Lecture # 20

Ballooning Modes
Indicial Equation

Balloon Equation

An equation along a field line

where

perpendicular variation is given by an eikonal.

$$\phi = \phi(l) \exp\left[i S(\alpha, \beta) \right]$$

$$\beta = \rho - q(\psi)(\theta - \theta_0)$$

(with appropriate choice of θ_0 we can set $\frac{\partial S}{\partial \alpha}(\alpha, \beta) = 0$)

$dl \equiv$ incremental distance along a field

$$\underset{\sim}{A} \simeq A_{\parallel} \underset{\sim}{b} \qquad \underset{\sim}{\xi} \simeq \frac{-\underset{\sim}{b} \times \underset{\sim}{\nabla} \phi}{\gamma B}$$

$$E_{\parallel} = -\frac{\partial \phi}{\partial l} - \gamma A_{\parallel} = 0$$

$$A_{\parallel} = -\frac{\partial \phi}{\partial l} \frac{1}{\gamma}$$

($\gamma = 1$ as it scales out)

$$\delta \underset{\sim}{B} = \underset{\sim}{\nabla} \times b A_{\parallel} \simeq -b \times \nabla A_{\parallel}$$

$$\simeq + \underset{\sim}{b} \times \underset{\sim}{\nabla} \frac{\partial \phi}{\partial l}$$

$$\frac{1}{2}\int d^3r \; \delta \underset{\sim}{B} \cdot \delta \underset{\sim}{B} = \int \frac{d^3r}{2} \left[b \times \nabla \frac{\partial \phi^*}{\partial l} \cdot b \times \nabla \frac{\partial \phi}{\partial l} \right]$$

Variation with respect to ϕ^* gives

$$\underset{\sim}{\nabla} S(\alpha, \beta) = \frac{\partial S}{\partial \beta} \nabla \beta$$

$$\frac{\partial}{\partial l} \; \underset{\sim}{b} \times \nabla \cdot \underset{\sim}{b} \times \underset{\sim}{\nabla} \frac{\partial \phi}{\partial l} =$$

$$\Rightarrow -\left(\frac{\partial S}{\partial \beta}\right)^2 \frac{\partial}{\partial l} \; \underset{\sim}{\nabla} \beta \cdot \underset{\sim}{\nabla} \beta \frac{\partial \phi}{\partial l}$$

energy drive $\quad \underset{\sim}{\xi} = -b \times \nabla \phi / B$

$$-\frac{1}{2}\int d^3r \;\; 2 \frac{\partial p}{\partial \psi} \underset{\sim}{\nabla} \psi \cdot \underset{\sim}{\xi} \left(\chi_\psi \underset{\sim}{\nabla} \psi + \chi_G \, \underset{\sim}{b} \times \underset{\sim}{\nabla} \psi \right) \cdot \xi^*$$

$$= \frac{-\left(\frac{\partial S}{\partial \beta}\right)^2}{2} \int d^3r \; \frac{2}{B^2} \frac{\partial p}{\partial \psi} \left(\underset{\sim}{\nabla} \psi \times b \right) \cdot \nabla \beta \; \phi$$

$$\cdot \left(\chi_\psi \underset{\sim}{\nabla} \psi \cdot \underset{\sim}{b} \times \nabla \beta + \chi_G \, \underset{\sim}{b} \times \underset{\sim}{\nabla} \psi \cdot b \times \underset{\sim}{\nabla} \beta \right) \phi$$

$$dl = B\, d\theta / B^\theta ; \quad \nabla\psi \times \nabla\beta \cdot \mathbf{b} = \mathbf{B} ,$$

Thus

$$\frac{\partial}{\partial l} \nabla\beta\cdot\nabla\beta \frac{\partial\phi}{\partial l} + 2\frac{\partial p}{\partial\psi} \left(\kappa_\psi + \frac{\kappa_a}{B^2}(\nabla\psi\cdot\nabla\beta)\right)\phi = 0$$

$$\nabla\beta = \nabla\left(\zeta - q(\psi)\,\theta\right)$$

$$= \nabla\zeta - q(\psi)\nabla\theta - \frac{\partial q(\psi)}{\partial\psi}\,\theta\,\nabla\psi$$

$$\nabla\beta\cdot\nabla\beta \simeq \frac{1}{B^2} + q^2|\nabla\theta|^2 + \left(\frac{\partial q}{\partial\psi}\right)^2|\nabla\psi|^2\,\theta^2$$

$$+ 2\theta q(\psi)\frac{\partial q}{\partial\psi}\nabla\theta\cdot\nabla\psi$$

$$\nabla\psi\cdot\nabla\beta = -\frac{\partial q}{\partial\psi}\,\theta\,|\nabla\psi|^2 - q\,\nabla\theta\cdot\nabla\psi$$

not important large aspect ratio limit

First let us look at this equation for the straight pinch, and we will see a "conjugate" way to view the Suydam criterion.

$$\beta = \varsigma - q(r)\theta$$

$$\underset{\sim}{\nabla}\beta = \frac{\hat{\varsigma}}{R} - q(r)\frac{\hat{\theta}}{r} - \frac{\partial q(r)}{\partial r}\theta\hat{r}$$

$$|\nabla\beta|^2 = \frac{1}{R^2} + \frac{q^2}{r^2} + \left(\frac{\partial q}{\partial r}\right)^2(\theta - \theta_0)^2$$

should be $(\theta-\theta_0)^2$

$$\frac{d}{dl}\left(\frac{1}{R^2} + \frac{q^2}{r^2} + \left(\frac{\partial q}{\partial r}\right)^2\theta^2\right)\frac{d}{dl}\phi + \frac{2}{r}\frac{\partial p}{\partial r}\frac{\phi}{B^2R^2} = 0$$

$$\frac{r\,d\theta}{B_\theta} = \frac{dl}{B} \qquad \theta - \theta_0 = \frac{(l-l_0)B_\theta}{rB}$$

For large l we look for a solution in the form $\left(q' = \frac{dq}{dr}\right)$

$$\phi = l^{-\nu}$$

$$\frac{d}{dl}\, q'^2\frac{(l-l_0)^2B_\theta^2}{B^2r^2}\frac{d}{dl}\phi = \frac{2}{r}\frac{\partial p}{\partial r}\frac{1}{B^2}\frac{1}{R^2}\phi$$

substituting $\phi = l^{-\nu} \rightarrow \nu(\nu+1) = \frac{2}{r}\frac{\partial p}{\partial r}\frac{r^2}{B_\theta^2 q'^2 R^2}$

Therefore with

$$D = -\frac{2\partial p}{r\partial r}\frac{r^2}{B_\theta^2 q'^2 B} = -\frac{2}{r}\frac{\partial \beta}{\partial r}\frac{q^2}{q'^2} = -\frac{2\partial\beta}{\partial r}\frac{r}{S^2}$$

$$\beta = \frac{2p}{B^2}$$

$$S = \frac{rq'}{q} \equiv \text{magnetic shear}$$

Then

$$\nu = \frac{1}{2} \pm \left[\frac{1}{4} - D_S\right]^{1/2}$$

instability if

$$D_S > \frac{1}{4} \quad \text{or}$$

$$-8r\frac{\partial\beta}{\partial r}\frac{1}{S^2} > 1$$

Same condition as obtained in a radial calculation, whereas this calculation is along the field line (A complementary picture)

In a tokamak the equations along the field line are more complicated

$(\theta-\theta_0)^2$ more general

$$\nabla\beta\cdot\nabla\beta \cong \frac{1}{B_0^2} + \frac{q^2}{r^2} - \left(\frac{dq}{dr}\theta\right)^2$$

straight pinch

tokamak

$$= \frac{1}{B^2} + q^2|\nabla\theta|^2 + \left(\frac{\partial q}{\partial\psi}\right)^2|\nabla\psi|^2(\theta-\theta_0)^2$$

modulation

$$+2\, q(\psi)\frac{\partial q}{\partial\psi}\nabla\theta\cdot\nabla\psi(\theta-\theta_0)$$

not too important for tokamak (nearly othogonal)

magnetic shear modulates in angle:

$$-\frac{\partial p}{\partial r}\frac{1}{r}\frac{B_\theta^2}{B^2} \rightarrow -\frac{\partial p}{\partial r}\frac{1}{r}\frac{B_\theta^2}{B^2}\left(1-q^2\right)$$

arises when shifted magnetic surfaces is considered

(demonstrated left for student research)

In a tokamak we need to treat effects from geodesic curvature term, which is much larger in magnitude than curvature of straight pinch.

(zero for straight pinch)

$$\frac{\partial p}{\partial \psi}\, \underset{\sim}{\kappa} \cdot \frac{\underset{\sim}{b} \times \nabla \underset{\sim}{\psi}}{|\nabla \psi|} \simeq \frac{\partial p}{\partial \psi} \frac{\sin\theta}{R}$$

$$\gg \frac{\partial p}{\partial \psi} \frac{B_\theta^2}{B^2} \frac{1}{r}(1-q^2)$$

$$\simeq \frac{\partial p}{\partial \psi} q^{-2}(1-q^2)\frac{r}{R^2}$$

drive of straight pinch and from average tokamak curvature

More careful work has to be done with indicial equation

Model

ballooning model equation:

Accounts for (1) modulation of the fields

(2) geodesic curvature

(3) Shafranov shifts from finite β shifting equilibrium

$$\underset{\sim}{B} = \underset{\sim}{\nabla}\psi \times \underset{\sim}{\nabla}\theta$$

$$\frac{\partial}{\partial\theta}\left[(1+\Lambda^2)\frac{\partial\phi}{\partial\theta}\right] + \left[\alpha(\Lambda\sin\theta + \cos\theta) + \lambda_m\right]\phi = 0$$

$$\lambda_m = -\frac{d\beta}{dr}(1-q^2)r \quad ; \quad \beta = \frac{2p}{B^2}$$

$$\alpha = -q^2 R_0 \frac{d\beta}{dr} \quad , \quad s = \frac{rq'}{q}$$

modulation of shear due to beta

$$\Lambda = s(\theta - \theta_0) - \alpha \sin\theta$$

To unravel the indicial equation we assume ($X_s(\theta)$ periodic in θ)

$$\phi = \frac{1}{(\theta-\theta_0)^{\nu}}\left[X_0(\theta) + \frac{X_1(\theta)}{\theta-\theta_0} + \frac{X_2(\theta)}{(\theta-\theta_0)^2} + \dots\right] \quad \text{for } \theta \to \infty \qquad (8)$$

Largest term of equation

$$\frac{s^2(\theta-\theta_0)^2}{(\theta-\theta_0)^{\nu}}\frac{d^2X_0}{d\theta^2} = \frac{\partial}{\partial\theta} s^2(\theta-\theta_0)^2\frac{\partial}{\partial\theta}X_0(\theta) = 0$$

$$\Rightarrow \quad X_0(\theta) = 1$$

$$\frac{s^2(\theta-\theta_0)^2}{(\theta-\theta_0)^{\nu+1}}\frac{d^2X_1}{d\theta^2} = \frac{-(\theta-\theta_0)}{(\theta-\theta_0)^{\nu}}\left[s^2\, 2\frac{dX_0}{d\theta}(1-\nu) + \alpha s \sin\theta\, X_0(\theta)\right]$$

$$\therefore \quad \frac{d^2X_1}{d\theta^2} = \frac{-\alpha}{s}\sin\theta; \quad \frac{dX_1}{d\theta} = \frac{+\alpha}{s}\cos\theta; \quad X_1 = \frac{\alpha}{s}\sin\theta$$

$$\frac{s^2(\theta-\theta_0)^2}{(\theta-\theta_0)^{\nu+2}}\frac{d^2X_2}{d\theta^2} = -\frac{1}{(\theta-\theta_0)^{\nu}}\left[2s^2\frac{dX_1}{d\theta}\nu + s^2\nu(\nu+1)X_0 + \frac{d^2}{d\theta^2}X_0(\theta) + \alpha\cos\theta X_0 - \frac{d\beta}{dr}(1-q^2)^{r}X_0 + \alpha s\sin\theta\, X_1(\theta) - \alpha(\sin\theta \quad)X_0\right]$$

$$X_1 = -\frac{\alpha}{s}\sin\theta \quad , \quad \frac{dX_1}{d\theta} = -\frac{\alpha}{s}\cos\theta$$

$$s^2\frac{d^2 X_2}{d\theta^2} = -\left[\begin{array}{l} 2\alpha s^{-}\nu\cos\theta \quad + s^2(\nu)(\nu+1) \\ +\alpha\cos\theta \quad - \frac{d\beta}{dr} r(1-q^2) \\ +\lambda m \quad + \alpha^2\sin^2\theta \end{array}\right]$$

We "annihale" the left hand side when we ~~[illegible]~~ average over a period to obtain indicial equation

$$s^2\nu(\nu+1) - \frac{d\beta}{dr}r(1-q^2] + \frac{\alpha^2}{2} = 0$$

$\nu(\nu+1)$

$$\nu = -\frac{1}{2} \pm \left[\frac{1}{4} - D_M\right]^{1/2}$$

$$D_M = -\frac{d\beta}{dr}\frac{r(1-q^2)}{s^2} + \frac{\alpha^2}{s^2} > \frac{1}{4}$$

is sufficient criterion for instability

$$\alpha = -q^2\frac{dp}{drB^2}R_0$$

Note: There should better model equation (α^2 dependence not quite right)

(10)