Lecture # 16

Homework # 1 solutions

Solutions HW#1

(1) Demonstrate that the time average over a gyro-period of the tensor

$$\overline{\underset{\sim}{u}\underset{\sim}{u}} = \int_{-\pi/\omega_c}^{\pi/\omega_c} \frac{dt}{T}\, \underset{\sim}{u}(t)\,\underset{\sim}{u}(t) \qquad T = \frac{2\pi}{\omega_c}$$

$$= \frac{u_\perp^2}{2}\left(\underset{\approx}{I} - \underset{\sim}{b}\underset{\sim}{b}\right) + u_\parallel^2$$

The velocity in th electric field frame is $\underset{\sim}{u} = u_\perp \cos(\phi - \omega_c t)\,\hat{x} + u_\perp \sin(\phi - \omega_c t)\,\hat{y} + u_\parallel \underset{\sim}{b}$

where $\hat{x}$ an $\hat{y}$ is $\perp$ to $\underset{\sim}{B}$, $\hat{z} \parallel \underset{\sim}{B}$

$$\underset{\sim}{u}\underset{\sim}{u} = \begin{pmatrix} u_\perp^2\cos^2(\phi-\omega_c t)\,\hat{x}\hat{x} & \hat{x}\hat{y}\,u_\perp^2\cos(\phi-\omega_c t)\sin(\phi-\omega_c t) & \hat{x}\hat{z}\,u_\perp u_\parallel \cos(\phi-\omega_c t) \\ u_\perp^2\cos(\phi-\omega_c t)\sin(\phi-\omega_c t)\,\hat{y}\hat{x} & \hat{y}\hat{y}\,u_\perp^2\sin^2(\phi-\omega_c t) & \hat{y}\hat{z}\,u_\perp u_\parallel \sin(\phi-\omega_c t) \\ u_\parallel u_\perp \cos(\phi-\omega_c t)\,\hat{z}\hat{x} & u_\parallel u_\perp \sin(\phi-\omega_c t)\,\hat{z}\hat{y} & \hat{z}\hat{z}\,u_\parallel^2 \end{pmatrix}$$

Now average $\underset{\sim}{u}\underset{\sim}{u}$ over a cyclotron period

$$\int_{-T/2}^{T/2} \frac{dt}{T}\begin{pmatrix}\cos^2(\omega_c t - \phi) \\ \sin^2(\omega_c t - \phi)\end{pmatrix} = \begin{pmatrix}1/2 \\ 1/2\end{pmatrix}$$

all terms linear in $\sin(\phi - \omega_c t)$ or $\cos \omega_c t$ vanish on average. And $\sin(\phi-\omega_c t)\cos(\phi-\omega_c t) = \frac{1}{2}\sin(2(\phi-\omega_c t)) = 0$

Thus we hav

$$\overline{u u} = \begin{pmatrix} \frac{u_\perp^2}{2}\,\hat{x}\hat{x} & 0\,\hat{x}\hat{y} & 0\,\hat{x}\hat{z} \\ 0\,\hat{y}\hat{x} & \frac{u_\perp^2}{2}\,\hat{y}\hat{y} & 0\,\hat{y}\hat{z} \\ 0\,\hat{z}\hat{x} & 0\,\hat{z}\hat{y} & \hat{z}\hat{z}\,u_\parallel^2 \end{pmatrix}$$

$$= \frac{u_\perp^2}{2}\left(\underset{\approx}{I} - \underset{\sim}{b}\underset{\sim}{b}\right) + u_\parallel^2\,\underset{\sim}{b}\underset{\sim}{b} \qquad \left[\underset{\approx}{I} = \begin{pmatrix}\hat{x}\hat{x} & 0 & 0 \\ 0 & \hat{y}\hat{y} & 0 \\ 0 & 0 & \hat{z}\hat{z}\end{pmatrix}\right] \quad \underset{\sim}{b}\underset{\sim}{b} = \begin{pmatrix}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hat{z}\hat{z}\end{pmatrix}$$

(2)

If $\underset{\sim}{A} = B_0 x \hat{y}$ $\quad \left(\underset{\sim}{\nabla} \times \underset{\sim}{A} = \frac{\partial}{\partial x} A_y \hat{z} = B_0 \hat{z} \right)$

we wish to evaluate

$$\oint p_x(E, x, p_y, p_z)\, \frac{dx}{2\pi} = cm u_\perp^2 / 2eB = \frac{u_\perp^2}{2\omega_c}$$

(let $\underset{\sim}{p}$ refer to momentum per unit mass)

$$H = \frac{p_x^2}{2} + \frac{p_z^2}{2} + \frac{1}{2}\left(p_y - \frac{eA_y(x)}{mc}\right)^2 \; ; \; \frac{eA_y(x)}{mc} = \omega_c x$$

$H \equiv$ energy E is a constant as the Hamiltonian is independent of time. p_z and p_y are constants, as H independent of y or z.

The

$$J = \oint p_x \frac{dx}{2\pi} = 2\int_{x_{min}}^{x_{max}} \frac{dx}{2\pi}\left[2E - p_z^2 - (p_y - \omega_c x)^2\right]^{1/2}$$

$$x_{max} = \left(\sqrt{2(E - p_z^2)} + p_y\right)/\omega_c$$

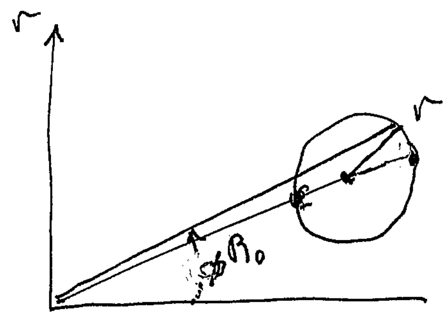
$$x_{min} = \left(-\sqrt{2(E - p_z^2)} + p_y\right)/\omega_c$$

let $-p_x + \omega_c x = z$, $u_\perp^2 = 2E - p_z^2$

$$J = \frac{2}{\omega_c}\int_{z_{min}}^{z_{max}} \frac{dz'}{2\pi}\left[u_\perp^2 - z^2\right]^{1/2} = \frac{u_\perp^2}{2\omega_c}$$

(3)

Now consider cylindrical coordinate system: Let $\hat{z} \parallel \underset{\sim}{B}$



$$2H = P_z^2 + P_r^2 + \left(P_\phi - \frac{re A_\phi}{mc}\right)^2 / r^2 \equiv 2E$$

Then $\frac{eA_\phi}{mc} = \frac{\omega_c r}{2}$

Note: $\nabla \times \hat{\phi} A_\phi = \hat{z} \frac{\partial (r A_\phi)}{r \partial r} = \underset{\sim}{B_0} \hat{z}$

In this coordinate system we wish to show

$$J = 2\int_{r_{min}}^{r_{max}} P_r(P_z, E, P_\phi, r) \frac{dr}{2\pi} = \frac{u_\perp^2}{2\omega_c} = \frac{E - P_z^2/2}{\omega_c}$$

with $P_r = \left[(2E - P_z^2) - \left(P_\phi - \frac{\omega_c r^2}{2}\right)^2 / r^2\right]^{1/2}$

Thus

$$J = \frac{2}{2\pi}\int_{r_{min}}^{r_{max}} dr \left[2E - P_z^2 - \left(P_\phi - \frac{\omega_c r^2}{2}\right)^2 / r^2\right]^{1/2}$$

$$= \frac{2}{2\pi}\int_{r_{min}}^{r_{max}} \frac{dr}{r} \left[(2E - P_z^2) r^2 - \left(P_\phi - \frac{\omega_c r^2}{2}\right)^2\right]^{1/2}$$

$$J = \frac{2}{2\pi} \int_{r_{min}}^{r_{max}} \frac{dr}{r} \left[u_\perp^2 r^2 - \left(P_\phi - \frac{\omega_c r^2}{2} \right)^2 \right]^{1/2}$$

$$= \frac{1}{\pi} \int_{r_{min}}^{r_{max}} \frac{dr}{r} \left[\left(P_\phi + \frac{u_\perp^2}{\omega_c} \right) \omega_c r^2 - P_\phi^2 - \frac{\omega_c^2 r^4}{4} \right]^{1/2}$$

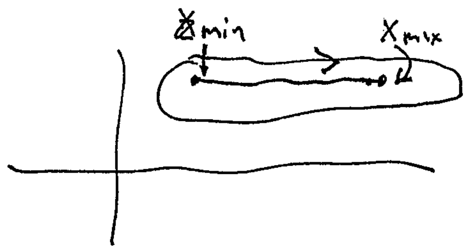
Let $\omega_c r^2 = x$, $\quad \frac{dr}{r} = \frac{1}{2} \frac{dx}{x}$

$$J = \frac{1}{\cancel{4}\, 4\pi} \int_{x_{min}}^{x_{max}} \frac{dx}{x} \left[4 \left(P_\phi + \frac{u_\perp^2}{\omega_c} \right) x - 4 P_\phi^2 - x^2 \right]^{1/2}$$

$$\left. \begin{matrix} x_{min} \\ x_{max} \end{matrix} \right\} = P_\phi + u_\perp^2/\omega_c \pm u_\perp \left[\frac{2 P_\phi}{\omega_c} + \frac{u_\perp^2}{2\omega_c^2} \right]^{1/2}$$

Now convert integral to a closed contour integral encircling x_{min} and x_{max}

$$J = \frac{1}{8\pi} \oint \frac{dz}{z} \left[4 \left(P_\phi + \frac{u_\perp^2}{\omega_c} \right) z - 4 P_\phi^2 - z^2 \right]^{1/2}$$



We distort contour to infinity, picking up pole contribution at $z = 0$, with residue $+i 2 P_\phi$

For large $|z|$ integral is

$$J = -\frac{P_\phi}{2} + \frac{i}{8\pi} \oint \frac{dz}{z}\, z \Big(\underbrace{1}_{\text{no contribution}} - \underbrace{\frac{4 \left(P_\phi + \frac{u_\perp^2}{\omega_c} \right)}{2z}}_{\text{pole contribution}} + \mathcal{O}\left(\frac{1}{z^2}\right) \Big)$$

$$= -\frac{P_\phi}{2} + \frac{1}{2} \left(P_\phi + \frac{u_\perp^2}{\omega_c} \right) = u_\perp^2 / 2\omega_c$$ as we wished to show

(4)

The proof of the constancy of adiabatic invariants goes like this:

Let us choose $\underset{\sim}{A} = B_0 x \hat{y}$ using cartesian coordinates.

We have

$$J = \oint \frac{dx}{2\pi} p_x(E, x, P_y, P_z)$$

To this we add $\int dy p_y + \int dz p_z$ and subtract the same thing, to form

$$J = \int d\underset{\sim}{r} \cdot \underset{\sim}{p} - \oint dz p_z - \oint dy p_y$$

The first integral can be written as $\oint \underset{\sim}{p} \cdot \underset{\sim}{v}\, dt$, where the time integration is over the true orbit. Then we have:

$$J = \oint dt\, \underset{\sim}{p} \cdot \underset{\sim}{v} - p_z \Delta z - p_y \Delta y$$

with Δz and Δy the distance moved in a period. We used that p_z and p_y are constants of motion. Also note that $\Delta y = 0$ after one period, but we keep this term for convenience. $\Delta z \neq 0$ in general.

Now the vector potential can change to $A \to A + \nabla \Phi$

In this case $\underset{\sim}{p} \to \underset{\sim}{p} + \frac{e\nabla\Phi}{mc}$

which follows from the Lagrangian formulation;

Thus, J, with the new momenta is in terms of the old coordinates, and momenta

$$J = \oint \underset{\sim}{p} \cdot \underset{\sim}{v}\, dt + \int v \cdot \nabla \Phi\, dt - P_z \Delta z - P_y \Delta y$$
$$- \int_{z_i}^{z_f} dz' \frac{\partial \Phi}{\partial z'}(x, y, z')$$
$$- \int_{y_i}^{y_f} dy' \frac{\partial \Phi}{\partial y'}(x, y', z)$$

where f & i subscripts are final and initial condition

Note $\int_{t_i}^{t_f} \underset{\sim}{v} \cdot \nabla \Phi\, dt' = \Phi(x_f, y_f, z_f) - \Phi(x_i, y_i, z_i)$

as $x_f = x_i$; $y_f = y_i$

$= \Phi(x_i, y_i, z_f) - \Phi(x_i, y_i, z_i)$

$\therefore$

$$\Delta J \equiv J\left(p + \frac{e\nabla\Phi}{mc}\right) - J(\underset{\sim}{p}) =$$
$$= \Phi(x_i, y_i, z_f) - \Phi(x_i, y_i, z_i)$$
$$- \int_{z_i}^{z_f} dz' \frac{\partial}{\partial z'} \Phi(x_i, y_i, z')$$
$$- \int_{y_i}^{y_f} dy' \frac{\partial}{\partial y'} \Phi(x_i, y', z_i)$$

(last term is zero since $y_f = y_i$)

$$= \Phi(x_i, y_i, z_f) - \Phi(x_i, y_i, z_i) - \left(\Phi(x_i, y_i, z_f) - \Phi(x_i, y_i, z_i)\right)$$
$$= 0, \quad \text{QED}$$

(3) Average drift velocity of a deeply trapped particle $(v_\perp^2 >> v_\parallel^2)$

$$\dot{\phi} \simeq \frac{B_\phi}{B} \frac{\sqrt{2(E - \mu B_0(1 - \frac{r}{R}\cos\theta)}}{R}$$

$$\simeq \sqrt{2(E - \mu B_0(1 - r/_R \cos\theta)} \,/\, R_0 \qquad (1)$$

Also note from conservation of momentum, the deviation of a particle from the flux surface $r = r_0$ where $v_\parallel = 0$ at $r = r_0$, is ~~found~~ to be:

$$P_\phi = \frac{e\psi}{mc} + R\sqrt{2(E - \mu B_0(1 - \frac{r}{R}\cos\theta)}$$

$$= \frac{e\psi(r_0)}{mc} + \frac{e\psi'(r_0)}{mc}(r - r_0) + R\sqrt{2(E - \mu B_0(1 - \frac{r_0}{R}\cos\theta)}$$

P_ϕ ''

$$\psi'(r_0) \simeq R B_\theta(r_0) = \frac{r_0 B_\phi}{q(r_0)}$$

$$\therefore \; r - r_0 = -\frac{qR}{r_0 \omega_c}\sqrt{2(E - \mu B_0(1 - \frac{r_0}{R}\cos\theta)} \qquad (2)$$

In Eq. (1) it turns out that a ~~most important~~ crucial variation is the variation in $r = r_0 + (r - r_0)$.

Substiting for r on the rhs of Eq. (1), and expanding, gives:

a

$$\dot{\phi} = \frac{\sqrt{2(E - \mu B_0(1 - \frac{r_0}{R}\cos\theta)}}{R_0} + \frac{\frac{\mu B_0(r - r_0)\cos\theta}{R^2}}{\sqrt{2(E - \mu B_0(1 - \cos\theta}}$$

(substitute for $r - r_0$ from Eq. 2)

$$\dot{\phi} = \frac{\sqrt{2(E - \mu B_0(1 - \frac{r_0}{R}\cos\theta)}}{R_0} - \frac{q\mu B_0}{\omega_c r_0 R_0}\cos\theta \qquad (3)$$

The last term already indicates the size of the answer when $\theta \ll 1$, so that $\cos\theta = 1$. The first term nearly does not contribute since $\sqrt{2(E - \mu B_0(1 - \frac{r_0}{R}\cos\theta)}$ oscillates and cancels over a period. So that it appears that $\bar{\dot{\phi}} \approx -q\mu B_0/\omega_c R_0$

However let us do this more systematically (we ~~may~~ pick up a significant correction term):

We note $\dot{\phi} = \dot{\theta}\frac{d\phi}{d\theta}$ and

$$\dot{\theta} \approx \frac{\sqrt{2(E - \mu B_0(1 - \frac{r}{R}\cos\theta)}}{q R_0} - \frac{\cos\theta\,(\mu B_0)}{\omega_c R^2 r_0} \qquad (4)$$

direct drift velocity contribution

expand in $r = r_0 + (r - r_0)$

$$\dot\theta = \sqrt{2(E - \mu B_0(1 - \frac{r_0}{R}\cos\theta)}\,/\,q R_0$$

$$+ \frac{(r - r_0)\mu B_0 \cos\theta}{q R_0^2 \sqrt{2(E - \mu B_0(1 - \frac{r_0}{R}\cos\theta)}}$$

$$- \frac{\mu B_0 \cos\theta}{\omega_c R_0 r}$$

Substitute for $r - r_0$ from Eq. (2)

$$\dot\theta = \sqrt{2(E - \mu B_0(1 - \frac{r_0}{R}\cos\theta)}\,/\,q R_0 \qquad (5)$$

$$- \frac{2\mu B_0}{r_0 R \omega_c}\cos\theta$$ ← (indeed an extra contribution)

Now using $\dot\phi = \dot\theta \frac{d\phi}{d\theta}$, and from (3) & (5)

$$\frac{d\phi}{d\theta} = \frac{\dot\phi}{\dot\theta} = \frac{\sqrt{2(E - \mu B_0(1 - \frac{r_0}{R}\cos\theta)}\,/\,R_0 - \frac{q\mu B_0}{\omega_c r_0 R_0}\cos\theta}{\frac{1}{qR_0}\left[\sqrt{2(E - \mu B_0)(1 - \frac{r_0}{R}\cos\theta)} - 2q\frac{\mu B_0}{\omega_c r_0 R_0}\cos\theta\right]}$$

$$\simeq q\left[1 + \frac{q\mu B_0}{\omega_c r_0 R}\frac{\cos\theta}{\sqrt{2(E - \mu B_0)(1 - \frac{r_0}{R}\cos\theta)}}\right]$$

Now θ oscillates periodically; Then, $\Delta\phi$ the change of ϕ in one period of poloidal oscillation, $T_b = \frac{2\pi}{\omega_b}$, $\omega_b = \frac{(\mu B_0)^{1/2}}{qR}\left(\frac{r_0}{R}\right)^{1/2}$, is given by,

↳ (see HW solution)

(c)

(d)

$$\Delta\phi \simeq \oint d\theta\, q + \oint d\theta \frac{q^2 \mu B_0 \cos\theta / \omega_c r_0}{\sqrt{2(E - \mu B_0(1 - \frac{r_0}{R}\cos\theta)}}$$

with $\cos\theta \simeq 1 - \theta^2/2$

Now, $\oint q\, d\theta = 0$

$$\therefore\ \Delta\phi \simeq q^2 \frac{(\mu_0 B_0)^{1/2} R^{1/2}}{\omega_c r_0^{3/2}} \oint \frac{d\theta}{\left[\frac{2(E - \mu B_0(1 - \frac{r_0}{R})}{\mu B_0} - \theta^2\right]^{1/2}}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel\ 2\pi$$

$$= 2\pi q^2 \frac{(\mu B_0)^{1/2}}{\omega_c r_0^{3/2}}$$

The mean toroidal drift velocity is (recall $T_b = \frac{2\pi}{(\mu B_0)^{1/2}} R_0 q \left(\frac{R}{r_0}\right)^{1/2}$)

$$\frac{\Delta\phi}{T_b} = \frac{2\pi q^2 (\mu B_0)^{1/2} R^{1/2}}{\omega_c r_0^{3/2}} \frac{(\mu B_0)^{1/2}}{2\pi R_0 q} \left(\frac{r_0}{R}\right)^{1/2}$$

$$= q \frac{(\mu B_0)}{\omega_c r_0 R} \equiv \overline{\dot\phi}$$

We see we did get a significant correction that changed sign of $\overline{\dot\phi}$.

Also not $\overline{\dot\phi}$ is a factor $q \frac{R}{r}$ larger than one might expect from instantaneous $\frac{\text{drift.}}{V_\parallel / R}$ The $\overline{\dot\phi}$ arises because the average of does not quite cancel during an oscillation period. (d)

Trapping Frequency of a trapped particle

$$\frac{d^2 s}{dt^2} = -\mu \underset{\sim}{b} \cdot \underset{\sim}{\nabla} B \qquad \mu = \frac{v_\perp^2}{2B}$$

$s \equiv$ distance along a field line

$$\frac{ds}{B} = \frac{r\,d\theta}{B_\theta}$$

$$ds = \frac{B}{B_\theta} r\,d\theta \simeq R\,q(r)\,d\theta$$

$$B = B_0 \left(1 - r/R_0 \cos\theta\right)$$

$$\underset{\sim}{b} \cdot \underset{\sim}{\nabla} B = \frac{\partial B}{\partial s} = \frac{1}{Rq} \frac{\partial}{\partial \theta} B_0 \left(1 - \frac{r}{R_0} \cos\theta\right)$$

$$= \frac{B_0}{Rq^2} \frac{r}{R_0} \sin\theta$$

$\therefore$

$$\frac{d^2 s}{dt^2} = Rq \frac{d^2\theta}{dt^2} = \frac{-\mu B_0 r}{Rq\,R} \sin\theta$$

$$\boxed{\frac{d^2\theta}{dt^2} = \frac{-\mu B_0}{(Rq)^2} \frac{r}{R} \sin\theta}$$

This is the equation of a pendulum.
The frequency of the deepest trapped particle is $(\sin\theta \simeq \theta)$

$$\omega_b^2 = \frac{(\mu B_0)^{1/2}}{qR} \left(\frac{r_0}{R}\right)^{1/2} = \frac{V_{\perp 0}}{qR_0} \left(\frac{r_0}{2R}\right)^{1/2}$$

To solve for the period note thus

$$\frac{d}{dt}\frac{d\theta}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta}\frac{d\theta}{dt} = \frac{1}{2}\frac{d}{d\theta}\left(\frac{d\theta}{dt}\right)^2$$

$$\frac{\mu B_0 r_0}{q^2 R_0^3}\sin\theta = \omega_b^2 \frac{\partial}{\partial\theta}(\cancel{E}-\cos\theta), \quad \text{wit } \mathcal{E} \text{ a constant}$$

$$\omega_{b0}^2 = \frac{\mu B_0 r_0}{q^2 R_0^3}$$

Thus

$$\frac{d}{d\theta}\left[\frac{\dot\theta^2}{2} + \omega_b^2(\cancel{E}\cos\theta)\right] = 0$$

$$\boxed{\frac{\dot\theta^2}{2} = \omega_{b0}^2\,(\cancel{E}\,\mathcal{E}\cos\theta)}\,; \quad \text{For trapping } -1 < \mathcal{E} < 1 \quad (1)$$

Also

$$\dot\theta = \frac{V_\parallel}{qR} = \frac{\sqrt{2(E - \mu B_0(1-\cos\theta\frac{r_0}{R}))}}{qR}$$

$$\frac{E}{\mu B_0} = \lambda \qquad 1-\epsilon < \lambda < 1+\epsilon$$

$$\epsilon = r/R$$

In comparison

$$\mathcal{E} = (\lambda - 1)\frac{R}{r};$$

The turning point, θ_T, is

$$\lambda - 1 + \cos\theta_T\, r/R$$

$$\cos\theta_T = \frac{R}{r}(1-\lambda), \quad \theta_T = \cos^{-1}\left(\frac{1-\lambda}{\epsilon}\right)$$

The solution of (1) is

$$\omega_{b0} t = \int^{\theta} \frac{d\theta'}{\sqrt{2}(\mathcal{E} - \cos\theta')^{1/2}}; \quad \omega_{b0} T = \frac{4}{\alpha}\int_0^{\theta_{max}} \frac{d\theta'}{[2(\mathcal{E} - \cos\theta')]^{1/2}} \quad (3)$$

$$\boxed{\omega_{b_0} T = 2\sqrt{2}\int_0^{\theta_{max}} \frac{d\theta'}{\left[(\lambda - 1)\frac{R}{r_0} + \cos\theta'\right]^{1/2}}}$$

$$\omega_{b_0}^2 = \frac{\mu B_0}{q^2 R_0^2}\frac{r_0}{R_0}, \quad \cos\theta_T = (-\lambda + 1)\frac{R}{r}$$

If $\theta \ll 1$

$$\boxed{\omega_{b_0} T \simeq 2\pi\left[1 - \frac{1}{16}\theta_T^2\right]} \qquad \omega_{b_0} = \frac{(\mu B_0)^{1/2}}{q R_0}\left(\frac{r_0}{R}\right)^{1/2}$$

if $\theta_T \ll 1$

as $\cos\theta_T \simeq 1 - \frac{\theta_T^2}{2} = \frac{R_0}{r_0}(1-\lambda)$

$$\theta_T \simeq \left[2\left(1 - \frac{R}{r_0}(1-\lambda)\right)\right]^{1/2}$$

$$\lambda = \left(1 - \frac{r_0}{R}\right) + \delta\lambda$$

$$\theta_T = \left(2\delta\lambda\frac{R}{r_0}\right)^{1/2}$$

$$1 - \frac{\theta_T^2}{2} = 1 - \delta\lambda\frac{R_0}{r_0}$$

$$\therefore \boxed{\theta_T = \left(2\frac{\delta\lambda R}{r_0}\right)^{1/2}}$$

if $\theta_T \ll 1$

or $\delta\lambda \ll \frac{r_0}{R_0}$

Also note as $\theta_T \to \pi$, $T \to \infty$ logarithmically: at a logarithmic rate

$$\omega_{b_0} T = 2\sqrt{2}\int_0^{\theta_{max}} \frac{d\theta'}{[-\cos\theta_T + \cos\theta]^{1/2}}$$

$$\propto 4\ln\left(\pi/\delta\theta_T\right) + O(1)$$

$\theta_T = \pi - \delta\theta_T$

$\theta = \pi - \delta\theta$

near $\theta \sim \pi$,

$\cos\theta = -\cos\delta\theta = -1 + \frac{\delta\theta^2}{2}$

$\cos\theta_T = -\cos\delta\theta_T = -1 + \frac{\delta\theta_T^2}{2}$

(C)

Solev'ev equilibrium (spherical boundary)

$$\psi = \psi_0 \frac{r^2}{r_0^4}\left(2r_0^2 - r^2 - z^2\right)$$

Solves equation

$$\nabla \cdot \frac{1}{r^2}\nabla\psi = - \frac{\partial p(\psi)}{\partial \psi} = \text{Const}; \quad \psi < \psi_0$$

$$\parallel$$

$$\frac{1}{r^2}\frac{\partial^2\psi}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}\psi = -\frac{\partial p}{\partial \psi}$$

$$\frac{1}{r}\frac{\partial \psi}{\partial r} = \psi_0\left[\frac{2}{r_0^4}\left(2r_0^2 - r^2 - z^2\right) - \frac{2r^2}{r_0^4}\right] +$$

$$\overbrace{-8\psi_0/r_0^4}^{} \quad = \psi_0\left[\frac{4r_0^2 - 4r^2 - 2z^2}{r_0^4}\right]$$

$$-2\psi_0/r_0^4 \quad \parallel$$

$$\frac{\partial^2\psi}{\partial z^2}\frac{1}{r^2} + \frac{\partial}{r\,\partial r}\frac{1}{r}\frac{\partial\psi}{\partial r} = -\frac{10\psi_0}{r_0^4} = -\frac{\partial p}{\partial \psi}$$

$$\therefore \quad \psi_0 = \frac{1}{10}\frac{\partial p}{\partial \psi} r_0^4, \quad \Delta p = 10\psi_0^2/r_0^4 = P(|\psi| = \psi_{max})$$

(Note $\psi_0 < 0$)
to have
$P(\psi = |\psi_{max}|) > 0$
$P(\psi = 0) = 0$

~~At~~

Now to match:
In spherical coordinates GS equation is

$$\frac{\partial^2\psi}{\partial \rho^2} + \frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\psi = 0 \quad (\text{in vaccuum})$$

$\rho = r^2 + z^2$, $\rho\sin\theta = r$

Solution in vacuum is

$$\psi = \sin^2\theta \left(A\rho^2 + \frac{B}{\rho} \right) \qquad \rho \geq \sqrt{2}\, r_0$$

~~(check this)~~

Solution from ~~vacuum~~ plasma at interface with vacuum is

$$\psi = \frac{\psi_0 r^2}{r_0^4} \left[2r_0^2 - \rho^2 \right]$$

$$= \psi_0 \frac{\rho^2}{r_0^4} \sin^2\theta \left[2r_0^2 - \rho^2 \right]$$

~~We n~~ At $\rho = \sqrt{2}\, r_0$ $\psi(\rho, \theta) = 0$ at plasma. Equate ψ from vacuum

(this is equivalent to requiring $\underline{B} \cdot \hat{\underline{\rho}}$ to be continuous)

$$A\, 2r_0^2 + B/\sqrt{2}\, r_0 = 0, \qquad \frac{A}{B} = \frac{-1}{2\sqrt{2}\, r_0^3}$$

Now we need $\frac{\partial \psi}{\partial \rho}$ continuous (i.e B_θ continuous)

$$\left.\frac{\partial \psi_v}{\partial \rho}\right|_{\rho = \sqrt{2} r_0} = \sin^2\theta \left[2A\sqrt{2}\, r_0 - \frac{B}{2r_0^2} \right] = A \sin^2\theta \left[2\sqrt{2}\, r_0 + \sqrt{2}\, r_0 \right] = 3A\sqrt{2}\, r_0 \sin^2\theta$$

$$\parallel$$

$$\left.\frac{\partial \psi_p}{\partial \rho}\right|_{\rho = \sqrt{2} r_0} = \sin^2\theta \frac{\psi_0}{r_0^4} \left[- 2\rho^3 \big|_{\rho = \sqrt{2} r_0} \right] = \left(-4\sqrt{2} \frac{r_0^3}{r_0^4} \psi_0 \right) \sin^2\theta = - \frac{4\sqrt{2}\psi_0 \sin^2\theta}{r_0}$$

$$\therefore \quad A = -\frac{4}{3r_0^2}\psi_0$$

$$\psi_v = -\frac{4\psi_0}{3r_0^2}\sin^2\theta\left(\rho^2 - \frac{2\sqrt{2}\, r_0^3}{\rho}\right)$$

A vertical field, $\underset{\sim}{B}_{ver}$, is given by

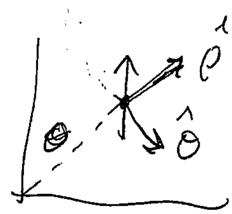
$$\underset{\sim}{B}_{ver} = \underset{\sim}{\nabla}\phi \times \underset{\sim}{\nabla}\psi_{vr}$$

with $\psi_{vr} = \dfrac{\mp B_{vr}\rho^2}{2}\sin^2\theta$

as $$\underset{\sim}{B}_{ver} = \frac{1}{\rho\sin\theta}\left(\hat\phi \times \hat r\frac{\partial}{\partial\rho}\psi_v + \hat\phi \times \frac{\hat\theta}{\rho}\frac{\partial\psi}{\partial\theta}\right)$$

$$= \frac{B_{vr}}{\rho\sin\theta}\left[-\hat\theta\,\rho\sin^2\theta + \hat\rho\,\rho\sin\theta\cos\theta\right]$$

$$= B_{vr}\left[\hat\rho\cos\theta - \hat\theta\sin\theta\right]$$



expression for vertical field

$$\therefore \quad \frac{B_v}{2} = -\frac{\cancel{4}}{3}\frac{\psi_0}{r_0^2} = \frac{\sqrt{P_0}\, r_0^2}{r_0^2\sqrt{10}}\,\frac{\cancel{4}}{3}$$

where we used $P_0 = 10\psi_0^2/r_0^4$ obtained earlier and that $\psi_0 < 0$

$$\beta_v = \frac{2P_0}{B_v^2} = \frac{180}{64} = \frac{90}{32} = \frac{45}{16}$$ (quite a large number)