Lecture # 19

Ballooning Mode Formalism

Pressure Driven Modes

At zero beta only a flute can be unstable, and if $\frac{d}{d\psi}\int \frac{dl}{B} < 0$, stability is guaranteed.

However, a "ballooning" mode, when field line bends in bad curvature region, Instability can arise if there is sufficient pressure gradient to drive the mode.

Basically, stabilization arises from bending energy, while destabilization arise from pressure drive where there is 'bad" curvature $\kappa_\psi \frac{\partial p}{\partial \psi} > 0$.

First let us look at a cylinder and derive a sufficient condition for instability (Suydam condition)

Consider the terms

$\underset{\sim}{Q_\perp}$ and $-2(\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}p)\,\underset{\sim}{\kappa}\cdot\underset{\sim}{\xi}$ in energy principle

$$\delta W_I \simeq \int d^3r\left[\frac{|Q_\perp|^2}{2} - (\underset{\sim}{\xi_\perp}\cdot\underset{\sim}{\nabla}p)(\underset{\sim}{\kappa}\cdot\underset{\sim}{\xi}^*)\right]$$

Let: $\underset{\sim}{Q_\perp} = \underset{\sim}{\nabla}\times(A_\parallel \underset{\sim}{b}) \simeq -\underset{\sim}{b}\times\nabla A_\parallel$

$$\frac{\partial \underset{\sim}{\xi_\perp}}{\partial t} = \frac{\underset{\sim}{b}\times\underset{\sim}{\nabla}\phi}{B} = \gamma\,\xi_\perp$$

$$\underset{\sim}{b}\cdot(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B}) = 0 = -\underbrace{\frac{\partial A_\parallel}{\partial t}}_{= -\gamma A_\parallel} - \frac{\partial\phi}{\partial s}, \quad ds = \text{distance along field line}$$

$$A_\parallel = \frac{-1}{\gamma}\frac{\partial\phi}{\partial s}$$

$$\therefore \quad \underset{\sim}{Q_\perp} = \frac{\underset{\sim}{b}\times\underset{\sim}{\nabla}}{\gamma}\frac{\partial\phi}{\partial s}$$

$$\underset{\sim}{\xi_\perp} = \frac{\underset{\sim}{b}\times\underset{\sim}{\nabla}\phi}{\gamma B}$$

(set $\gamma = 1$ as common factor of all terms)

$$\delta W_T \simeq \frac{1}{2}\int d^3r\left[\frac{|b\times\nabla\partial\phi/\partial s|^2}{B^2} - \frac{2(b\times\nabla\phi\cdot\underset{\sim}{\nabla}p)(\underset{\sim}{\kappa}\cdot\underset{\sim}{b}\times\underset{\sim}{\nabla}\phi)^*}{B^2}\right]$$

Near resonance we look for solution where $\frac{\partial}{\partial r} \gg \frac{m}{r}$, $\underset{\sim}{\kappa} = -\frac{\hat{B}_\theta}{B^2}\frac{1}{r}\underset{\sim}{\hat{r}}$

$B_z \gg B_\theta$

$$\frac{\partial \phi}{\partial s} = \frac{B_\theta}{rB}\left(m - n q(r)\right)$$

resonance condition $m - n q(r_s) = 0$

Euler - Lagrange equation

$$\frac{\partial}{\partial r} \frac{B_\theta^2(r) m^2}{r^2 B^2}\left(1 - \frac{n}{m} q(r)\right)^2 \frac{\partial \phi}{\partial r} - \frac{2 m^2}{r^3} \frac{\partial p}{\partial r} \cdot \frac{B_\theta^2}{B^2} \phi = 0$$

near resonance surface

$$\frac{m}{n} = q(r_s) \quad , \quad r = r_s + \delta r$$

$$\left(1 - \frac{n}{m} q(r)\right) \simeq \delta r\, q'(r_s) / q(r_s)$$

$$\frac{B_\theta^2(r_s)}{r_s^2} \frac{q'(r_s)^2}{q^2(r_s)} \frac{\partial}{\partial r} \delta r^2 \frac{\partial \phi}{\partial r} - \frac{2}{r_s^3} \frac{\partial p(r_s)}{\partial r} \frac{B_\theta^2(r_s)}{B^2(r_s)} \phi = 0$$

search for a solution of

equation $\frac{\partial}{\partial r} \delta r^2 \frac{\partial \phi}{\partial r} + D \phi = 0$; $D = \frac{2}{r_s} \frac{\partial p(r_s)}{\partial r} \frac{q^2(r)}{B^2 \left(\frac{\partial q(r_s)}{\partial r}\right)^2}$

$\phi = r^\nu$, upon substitution

$$\nu(\nu + 1) + D = 0 \quad ; \quad \nu = -\frac{1}{2} \pm \left[\frac{1}{4} - D\right]^{1/2}$$

complex, oscillatary solution if $D > \frac{1}{4}$
as $\delta r \to 0$

$$\delta r^{-1/2 \pm i[D-1/4]^{1/2}} = \frac{\exp\left[\pm i(D-1/4)^{1/2}\ln|\delta r|\right]}{\delta r^{1/2}}$$

oscillatory as $\delta r \to 0$

Therefore instability guaranteed if

$$\boxed{4D = \frac{8}{r_s B_\theta^2}\frac{\partial p}{\partial r}\frac{q^2(r_s)}{\left(\frac{dq(r_s)}{dr}\right)^2} > 1}$$

Suydam Criterion

Recall Newcomb's theorem that notes if Euler-Lagrange Equation of δW is oscillatory, system is unstable

(4) If Newco Suydam criterion gives stability ($D < 1/4$), instability still possible from non-local source. However Euler equation has to be integrated over entire radius (or at least within region of ~~with~~ rational surface)

Ballooning Mode Equation

(1) Leads to a more general sufficient instability condition for a tokamak with shifted magnetic surfaces (Mercier condition)

$$4D_M > 1, \quad D_M = -\frac{8r_s}{B_0^2(r_s)}\frac{\partial p(r_s)}{\partial r}\frac{1}{q'^2(r_s)}\left(1 - q^2(r_s)\right)$$

The extra q^2 is indicative that a magnetic well forms if $q^2 > 1$. For $q < 1$, (no magnetic well), the extra q^2 factor difficult to calculate simply (not here)

However, we will outline the ballooning formalism that has been found to be very informative for instability analysis in confined system:

The δW energy integral has the following terms (when magnetic compressional terms are removed)

$$\delta W = \frac{1}{2}\int d^3r \left[|Q_\perp|^2 - 2(\underset{\sim}{\xi_\perp}\cdot\nabla p)(\underset{\sim}{\xi_\perp^*}\cdot\underset{\sim}{\kappa}) \right]$$

For short $\perp$ wavelengths, let us represent these perturbations in the form (as before)

$$\underset{\sim}{\xi_\perp} = \frac{\underset{\sim}{b}\times\underset{\sim}{\nabla}\phi}{\gamma B}, \quad \underset{\sim}{Q_\perp} = \nabla\times(\underset{\sim}{b}\, A_\parallel) \simeq -\underset{\sim}{b}\times\underset{\sim}{\nabla} A_\parallel$$

ideal Ohm's law.

$$\underset{\sim}{E_\parallel} = -\frac{\partial A_\parallel}{\partial t} - \nabla\phi\cdot\underset{\sim}{b} = 0$$

$$\therefore\ A_\parallel = \frac{-\underset{\sim}{b}\cdot\nabla\phi}{\gamma}$$

$$\therefore\ \underset{\sim}{\xi_\perp'} \equiv \gamma\underset{\sim}{\xi_\perp} = \underset{\sim}{b}\times\underset{\sim}{\nabla}\phi/B$$

$$\underset{\sim}{Q_\perp'} = \gamma Q_\perp = \underset{\sim}{b}\times\underset{\sim}{\nabla}\frac{\partial\phi}{\partial s}$$

(now γ can be set to unity without loss of generality)

ds : distance along field line

$$\delta W = \frac{1}{2}\int d^3r \left|\frac{\partial S}{\partial \beta}\right|^2 \left[\left(\left(\underset{\sim}{b}\times\nabla\rho - q(\psi)\,\underset{\sim}{b}\times\nabla\theta - \underset{\sim}{b}\times\nabla\psi\, q'\theta \right) \cdot \left(\underset{\sim}{b}\times\nabla\rho - q(\psi)\,\underset{\sim}{b}\times\nabla\theta - \underset{\sim}{b}\times\nabla\psi\, q'\theta \right) \right) \left|\frac{\partial \phi}{\partial S}\right|^2 - 2\,|\phi|^2 \frac{\partial p}{\partial \psi}\left[\kappa_\psi + \frac{\kappa_G}{B}\left(\nabla\psi\cdot\left(-\nabla\rho + q(\psi)\nabla\theta\right) + q'\theta|\nabla\psi|^2 \right) \right] \right]$$

$$q' \equiv \frac{\partial q}{\partial \psi}$$

Aside:

It is interesting that parallel current drive (kink) does not appear in this limit.

$$J_\parallel \left(\xi_\perp^* \times b\right)\cdot Q_\perp \simeq J_\parallel \left(\underset{\sim}{b}\times\nabla\phi\right)\times \underset{\sim}{b} \cdot \underset{\sim}{b}\times\nabla\frac{\partial\phi}{\partial S}$$

$$= \underset{\sim}{J}\ \nabla_\perp\phi\cdot\left(\underset{\sim}{b}\times\nabla_\perp\right)\frac{\partial\phi}{\partial S}$$

$$= \underset{\sim}{J}\left(\frac{\partial S}{\partial \beta}\right)^2\left[\nabla_\perp\beta\cdot \underset{\sim}{b}\times\nabla_\perp\beta\right]\frac{\partial\phi}{\partial S} = 0$$

The Euler-Lagrange Equation

The Euler Lagrange Equation is $[\nabla\rho\cdot\nabla\theta = 0]$

$$\frac{\partial}{\partial S}\left[\nabla_\perp\rho\cdot\nabla_\perp\rho + q^2(\psi)\nabla_\perp\theta\cdot\nabla_\perp\theta + |\nabla\psi|^2 q'^2\theta^2 + 2q(\psi)\nabla\theta\cdot\nabla\psi \right]\frac{\partial\phi}{\partial S}$$

$$+ 2\left[\frac{\partial p}{\partial\psi}\left[\kappa_\psi + \frac{\kappa_G}{B}\left(q(\psi)\nabla\psi\cdot\nabla\theta + q'\theta|\nabla\psi|^2\right)\right]\right]\phi = 0$$

(more generally $\theta \to \theta - \theta_0$)