Lecture # 22

Electrostatic Oscillations

Waves in kinetic Theory

We start simply.
Homogeneous plasma, without a magnetic field. We look at electrostatic oscillations of a fluid: <u>1-D</u> $\quad e > 0$

$$n_0 m_i \frac{d\delta v_i}{dt} = -e n_0 \frac{\partial \phi}{\partial x} - \frac{\partial \delta p_i}{\partial x}$$

$$n_0 m_e \frac{d\delta v_e}{dt} = + n_0 e \frac{\partial \phi}{\partial x} - \frac{\partial \delta P_e}{\partial x}$$

momenta equation

$$\frac{\partial \delta n_i}{\partial t} = -\frac{\partial n_i \delta v_i}{\partial x}$$

$$\frac{\partial \delta n_e}{\partial t} = -\frac{\partial n_e \delta v_e}{\partial x}$$

continuity equation

entropy conservation

$$\frac{\partial \delta P_e}{\partial t} = -\gamma_e P_{e0} \frac{\partial \delta v_e}{\partial x} \quad , \quad \frac{\partial^2 \phi}{\partial x^2} = (\delta n_e - \delta n_i) e 4\pi$$

$$\frac{\partial \delta P_i}{\partial t} = -\gamma_i P_{i0} \frac{\partial \delta v_i}{\partial x} = \frac{\partial \delta n_i}{\partial t} \gamma_i P_{i0} / n_0$$

Field Eq.

Equilibrium $\quad n_{e0} = n_{i0} = \text{const.}$

$V_{e0} = V_{i0} = 0$

$P_e = P_{e0}$, $P_i = P_{i0}$ $\quad \frac{\partial}{\partial x} P_{e,i0} = 0$

We look for solutions of the form

$$\begin{pmatrix} \delta n_e \\ \delta n_i \\ \delta V_e \\ \delta V_i \\ \phi \end{pmatrix} = \begin{pmatrix} \hat{\delta n_e} \\ \delta n_i \\ \delta V_e \\ \delta V_i \\ \phi \end{pmatrix} \left(\exp[-i\omega t + ikx] + c.c. \right)$$

$$\therefore \quad \delta P_{e,i} = \gamma_{\binom{e}{i}} P_{\binom{e0}{i}} \frac{k}{\omega} \delta V_{e,i}$$

$$\delta V_{i,e} = + \frac{\delta n_{i,e}\,\omega}{n_{0i,e}\,k}$$

$$\therefore \quad \delta P_{e,i} = \gamma_{\binom{e}{i}} P_{\binom{e0}{i0}} \delta n_{e,i}/n_0$$

In momentum equation solve in terms of δn & ϕ

$$n_0 m_{i,e} \frac{\omega^2}{k} \delta n_{i,e} = n_0 e k \phi + k \gamma_{\binom{e}{i}} P_{\binom{e,0}{i,0}} \delta n_{e,i}/n_0$$

$$P_{0j}/n_0 = T_j \quad , \quad V_{th_j}^2 = T_j/m_j$$

Gathering $\delta n_{e,i}$ together

$$\delta n_{i,e} \left(1 - \frac{k^2 V_{th\,i,e}^2}{\omega^2} \right) = \frac{k^2}{\omega^2} \frac{n_0 e \phi}{m_{i,e}}$$

Substitute in Poisson equation

$$k^2 \phi = \frac{\omega_{pe}^2 k^2 \phi}{\omega^2 - \gamma_e k^2 V_{the}^2} + \frac{k^2 \omega_{pi}^2 \phi}{\omega^2 - \gamma_i k^2 V_{thi}^2}$$

Dispersion Relation

$$1 = \frac{\omega_{pe}^2}{\omega^2 - \gamma_e k^2 V_{the}^2} + \frac{\omega_{pi}^2}{\omega^2 - \gamma_i k^2 V_{thi}^2}$$

solutions:

electron plasma wave

$$\omega^2 - \gamma_e k^2 V_{the}^2 = \omega_{pe}^2 + \omega_{pi}^2 \frac{(\omega^2 - \gamma_e k^2 V_e^2)}{(\omega^2 - \gamma_i k^2 V_i^2)}$$

$$\boxed{\omega^2 = \omega_{pe}^2 + \gamma_e k^2 V_{the}^2 + \mathcal{O}\left(\frac{m}{M}\right)}$$

electron plasma waves

Now, Ion acoustic waves $\omega \lesssim \omega_{pi}$

$k V_{the} >> \omega$

$$1 = -\underbrace{\frac{\omega_{pe}^2}{\gamma_e k^2 V_{the}^2}}_{= \frac{\omega_{pi}^2}{\gamma_e k^2 V_{thi}^2} \frac{T_i}{T_e}} + \frac{\omega_{pi}^2}{\omega^2 - \gamma_i k^2 V_{thi}^2}$$

$\therefore$

(4)

$$C_s^2 = \gamma_e V_{thi}^2 \frac{T_e}{T_i}$$

$$\omega^2 = k^2 \left(\gamma_i V_{thi}^2 + \frac{C_s^2}{1 + k^2 C_s^2/\omega_{pi}^2} \right)$$

$$\longrightarrow \left(\begin{array}{ll} k^2 (\gamma_i V_{thi}^2 + C_s^2) , & k^2 C_s^2/\omega_{pi}^2 \ll 1 \\ \gamma_i k^2 V_{thi}^2 + \omega_{pi}^2 , & k^2 C_s^2/\omega_{pi}^2 \gg 1 \end{array} \right)$$

Finally if $\omega^2 \ll \gamma_i k^2 V_{thi}^2, \gamma_e k^2 V_{the}^2$

$$\epsilon(\omega = 0) = 1 + \frac{\omega_{pe}^2}{\gamma_e k^2 V_{the}^2} + \frac{\omega_{pi}^2}{\gamma_i k^2 V_{thi}^2}$$

$$= 1 + \frac{1}{\gamma_e k^2 \lambda_{De}^2} \left(1 + \frac{\gamma_e T_e}{\gamma_i T_i} \right)$$

How do these results change with kinetic theory?

$$\lambda_{De} = \frac{V_{the}}{\omega_{pe}}$$

We start with N-body Liouville Equation $N \to \infty$

$$\frac{\partial f_N}{\partial t} + \left(\sum_{\substack{k=1 \\ j=e,i}}^{K} \underset{\sim}{v}_k \cdot \frac{\partial}{\partial \underset{\sim}{x}_k} - \frac{e_j}{m_j} \frac{\partial \phi(X_k)}{\partial X_k} \frac{\partial}{\partial v_j} \right) f_N(X_k) = 0$$

$$-\nabla^2 \phi(\underset{\sim}{x}) = \sum_j \int d^N v\, F_N\, 4\pi e_j$$

One approximate solution of N-body distribution function is to take

$$F_{Nj} = \prod_{k=1}^{K} f_{1j}(X_k) \ldots\ldots$$

and find to order $\left(\frac{1}{n \lambda_p^3}\right)$ that each 1-particle distribution satisfies 'collisionless Boltzman" equation, or Vlasov equation

$$\frac{\partial f_{1j}}{\partial t} + \underset{\sim}{v}_j \cdot \frac{\partial f_{1j}}{\partial \underset{\sim}{x}_j} - e \underset{\sim}{\nabla} \phi \cdot \frac{\partial f_{1j}}{\partial \underset{\sim}{p}_j} = 0$$

$$f_{1j}(\underset{\sim}{x}, \underset{\sim}{p}, t)$$

$$\underset{\sim}{\nabla}^2 \phi = 4\pi e \left[\int d^3 p_e \left(f_{1e}(\underset{\sim}{r}, \underset{\sim}{p}_e) - \int d^3 p_i\, f_{1i}(\underset{\sim}{r}, p_i) \right) \right]$$

Now consider 1-D spatially homogeneous equilibrium, and its linear perturbation,

$$\frac{\partial f_j}{\partial t} + v\frac{\partial f_j}{\partial x} - \frac{e}{m}\nabla\phi\frac{\partial f_j(x,v,t)}{\partial v} = 0$$

without perturbation

$$f = f_0(v_0) \quad n_{i0} \equiv \int d^3v\, f_{i0}(v) = \int d^3v\, f_{e0}(v) = n_{e0}$$

$$n_{e0} = n_{i0};\ \nabla \Phi_0 = 0$$

First order perturbation

$$\frac{\partial \delta f_j}{\partial t} + v\frac{\partial \delta f_j}{\partial x} = \frac{e}{m}\nabla\phi\frac{\partial f_{0j}(v)}{\partial v}$$

Let us integrate this equation by the method of characteristics:

suppose initially $\delta f(x,v,t=0) = \delta f_0(x,v)$

$$\frac{\partial^2\phi}{\partial x^2} = 4\pi e\left(\int d\delta f_e - \int d\delta f_i\right)$$

$$\delta f_j(x,v,t) = \delta f_{0j}(x - vt,\ v) + \frac{e}{m}\int_0^t dt'\,\frac{\partial\phi}{\partial x'}(x+v(t'-t),t')\frac{\partial f_{0j}(v)}{\partial v}$$

$$\parallel$$
$$\widetilde{\delta f}$$

(C

Proof of solution

$$\frac{\partial \delta f_j}{\partial t} + v_i \frac{\partial}{\partial x} \delta f_j = 0$$

because $\delta f_j = \delta f_j (x - vt, v)$

$$\left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} \cancel{\boxed{+\frac{e\,\partial\phi/\partial x}{m}\frac{\partial}{\partial v}}}\right]\tilde{\delta f}$$

$$= \cancel{\boxed{\frac{e}{m}\frac{\partial\phi(x,t)}{\partial x}\frac{\partial f_{oj}}{\partial v}}} + \frac{e}{m}\frac{\partial\phi(x,t)}{\partial x}\frac{\partial f_{oj}}{\partial v}$$

$$+ \frac{e}{m}\int_0^t dt' \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\frac{\partial}{\partial x'}\phi(x+v(t'-t),t')\frac{\partial f_o(v)}{\partial v}$$

$$\underbrace{}_{=0}$$

Therefore, linear Vlasov equation satisfied

---

key conditions:

$x(t')$, $v(t')$ satisfy $\frac{dx(t')}{dt'} = v(t')$

$\frac{dv(t')}{dt'} = a(t') \leftarrow$ (acceleration over unperturbed orbits

with conditions:

$x(t'=t) = x$

$v(t'=t) = v$

If we Fourier Analyze spatial solution

$$\phi(x) = \sum_k \phi_k(t) \exp(ikx); \quad \phi(x-vt) = \sum_k \phi_k(t)\, e^{ik(x-vt)}$$

$$\frac{\partial}{\partial t}\delta f_k + ikv\,\delta f_k - \frac{iek}{m}\phi_k \frac{\partial f_0}{\partial v} = 0$$

$$k^2 \phi_k = 4\pi e \int d^3v \left(\delta f_{ik} - \delta f_{ek}\right)$$

$$\delta f(t) = \exp\left[-ik\,vt\right]\delta f_k(v)$$

$$\tau = t' - t$$

$$+ \frac{ie}{m} k \int_{-t}^{0} d\tau\, \phi_k(t+\tau)\, e^{ikv\tau} \frac{\partial f_{0j}(v)}{\partial v}$$

If we now Laplace Transform in time

$$\boxed{g_\omega = \int_{-\infty}^{\infty} d\tau\, e^{+i\omega\tau} g(\tau)} \qquad g(\tau) = 0 \quad \tau < 0$$

$$g(t) = \int_C \frac{d\omega}{2\pi i} e^{-i\omega t} g_\omega$$

C contour in upper half plane so that $g(t) = 0$ for $t < 0$

causality; $\tilde{g}(\omega)$ analytic in upper half plane

$$\delta f_\omega = \frac{i\,\delta f_k(0)}{\omega - kv} + i\frac{e}{m}\int_0^\infty dt\, e^{i\omega t}\int_{-t}^{0} d\tau\, e^{ikv\tau}\,\phi_k(t+\tau)\,\frac{\partial f_0(v)}{\partial v}$$

$\uparrow$ reverse integration order

let $t' = t + \tau$

$$\delta f_\omega - \frac{i\,\delta f_k(0)}{\omega - kv} = \frac{ie}{m}\int_{-\infty}^{0} d\tau\, e^{i(kv-\omega)\tau}\int_0^\infty dt'\,\phi_k(t')\,e^{i\omega t'}\frac{\partial f_0}{\partial v}$$

$$= -\frac{e}{m}\frac{k\phi_\omega}{\omega - kv}\frac{\partial f_0(v)}{\partial v}$$

If we neglect initial condition

$$\delta f_\omega = -\frac{e}{m}\frac{k\phi_\omega}{\omega - kv}\frac{\partial f_0}{\partial v}$$

$$\sum_j e n_\omega = 4\pi e \sum_j \int dv\, \delta f_\omega \rightarrow -4\pi e^2 k \sum_j \int dv \frac{\frac{\partial f_{0j}}{\partial v}}{\omega - kv}\frac{\phi_{k,\omega}}{m_j} = k^2\phi_k$$

Kinetic Dispersion Relation

$$D(\omega,k) \equiv 1 + \sum_j \frac{\omega_{pj}^2}{n_{j0}} \frac{k}{k^2} \int \frac{dv\, \partial f_j(v)/\partial v}{\omega - kv} = 0$$

If $\omega \approx \omega_{pe}$

and $\dfrac{\omega}{k v_{the}} >> 1$

$$\underbrace{f_j = \frac{n_{oj}}{(2\pi v_{th_j}^2)^{1/2}} \exp\left(-\frac{v^2}{2 v_{th}^2}\right)}_{\equiv \text{ Maxwellian}}$$

$$\frac{1}{\omega - kv} = \frac{1}{\omega}\left(1 + \frac{kv}{\omega} + \frac{k^2v^2}{\omega^2} + \frac{k^3v^3}{\omega^2} + \frac{k^4v^4}{\omega^4} \cdots\right)$$

$$D(\omega,k) \simeq 1 + \sum_j \frac{\omega_{pj}^2}{k^2} \frac{k}{\omega} \int dv \frac{\partial f_j}{\partial v}\left(\underset{0}{\underset{\|}{1}} + \frac{kv}{\omega} + \underset{0}{\underset{\|}{\frac{k^2v^2}{\omega^2}}} + \frac{k^3v^3}{\omega^3} + \cdots\right)$$

$$= 1 + \frac{\omega_{pe}^2}{\omega^2}\left[\frac{1}{n_0} \int dv\, v \frac{\partial f_e}{\partial v}\left(1 + \frac{k^2v^2}{\omega^2}\right)\right]$$

$$= 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + 3\frac{k^2 v_{the}^2}{\omega^2}\right) = 0$$

Solving for ω^2, assuming

$$\frac{k^2 V_{the}^2}{\omega^2} << 1$$

$$\omega^2 = \omega_{pe}^2 \left(1 + \frac{3 k^2 V_{the}^2}{\omega^2}\right)$$

$$\simeq \omega_{pe}^2 \left(1 + 3 \frac{k^2 V_{the}^2}{\omega_{pe}^2}\right)$$

compare with fluid result

$$\omega^2 \simeq \omega_{pe}^2 + 3 k^2 V_{the}^2 \quad \text{(kinetic)}$$

$$\omega^2 = \omega_{pe}^2 + \gamma_e k^2 V_{the}^2 \quad \text{(fluid)}$$

$$\gamma_e = 3$$

(this is sensible, $\gamma = \frac{N+2}{N}$.

N # of degrees of freedom

In collisionless plasma $N = 1$

Now consider acoustic wave

$\omega \lessapprox \omega_{pi}$