Lecture 18

Tearing Modes

Δ' Calculation

Resistive Instability

$\psi = \psi(r) e^{-im\theta + in\phi}$ cylinder (conducting wall at $r=b$)

$$\delta W_R = \int_0^b dr \left[H\left(\frac{d\psi}{dr}\right)^2 + \psi^2\left(\frac{g}{F^2} + F^{-1}\frac{d}{dr}\left(H\frac{dF}{dr}\right)\right)\right]$$

$$= -H\psi\frac{d\psi}{dr}\Bigg|_{r_s-\epsilon}^{r_s+\epsilon} + \int_0^b dr\,\psi\left[-\frac{d}{dr}\left(H\frac{d\psi}{dr}\right) + \frac{g}{F^2} + F^{-1}\frac{d}{dr}H\frac{dF}{dr}\right]$$

Euler-Lagrange Equation "0"

$$\delta W_R = -H(r_s)\psi(r_s)\left(\frac{d\psi(r_s+\epsilon)}{dr} - \frac{d\psi(r_s-\epsilon)}{dr}\right)$$

$$\equiv \Delta'\psi(r_s)$$

$$= -H(r_s)\psi^2(r_s)\Delta'$$

If $\Delta' > 0$, $\delta W_R < 0 \rightarrow$ instability

$k \equiv \frac{n}{R_0}$

$$g(r) = \frac{(m^2-1)rF^2}{k^2r^2+m^2} + \frac{k^2r^2}{k^2r^2+m^2}\left(2\frac{dP}{dr} + rF^2 - 2F\frac{(krB_z + mB_\theta)}{k^2r^2+m^2}\right)$$

$$F(r) = \underset{\sim}{k}\cdot\underset{\sim}{B} \equiv \frac{mB_\theta}{r} - \frac{nB_\phi r}{R_0}, \quad H = \frac{r^3}{k^2r^2+m^2}$$

Form of δW in the zero beta and $\frac{nb}{R_0} \ll 1$ limit

$$\frac{\delta W}{R} = \int_0^b dr \left[r^3 \left(\frac{d\psi}{dr}\right)^2 + (m^2-1) r \psi + \frac{dj}{dr} \frac{r^3 \psi^2}{B_\theta \left(1 - \frac{n}{m} q(r)\right)} \right]$$

(1) Home Work: Show that the above expression emerges from the general cylindrical expression (point out where the approximations are made).

(2) For the above expression, calculate the numerical values of Δ' as a function of r, for a current profile of the form $j_\theta = j_0 \left(1 - \frac{r^2}{a^2}\right)^\nu$, with $m=2$, $\nu=2$.

(Note that the current profile allows you to calculate $B_\theta(r)$, using the equilibrium $\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \psi}{\partial r} = j(r)$, and for the you can then calculate $q(a)/q(0)$, and $q(r)/q(0)$. $q(r_s) = m/n$ and for any r_s an n can be found. Thus, replace $\frac{m}{n}$ by $q(r_s)$, and solve tearing mode equation for $0 < r_s < a$. Assume conducting wall at $r=a$. See Fig 6.7.1 for solution with $\nu = 1/2 q(a)$

Note, tearing mode equation is valid in the vacuum (where $\Delta' = 0$), and if wall is absent note there, one can integrate to infinity.

Also note in vacuum

$$\psi = A r^2 + B/r^2$$

If $b \to \infty$, then $A = 0$
$B = \psi(a)\, a^2$

If b finite (conducting wall at $r = b$)

$$r > a, \quad \psi = B\left(\frac{b^2}{r^2} - \frac{r^2}{b^2}\right)$$

$$= \frac{\psi(a)}{\left(\frac{b^2}{a^2} - a^2/b^2\right)} \left(\frac{b^2}{r^2} - \frac{r^2}{b^2}\right)$$

Example

$$j = j_0\left(1 - \frac{r^2}{a^2}\right)$$

Equilibrium

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \psi_0}{\partial r} = -j_0\left(1 - \frac{r^2}{a^2}\right)$$

$$\frac{\partial}{\partial r} r \frac{\partial \psi_0}{\partial r} = -j_0 r\left(1 - \frac{r^2}{a^2}\right)$$

$$r\frac{\partial \psi_0}{\partial r} = -j_0 \frac{r^2}{2}\left(1 - \frac{r^2}{2a^2}\right)$$

$$B_\theta = \frac{\partial \psi_0}{\partial r} = -j_0\left(\frac{r}{2} - \frac{r^3}{4a^2}\right)$$

$$\psi_0 = -j_0\left(\frac{r^2}{4} - \frac{r^4}{16a^2}\right)$$

$$\frac{q(r)}{q(0)} = \lim_{\epsilon \to 0} \frac{B_\theta(\epsilon)\, r}{B_\theta(r)\, \epsilon} = \frac{r}{2\left(\frac{r}{2} - \frac{r^3}{4a^2}\right)} = \frac{1}{1 - \frac{r^2}{2a^2}}$$

$$\therefore \quad 1 - \frac{n}{m} q(r) = 1 - \frac{1}{q(r_s)} \frac{1}{(1 - r^2/2a^2)}$$

with $q(r_s) = m/n$

Now with m given we can solve for Δ'.

Growth Rate is determined by

bringing in Ohm's Law

(1) $E + \underset{\sim}{V} \times \underset{\sim}{B} = \eta \underset{\sim}{j}$

(2) Assuming incompressible flow in boundary layer

$\underset{\sim}{u}_1 = \nabla f \times \hat{\underset{\sim}{b}}(r_s)$ (f is stream function)

vorticity $\equiv \Omega = \hat{\xi}' \cdot \nabla \times \underset{\sim}{V} = -\nabla_\perp^2 f$; $\hat{\xi}' = \underset{\sim}{b}(r_s)$

(3) $\underset{\sim}{E}$ and $\underset{\sim}{B}$

generated by a vector potential

$\underset{\sim}{A} = A_\rho \hat{\xi}$, $\quad \underset{\sim}{E}_1 = -\dfrac{\partial A_\rho \hat{\xi}}{\partial t}$, $\quad B_1 = \underset{\sim}{\nabla} \times A_\rho \hat{\xi}$

Ohms law becomes

$$-\frac{\partial A_{\rho_1}}{\partial t} + \underset{\sim}{\hat{y}} \times \underset{\sim}{\hat{\xi}} \cdot \underset{\sim}{u}_1\, \underset{\sim}{b}(x) = \eta J_{\rho_1} = -\eta \nabla_\perp^2 A_{\rho_1}$$

$$b(x) = \underset{\sim}{B} - \underset{\sim}{B}(x_s)$$

Leads to dimensional result

$$\gamma \approx (\Delta' r_s)^{4/5}\, \tau_R^{-3/5}\, \tau_A^{-2/5} \left(\frac{n\, r_s\, q'(r_s)}{R_0\, q(r_s)} \right)^{2/5} ; \quad \tau_A^{-1} = \frac{B_\rho}{r_s \sqrt{\rho}}$$

$$\Delta_{Resis} \approx (r_s \Delta')^{1/5} \left(\frac{\tau_A}{\tau_R}\right)^{2/5} \left(\frac{q}{r_s q'}\right)^{2/5} ; \qquad \tau_R^{-1} = \eta / r_s^2$$

(3)

Analysis of tearing modes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) - \frac{m^2}{r^2}\psi - \frac{dj/dr \ \psi}{B_\theta(1-nq/m)} = 0$$

As $q(r) \to \frac{m}{n}$ at $r = r_s$

Let $\psi_1 = 1 + \delta\psi_1$, and iterate

$$\frac{1}{r_s}\frac{d}{dr} r_s \frac{d\delta\psi_1}{dr} = \frac{\frac{dj}{dr}(r_s)}{B_\theta\,(nq'(r_s)\,\delta r/m)} \quad , \quad \delta r = r - r_s$$

$$\frac{d\delta\psi_1}{dr} = x \ln \delta r \qquad x = \frac{j'(r_s)\, q(r_s)}{B_\theta(r_s)\, q'(r_s)}$$

$$\delta\psi_1 = x\,(\delta r(\ln\delta r - 1))$$

One solution

$1 + x\,[\delta r\,(\ln\delta r - 1)] + \ldots \qquad r \lessgtr r_s$

dominant solution

Other solution (sub-dominant), $\psi_2 = \cancel{\delta r} + \delta\psi_2$

$$\frac{d^2\delta\psi_2}{dr^2} = \frac{x\psi}{\delta r} \quad , \quad \psi_2 = \delta r + \delta\psi_2$$

$$\frac{d^2\delta\psi_2}{dr^2} = x \ , \qquad \delta\psi_2 = x\,\delta r^2/2 \ \ldots \ \text{(no divergences)}$$

Solutions needs ~~to~~ be continuous at either side of discontinuity

Thus solution when going from the origin, with boundary condition of regularity

$\xi(r) = r^m$ ($\to \frac{1}{r^m}$ discarded)

produce a solution as $r \to r_s$

$$\xi^-(r) = \alpha^-\left[1 + x(\delta r \ln \delta r - 1) + \dots\right] + A^- \delta r$$

$\xi^+(r)$ from $r = a$, with $\xi(a) = 0$

any say $\frac{\partial \xi}{\partial r}(a) = 1$

will give

$$\xi^+(r) = \alpha^+\left[1 + x(\delta r \ln \delta r - 1) + \dots\right] + A^+ \delta r$$

To make $\xi(r)$ continuous, multiply $\xi^+(r)$ by $\frac{\xi^-(r_s)}{\xi^+(r_s)} \equiv \beta$ and then subtract $\beta \frac{d\xi^+}{dr}(r_s + \epsilon) - \frac{d\xi^-}{dr}(r_s - \epsilon) =$

$$= \beta A^+ - A^- \equiv \Delta' \psi(r_s)$$

If Integration cuts-off before $r = r_s$, select a

$\delta r = \epsilon' \ll \epsilon$ for cut-off where

$\xi(r_s \pm \epsilon')$ is the closest value

of $\xi(r)$ that integration gets to $r = r_s$.

Necessary Condition for stability to pressure driven modes (Suydam criterion)

One needs to balance line bending terms with pressure gradient drive. At short wavelengths system is most unstable (Taylor); $\underset{\sim}{k} = -\frac{n}{R_0}\hat{\phi} + \frac{m}{r}\hat{\theta}$

$$\delta W = \int \frac{r^3}{m^2} dr \left[\frac{1}{2} \left(\underset{\sim}{k} \cdot \underset{\sim}{B} \frac{\partial \xi}{dr} \right)^2 + \frac{n^2}{r} \frac{dp}{dr} \xi^2 \right]$$

The Euler Equation is

when $\underset{\sim}{k} \cdot \underset{\sim}{B} \sim 0$

(Line bending weakest at the point, $r = r_s$) 9