}^2}{\omega^2} \ll 1$

$$\omega^2 = \frac{k^2 C_s^2\left(1 + 3\frac{T_i}{T_e}\right)}{1 + k^2\lambda_{De}^2}$$

$$\rightarrow \begin{cases} k^2 C_s^2\left(1 + \frac{3T_i}{T_e}\right), & (k^2\lambda_{De}^2 \ll 1) \\ \omega_{pi}^2, & (k^2\lambda_{De}^2 \gg 1, \ \frac{3T_i}{T_e} \ll 1) \end{cases}$$

Looking at sound wave we have, when comparing with fluid result $\gamma_i = 3$, $\gamma_e = 1$

$$\omega^2 = \gamma_i k^2 V_{thi}^2 + \gamma_e k^2 V_{the}^2 \quad \text{(fluid result)}$$

Caution about acoustic mode

We assumed $\frac{k V_{thi}}{\omega} << 1$;

is this the case?

$$\frac{\omega^2}{k^2 V_{thi}^2} = \frac{k^2 (V_{thi}^2 T_e/T_i + 3 V_{thi}^2)}{k^2 V_{thi}^2}$$

$$= \frac{T_e}{T_i} + 3$$

(3 is somewhat large, but not really

$\frac{T_e}{T_i}$ needs to be large ~~to~~ have unequivacable validity of acoustic wave solution)

When $\frac{k v_{th}}{\omega} \simeq 1$ we have to treat dispersion relation more precisely

Let's look at electron plasma oscillations

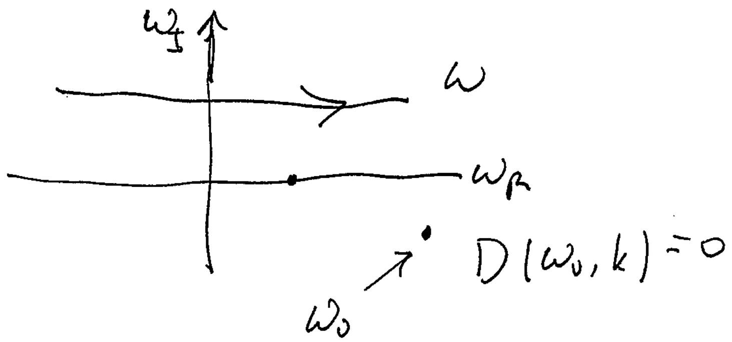
$$D(\omega, k) \equiv 1 + \frac{\omega_{pe}^2}{n_0} \int dv \frac{\frac{\partial F_e}{\partial v}}{\omega - kv} = 0$$

How should one treat denominator that has zero $\frac{1}{\omega - kv}$.

For this we need to remember that in the initial value problem, with $\delta f(v, t=0)$ given

$$\phi(k,t) = \int_C d\omega \frac{e^{-i\omega t} \int dv \frac{\delta f_k(v, t=0)}{\omega - kv}}{D(\omega, k)}$$

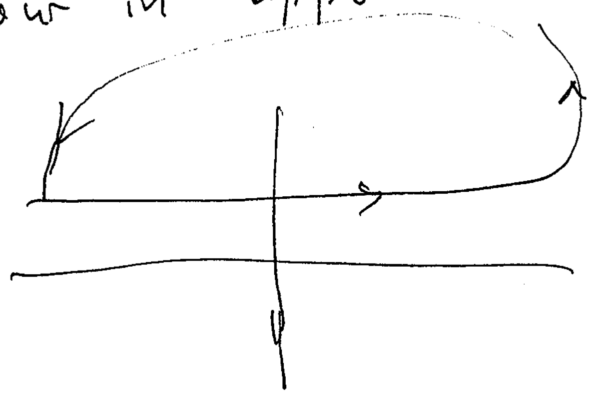
With the contour C required to be in the upper half plane



Note $D^{-1}(\omega, k)$ and $\int \frac{dv\, \delta f_k(v, t=0)}{\omega - kv}$

are analytic in upper half plane when $\operatorname{Im}\omega > 0$, if $\operatorname{Im}\omega$ sufficiently large

We note that for $t<0$, $\phi_k(t) = 0$ which is obtain by closing contour in upper half plane



$e^{-i\omega t} \to 0$ if $\operatorname{Im}\omega > 0$, $t<0$

However integral cannot be closed in upper half plane if $t>0$

We can close in lower half plane, but then there are singularities to pick up, especially zero of $D(\omega, k)$, common to any initial perturbation

When we consider the dispersion function

$$\int_{-\infty}^{\infty} dv \, \frac{\dfrac{\partial F_e(v)}{\partial v}}{\dfrac{\omega}{k} - v}$$

(take $k > 0$ for simplicity)

The original ω contour integral is ω in upper half plane. As we lower contour we hit poles of the real axis. If we lower ω into lower half plane we either have to pick up a continuum of poles, $\omega = kv$, or if $\dfrac{\partial F(v)}{\partial v}$ is analytic on the real v-axis, we can distort v-contour into the complex plane so that $\operatorname{Im} \dfrac{\omega}{k} > \operatorname{Im} v$, and no pole need ~~to~~ be picked up.

This analytic continuation procedure must be employed in order to evaluate the dispersion relation.

electron plasma waves

$$D(\omega, k) = 1 + \frac{\omega_p^2}{k^2} \int_{C(v)} dv \, \frac{\frac{\partial F_e}{\partial v}}{\frac{\omega}{k} - v} = 0$$

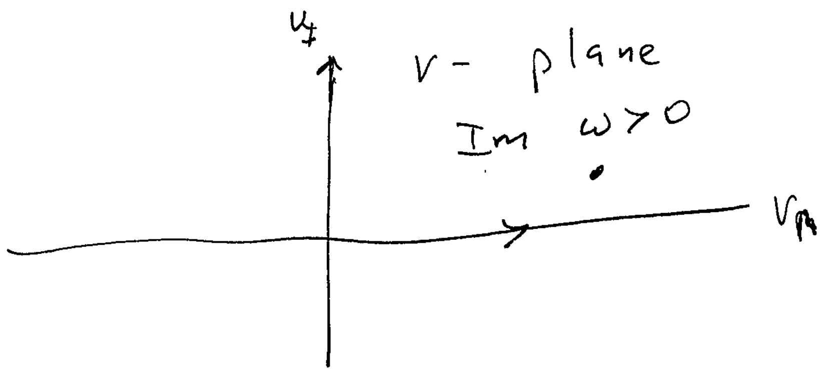
When $\frac{\omega}{k} \gg v_{the}$

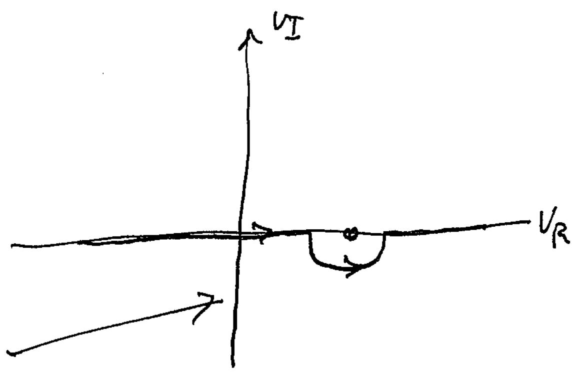
Let extract imaginary part of D when ω is real

$$Im = \frac{\omega_{pe}^2}{k^2} \int_{C(v)} dv \, \frac{\frac{\partial F}{\partial v}(v)}{\frac{\omega}{k} - v}$$

~~$Im \frac{1}{\frac{\omega}{k} - v} < 0$, because $= -\frac{Im\,\omega}{k} / (\omega/k - v)^2$~~

Let $Im\,\omega \to 0$





Distortion of v-contour

leads to rule $\frac{P}{\frac{\omega}{k} - v} - i\pi\,\delta\!\left(\frac{\omega}{k} - v\right) = \frac{1}{\frac{\omega}{k_0} - v}$

take
$$\boxed{\begin{aligned} v &= \frac{\omega}{k} + |\epsilon| e^{i\theta} \\ dv &= i|\epsilon| e^{i\theta}\, d\theta \end{aligned}}$$

$$\int_{-\pi}^{0} \frac{d(|\epsilon| e^{i\theta})}{-|\epsilon| e^{i\theta}} = -i\int_{-\pi}^{0} d\theta = -i\pi$$

Now if $\frac{\omega}{k v_{the}} \ll 1$

$$D(\omega,k) \simeq 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{3k^2 V_{the}^2}{k^2}\right) - i\pi \frac{\omega_{pe}^2}{k^2}\frac{\partial F(\frac{\omega}{k})}{\partial v} \simeq 0$$

$$D_R(\omega_0,k) = 0 \quad ; \quad D = D_R(\omega) + i\, D_I(\omega)$$

$$\omega = \omega_0 + \delta\omega$$

$$D_R(\omega_0 + \delta\omega, k) + i\, D_I(\omega_0, k) \simeq 0$$

↑ expand

$$\delta\omega \frac{\partial D_R}{\partial \omega} = -i\, D_I(\omega_0, k)$$

$$\frac{\partial D_R}{\partial \omega} \simeq + \frac{2\omega_{pe}^2}{\omega^3}\left(1 + \frac{3k^2 V_{the}^2}{\omega_{pe}^2}\right)$$

$$D_I(\omega,k) = i\pi \frac{\omega_{pe}^2}{k^2}\frac{\partial F(\frac{\omega}{k})}{\partial v}$$

$$\delta\omega = -\frac{\left(\frac{\pi}{2}\right)^{1/2} \omega_{pe}^2 \frac{\omega_0}{k^3} \exp\left(-\frac{\omega_0^2}{2k^2 V_{the}^2}\right)}{2\, \frac{\partial D_R}{\partial \omega}(\omega_0)}$$

with $\omega_0^2 \simeq \omega_{pe}^2\left(1 + \frac{3k^2 V_{the}^2}{\omega_{pe}^2}\right)$

$$\frac{\partial D(\omega_R)}{\partial \omega} = \frac{2\omega_{pe}^2}{\omega^3} + 12\frac{\omega_{pe}^2 k^2 V_{the}^2}{\omega^5} \quad ; \quad \simeq \frac{2\left(1 + 4.5\frac{k^2 V_{the}^2}{\omega_{pe}^2}\right)}{\omega_{pe}}$$