Lecture 18
Tearing Modes
A' Calculation
Resistive Instability

\[ \psi = \psi(r) e^{-i m \phi + i n \phi} \]

Cylinder (conducting \( \Gamma = 1 \) \( \theta = \theta \)) \( r = 6 \)

\[ \delta W_p = \delta \int_0^b dr \left[ H \left( \frac{d\psi}{dr} \right)^2 + \psi^2 \left( \frac{q}{F^2} + F^{-1} \frac{d}{dr} \left[ \frac{d}{dt} \left( \frac{d}{dr} F \right) \right] \right) \right] \]

\[ = -H \psi A \frac{d\psi}{dr} \bigg|_{r_s = 4}^{r_b} + \int_{r_s}^{r_b} dr \psi \left[ -\frac{d}{dr} \left( \frac{H \psi}{F^2} \right) + \frac{q}{F^2} + F^{-1} \frac{d}{dr} \left( \frac{d}{dt} F \right) \right] \]

\[ \delta W_R = -H(\delta) \psi(\delta) \left( \frac{d\psi(\delta_s = 4)}{dr} - \frac{d\psi(\delta_s = 4)}{dr} \right) \]

\[ = -H(\delta) \psi(\delta) \Delta' \]

\[ \text{If } \Delta' > 0, \delta W_R < 0 \text{ instability} \]

\[ k = \frac{n}{p_0} \]

\[ g(r) = \frac{(m^2 - 1) r F^2}{k^2 r^2 + m^2} + \frac{k^2 r^2}{k^2 r^2 + m^2} \left( 2 \frac{d\psi}{dr} + q F^2 - 2F \frac{k^2 r^2 + m^2}{k^2 r^2 + m^2} \right) \]

\[ F(r) = k \cdot \psi = \frac{m \psi}{r} - n \psi \frac{\frac{d\psi}{dr}}{r} \]

\[ H = \frac{r^3}{k^2 r^2 + m^2} \]
Form of $SW$ in the zero beta and $\frac{Wb}{R_0} \ll 1$ limit

$$
\phi \equiv \int_0^b \left[ \frac{3}{2} \left( \frac{d\psi}{dt} \right)^2 + \left( n^2 - 1 \right) \psi^4 + \frac{d^4}{dt^4} \left( \psi^4 - \frac{4 \psi^2}{13 \left( 1 - \frac{n}{m} \psi^2 \right)} \right) \right]
$$

(1) **Home Work:** Show that the above expression emerges from the general cylindrical expression (point out the approximations are made).

(2) For the above expression, calculate the numerical values of $\Delta'$ as a function of $\tau$ for a current profile of the form $\psi = \psi_0 \left( 1 - \frac{r}{a} \right)^m$ with $m = 2$, $n = 2$. (Note that the current profile allows you to calculate $B_0(\tau)$, using $\frac{d}{dt} \frac{\partial \psi}{\partial \tau} = \frac{1}{2} \pi \frac{\partial \psi}{\partial r}$, and for the equilibrium $q(0) / q(1)$, and $q'(0) / q(1)$. You can then calculate $q(0) / q(1)$, and for any $r$, as $n \psi(13) = m/n$ and for any $r$, $n$ can be found. Thus, replace $m/n$ by $\psi(13)$, and solve tearing mode equation for $0 < \psi < a$. Assume conducting walls at $r = a$, see Fig 6.7.6 for solution with $\psi(13)$.)
Note, tearing mode equation is valid in the vacuum (where \( \Delta^{(')} \)) and if wall is infinite, one can integrate to infinity.

Also, note in vacuum

\[
\psi = A \frac{1}{r^2} + B/r^2
\]

If \( b \to \infty \), then \( A = 0 \)

\[
B = \psi(a) a^2
\]

If \( b \) finite (conducting wall at \( r = b \))

\( r > a \), \( \psi = B \left( \frac{b^2}{r^2} - \frac{a^2}{b^2} \right) \)

\[
= \frac{\psi(a)}{\left( \frac{b^1}{a^2} - \frac{a^2}{b^2} \right)} \left( \frac{b^1}{r^2} - \frac{a^2}{b^2} \right)
\]
Example

\[ j = j_0 \left( 1 - \frac{r^2}{a^2} \right) \]

Equilibrium

\[ \frac{1}{r} \frac{d}{dr} \left( r^2 \frac{d\psi_0}{dr} \right) = -j_0 \left( 1 - \frac{r^2}{a^2} \right) \]

\[ \frac{2}{r} \frac{d\psi_0}{dr} = -j_0 \left( 1 - \frac{r^2}{a^2} \right) \]

\[ \frac{\rho^2 \psi_0}{r} = -j_0 \frac{r^2}{2} \left( 1 - \frac{r^2}{2a^2} \right) \]

\[ B_\theta = \frac{\rho \psi_0}{r} = -j_0 \left( \frac{r^4}{2} - \frac{r^6}{4a^2} \right) \]

\[ \psi_0 = -j_0 \left( \frac{r^2}{4} - \frac{r^4}{16a^2} \right) \]

\[ \frac{q(r)}{q(0)} = \frac{B_\theta(r)}{B_\theta(0)} e^{-\frac{1}{2} \left( \frac{r^2}{2} - \frac{r^3}{4a^2} \right)} = 1 - \frac{r^2}{2a^2} \]

\[ 1 - \frac{n}{m} q(r) = 1 - \frac{l}{q(0)} \frac{1}{q(r)} \left( 1 - \frac{r^2}{2a^2} \right) \]

with \( q(r) = m/n \)

Now with \( m \) given we can solve for \( \Delta' \).
Growth rate is determined by

\[ \mathbf{E} + \nu \times \mathbf{B} = \frac{\partial \mathbf{J}}{\partial t} \]

(1)

(2) Assuming incompressible flow in boundary layer,

\[ \mathbf{u}_t = \nabla f \times \mathbf{b}(x) \quad (f \text{ is stream function}) \]

\[ \text{Vorticity } \nabla \times \mathbf{u} = \mathbf{S} \cdot \nabla \times \mathbf{u} = -\nabla^2 f \quad \mathbf{S} = \frac{1}{2} \mathbf{b}(x) \]

(3) \( \mathbf{E} \) and \( \mathbf{B} \)

generated by a vector potential

\[ A = A_0 \hat{\mathbf{B}} \quad \mathbf{S} = -\frac{\partial A_0}{\partial t} \quad \mathbf{B} = \nabla \times A_0 \hat{\mathbf{B}} \]

Ohms law becomes

\[ -\frac{\partial A_0}{\partial t} + \nabla \times \mathbf{S} \cdot \mathbf{u}_t \mathbf{b}(x) = \nabla \cdot \mathbf{J}_\mathbf{b} = -\nabla \cdot \mathbf{A}_0 \hat{\mathbf{B}} \]

\[ \mathbf{b}(x) = \mathbf{B} - \mathbf{B}(x) \]

Leads to dimensional result

\[ \gamma \approx \left( \frac{\Delta t}{\xi} \right)^{2/5} \mathcal{Z}_R^{-3/5} \mathcal{Z}_A^{-2/5} \left( \frac{\nu \delta_s}{\nu_s \delta_s} \right)^{2/5} \]

\[ \mathcal{Z}_R = \frac{B_0}{v_3 \Delta} \]

\[ \Delta_{\text{Re}} \approx \left( \frac{\mathcal{Z}_A}{\mathcal{Z}_R} \right)^{4/5} \left( \frac{\nu_s \delta_s}{\nu_s \delta_s} \right)^{2/5} \]
Analysis of tearing modes

\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{d\psi}{dr} \right) - \frac{m^2}{r} \psi - \frac{dj/dr}{Bo(1 - \nu q/m)} \psi = 0 \]

As \( q(r) \to \frac{m}{n} \) at \( r \to R_s \)

Let \( \psi = 1 + \delta \psi \), and iterate

\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{d\delta \psi}{dr} \right) = \frac{\frac{dj}{dr}}{Bo \left( \frac{n q(R_s)}{R_s} \right) \delta \psi / m} \]

\[ \frac{d\delta \psi}{dr} = \chi \ln \delta r \]

\[ \delta \psi = \chi (\delta r (\ln \delta r - 1)) \]

One solution

\[ l + \chi (\delta r (\ln \delta r - 1)) \ldots \]

\[ R \leq R_s \]

Other solution (sub-dominant), \( \psi = \delta r + \delta \psi \)

\[ \frac{d^2 \delta \psi_2}{dr^2} = \chi \delta \psi_2 \]

\[ \delta \psi_2 = \chi \delta r \psi_2 \]

Solutions need to be continuous at either side of discontinuity.
Thus solution when going from the origin, with boundary condition of regularity

\[ g(r) = r^m \quad (m \to \frac{1}{m} \text{ discarded}) \]

produce a solution as \( r \to R \)

\[ g(r) \bigg|_{r=0} \quad \text{and say} \quad \frac{\partial g}{\partial r} \bigg|_{r=0} = 1 \]

will give

\[ g(r) = \alpha^+ \left[ 1 + \chi (\delta r \ln \delta r - 1) + \ldots \right] + A^+ \delta r \]

To make \( g(r) \) continuous, multiply \( g^+(r) \) by

\[ \frac{g^-(r_s)}{g^+(r_s)} = \beta \quad \text{and then subtract} \]

\[ \frac{\delta g^+(r_s + \epsilon) - \delta g^+(r_s - \epsilon)}{2} = \beta A^+ - A^- = \Delta A^+ \delta g^+(r_s) \]

If integration cuts-off before \( r = R_s \), select \( \alpha \)
\( \delta \leq \delta' < \epsilon \) for cut-off where
\( \phi \left( \frac{\delta}{\delta'} + \epsilon' \right) \) is the closest value of \( \phi(r) \) that integration gets to \( r = \delta' \).

Necessary Condition for stability to pressure driven modes (Suynclam criterion)

One needs to balance line bending terms with pressure gradient drive. At short wavelengths, the system is most unstable (Taylor):
\[ k = -\frac{m}{R_0} \frac{\phi}{r} \]

\[ SW = \int \frac{r^3}{m^2} \, dr \left[ \frac{1}{2} \left( k \cdot \beta \frac{d\phi}{dr} \right)^2 + \frac{\phi}{r} \frac{d\phi}{dr} \right] \]

The Euler equation is
when \( k \cdot B = 0 \) (line bending weakest at the point, \( r = \delta' \)).